# DISPERSING COCYCLES AND MIXING FLOWS UNDER FUNCTIONS

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ABSTRACT. Let $T$ be a measure-preserving and mixing action of a countable abelian group $G$ on a probability space $(X, \mathcal{S}, \mu)$ and $A$ a locally compact second countable abelian group. A cocycle $c \colon G \times X \longmapsto A$ for $T$ *disperses* if $\lim_{g \to \infty} c(g, \cdot) - \alpha(g) = \infty$ in measure for every map $\alpha \colon G \longrightarrow A$. We prove that such a cocycle $c$ does not disperse if and only if there exists a compact subgroup $A_0 \subset A$ such that the composition $\theta \circ c \colon G \times X \longrightarrow A/A_0$ of $c$ with the quotient map $\theta \colon A \longrightarrow A/A_0$ is *trivial* (i.e. cohomologous to a homomorphism $\eta \colon G \longrightarrow A/A_0$).

This result extends a number of earlier characterizations of coboundaries and trivial cocycles by tightness conditions on the distributions of the maps $\{c(g, \cdot) : g \in G\}$ and has implications for flows under functions: let $T$ be a measure-preserving and ergodic automorphism of a probability space $(X, \mathcal{S}, \mu)$, $f \colon X \longrightarrow \mathbb{R}$ a positive Borel map with $\int f \, d\mu = 1$, and let $T^f$ be the flow under the function $f$ with base $T$. Our main result implies that, if $T$ is mixing and $T^f$ is weakly mixing, or if $T$ is ergodic and $T^f$ is mixing, then the cocycle $\mathbf{f} \colon \mathbb{Z} \times X \longrightarrow \mathbb{R}$ defined by $f$ disperses. The latter statement answers a question raised by Mariusz Lemańczyk in [7].

## 1. DISPERSION OF COCYCLES

**Definition 1.1.** Let $T \colon g \mapsto T_g$ be a measure-preserving action of a countable additive abelian group $G$ on a standard probability space $(X, \mathcal{S}, \mu)$, and let $A$ be a locally compact second countable additive abelian group with identity element 0. A Borel map $c \colon G \times X \longrightarrow A$ is a *cocycle* for $T$ if

$$c(g, T_h x) + c(h, x) = c(g + h, x)$$

for every $g, h \in G$ and $x \in X$. Two cocycles $c, c' \colon G \times X \longrightarrow A$ are *cohomologous* if there exists a Borel map $b \colon X \longrightarrow A$ such that

$$c(g, x) = c'(g, x) + b(T_g x) - b(x) \tag{1.1}$$

for every $g \in G$ and $\mu$-a.e. $x \in X$. The map $b$ in (1.1) is called a *transfer function*. If $c$ is cohomologous to the zero cocycle $c' \equiv 0$ then $c$ is a *coboundary* with *transfer* (or *cobounding*) function $b$.

Let $c \colon G \times X \longrightarrow A$ be a cocycle. The cocycle $c$ is a *homomorphism* if the map $c(g, \cdot) \colon X \longrightarrow A$ is constant for every $g \in G$, and $c$ is *trivial* if it is cohomologous to a homomorphism.

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The cocycle $c$ is *bounded (in measure) on a subset* $H \subset G$ if there exists, for every $\varepsilon > 0$, a compact subset $C \subset A$ with

$$\mu(\{x : c(g, x) \in C\}) > 1 - \varepsilon \qquad (1.2)$$

for every $g \in H$.

The cocycle $c$ is *translation-bounded on a subset* $H \subset G$ if there exist, for every $\varepsilon > 0$, a map $\alpha \colon H \longrightarrow A$ and a compact subset $C \subset A$ with

$$\mu(\{x : c(g, x) - \alpha(g) \in C\}) > 1 - \varepsilon \qquad (1.3)$$

for every $g \in H$.

If $H = G$ in (1.2) or (1.3) then $c$ is said to be *bounded* or *translation-bounded*, respectively.

Finally, the cocycle $c$ *disperses* if $\lim_{g \to \infty} c(g, \cdot) - \alpha(g) = \infty$ in measure for every map $\alpha \colon G \longrightarrow A$ or, equivalently, if

$$\limsup_{g \to \infty} \mu(\{x \in X : c(g, x) - a \in C\}) = 0 \qquad (1.4)$$

for every compact set $C \subset A$.

It has long been known that a cocycle $c \colon G \times X \longrightarrow \mathbb{R}$ is a coboundary if and only if it is bounded in the sense of Definition 1.1 (cf. [9, Theorem 11.8]). More generally, if $A$ is a locally compact second countable abelian group and $c \colon G \times X \longrightarrow A$ a bounded cocycle, then $c$ is cohomologous to a cocycle taking values in a compact subgroup $A_0 \subset A$ (for extensions of this result with varying degrees of generality see [8, Theorem 5.2], [10, Theorem 4.7] and [1]). Furthermore, if $A = \mathbb{R}$, then $c$ is trivial if and only if it is translation-bounded (cf. [8, Theorem 6.2]).

More recently it was shown that, if $T$ is mixing, then boundedness (or translation-boundedness) of a cocycle $c \colon G \times X \longrightarrow \mathbb{R}$ on an infinite subset $H \subset G$ also implies triviality. The first published result in this direction is [1, Theorem 2], where it is proved that, for a mixing action of $G = \mathbb{Z}$, translation-boundedness of a cocycle $c \colon \mathbb{Z} \times X \longrightarrow \mathbb{R}$ on some infinite subset $H \subset G$ implies triviality of $c$, and boundedness of $c$ on $H$ implies that $c$ is a coboundary. These results can break down for $\mathbb{Z}$-actions which are only mildly mixing (cf. [1]).

In this note we prove the following extension of [1, Theorem 2].

**Theorem 1.2.** *Let $T$ be a measure-preserving and mixing action of a countable abelian group $G$ on a standard probability space $(X, \mathcal{S}, \mu)$, $A$ a locally compact second countable abelian group and $c \colon G \times X \longmapsto A$ a cocycle for $T$. The following conditions are equivalent.*

(1) *The cocycle $c$ does not disperse (cf. (1.4));*
(2) *There exists a compact subgroup $A_0 \subset A$ such that the composition $\theta \circ c \colon G \times X \longrightarrow A/A_0$ of $c$ with the quotient map $\theta \colon A \longrightarrow A/A_0$ is a trivial cocycle.*

For the proof of Theorem 1.2 we need a little bit of notation. Let $T$ be a continuous action of a countable abelian group $G$ on a compact metrizable space $X$ and $\mu$ be a $T$-invariant Borel probability measure on $X$. We denote by $\Delta = \{(x, x) : x \in X\}$ the diagonal in $X \times X$ and define the 'diagonal'

probability measure $\mu_\Delta$ on $X \times X$ by setting

$$\mu_\Delta(\{(x,x) : x \in B\}) = \mu(B)$$

for every Borel set $B \subset X$. For every $g \in G$, the 'off-diagonal' probability measure

$$\nu_g = (T_g \times \mathrm{Id}_X)_*(\mu_\Delta) \tag{1.5}$$

is the self-joining of $\mu$ supported on the graph $\{(T_gx, x) : x \in X\}$ of $T_g$.

Theorem 1.2 is an easy consequence of the following proposition.

**Proposition 1.3.** *Let $T$ be a continuous action of a countable abelian group $G$ on a compact metrizable space $X$, $\mu$ a $T$-invariant and weakly mixing Borel probability measure on $X$, $A$ a locally compact second countable abelian group and $c\colon G \times X \longrightarrow A$ a cocycle for $T$. Suppose that there exists a sequence $(h_n, \ n \geq 1)$ in $G$ with the following properties.*

(1) $\lim_{n\to\infty} \nu_{h_n} = \mu \times \mu$ *in the topology of weak convergence;*
(2) *There exist an $\varepsilon > 0$, a compact set $C \subset A$ and elements $\alpha_n \in A$, $n \geq 1$, with*

$$\mu(\{x \in X : c(h_n, x) - \alpha_n \in C\}) \geq \varepsilon \tag{1.6}$$

  *for every $n \geq 1$.*

*Then there exists a compact subgroup $A_0 \subset A$ such that the composition $\theta \circ c\colon G \times X \longrightarrow A/A_0$ of $c$ with the quotient map $\theta\colon A \longrightarrow A/A_0$ is trivial.*

For the proof of Proposition 1.3 we require an elementary lemma closely related to [10, Lemma 4.4].

**Lemma 1.4.** *Let $T$ be a measure-preserving and ergodic action of a countable abelian group $G$ on a standard probability space $(X, \mathcal{S}, \mu)$, $A$ a locally compact second countable abelian group and $c\colon G \times X \longrightarrow A$ a cocycle for $T$. We define the skew-product action $T^{(c)}$ of $G$ on $Y = X \times A$ by setting*

$$T_g^{(c)}(x, a) = (T_gx, c(g, x) + a)$$

*for every $g \in G$ and $(x, a) \in Y$.*

*If there exists a $T^{(c)}$-invariant probability measure $\rho$ on $Y$ with $\pi_*(\rho) = \mu$ (where $\pi\colon Y \longrightarrow X$ is the first coordinate projection), then $c$ is cohomologous to a cocycle $c'$ taking values in some compact subgroup $A_0 \subset A$.*

*Conversely, if $c$ is cohomologous — with transfer function $b$ — to a cocycle $c'\colon G \times X \longrightarrow A_0$, where $A_0 \subset A$ is a compact subgroup with normalized Haar measure $\lambda_{A_0}$, then the probability measure $\rho$ on $Y$, defined by*

$$\int f \, d\rho = \iint f(x, a + b(x)) \, d\lambda_{A_0}(a) \, d\mu(x)$$

*for every bounded Borel map $f\colon Y \longrightarrow \mathbb{R}$, is $T^{(c)}$-invariant and $\pi_*(\rho) = \mu$.*

*Proof.* Choose a Borel measurable family of probability measures $\{\rho_x : x \in X\}$ on $A$ such that

$$\int f \, d\rho = \iint f(x, t) \, d\rho_x(t) \, d\rho(x)$$

for every bounded Borel map $f\colon Y \longrightarrow A$. Since $\rho$ is $T^{(c)}$-invariant,

$$\rho_{T_gx}(B + c(g, x)) = \rho_x(B) \quad \text{for } \mu\text{-a.e. } x \in X,$$

for every Borel set $B \subset A$ and every $g \in G$. We fix a nonnegative continuous map $\phi \colon A \longrightarrow \mathbb{R}$ with compact support such that $\phi(0) > 0$. For every $x \in X$, the map $a \mapsto \int \phi(a + s) \, d\rho_x(s) = \psi(x, a)$ from $A$ to $\mathbb{R}$ is continuous, not everywhere equal to zero, and vanishes at infinity. Furthermore, the resulting Borel map $\psi \colon X \times A \longrightarrow \mathbb{R}$ is $T^{(c)}$-invariant, and for some $\varepsilon > 0$ the Borel set

$$K = \{(x, a) \in Y : \psi(x, a) \geq \varepsilon\}$$

is nonempty and again $T^{(c)}$-invariant. For every $x \in X$, the set

$$K_x = \{a \in A : (x, a) \in K\}$$

and the subgroup

$$A_x = \{a \in A : a + K_x = K_x\}$$

are both compact, and the ergodicity of $T$ and the $T^{(c)}$-invariance of $K$ imply that there exists a compact subgroup $A_0 \subset A$ with $A_x = A_0$ for $\mu$-a.e. $x \in X$. By using one of the standard selection theorems (cf. e.g. Kunugui's theorem in [5]–[6]) we can choose a Borel map $b \colon X \longrightarrow A$ with $b(x) \in K_x$ for $\mu$-a.e. $x \in X$ and obtain that $c(g, x) + b(x) - b(T_g x) \in A_0$ for every $g \in G$ and $\mu$-a.e. $x \in X$.

The final statement of the lemma is obvious. $\qquad\square$

*Proof of Proposition* 1.3. Let $\bar{T} \colon g \mapsto T_g \times T_g$ denote the diagonal action of $G$ on $\bar{X} = X \times X$, and let $\bar{c} \colon G \times \bar{X} \longrightarrow A$ be the cocycle

$$\bar{c}(g, (x_1, x_2)) = c(g, x_1) - c(g, x_2)$$

for $\bar{T}$. The cocycle equation (2.1) yields that

$$c(g, T_{h_n} x) - c(g, x) = c(h_n, T_g x) - c(h_n, x) \tag{1.7}$$

for every $g \in G$ and $n \geq 1$. For every $n \geq 1$ we define $b_n \colon \bar{X} \longrightarrow A$ by $b_n(x_1, x_2) = c(h_n, x_2) - \alpha_n$ and conclude from (1.7) that

$$\bar{c}(g, (x_1, x_2)) = b_n \circ \bar{T}_g(x_1, x_2) - b_n(x_1, x_2)$$

for every $g \in G$ and $(x_1, x_2) \in (T_{h_n} \times \mathrm{Id}_X)(\Delta)$, i.e. that $\bar{c}$ is a coboundary with cobounding function $b_n$ with respect to the $\bar{T}$-invariant measure $\nu_n = \nu_{h_n}$. We denote by $\rho_n$ the probability measure on $Y = \bar{X} \times A$ with

$$\rho_n(\{(T_{h_n} x, x, b_n(T_{h_n} x, x)) : x \in B\}) = \mu(B)$$

for every Borel set $B \subset X$ and observe as in Lemma 1.4 that $\rho_n$ is the unique $T^{(c)}$-invariant probability measure supported on the graph of $b_n$ with $\pi_*(\rho_n) = \nu_n$, where $\pi \colon \bar{X} \times A \longrightarrow \bar{X}$ is the projection map. If

$$A_n = \{x \in X : b_n(x) \in C\},$$

where $C \subset A$ is the compact set appearing in (1.6), then

$$\rho_n(\bar{X} \times C) = \nu_n(T_h^{-1} A_n \times A_n) = \mu(A_n) \geq \varepsilon \tag{1.8}$$

for every $n \geq 1$.

By going over to a subsequence of $(h_n)$, if necessary, we may take it that the sequence of probability measures $(\rho_n)$ converges vaguely to a finite measure $\rho$ on $Y$, i.e. that $\lim_{n \to \infty} \int f \, d\rho_n = \int f \, d\rho$ for every continuous function $f \colon Y \longrightarrow A$ with compact support. According to (1.8),

$$\rho(\bar{X} \times C) \geq \limsup_{n \to \infty} \rho_n(\bar{X} \times C) \geq \varepsilon,$$

which implies that $\rho$ is nonzero. We set $\nu = \pi_*(\rho)$ and claim that

$$\nu(B) \leq (\mu \times \mu)(B) \tag{1.9}$$

for every Borel set $B \subset \bar{X}$.

Indeed, let $f: \bar{X} \longrightarrow A$ be a continuous function, $\mathcal{U} \subset A$ an open neighbourhood of the identity with compact closure, $(D_m, m \geq 1)$ a sequence of compact subsets of $A$ with $D_{m+1} \supset D_m + \mathcal{U}$ for every $m \geq 1$ and $\bigcup_{m \geq 1} D_m = A$, and let, for every $m \geq 1$, $\phi_m: A \longrightarrow A$ be a continuous map with $\phi_m(a) = 1$ for $a \in D_m$ and $\phi_m(a) = 0$ for $a \notin (D_m + \mathcal{U})$. We set $f_m(x_1, x_2, a) = f(x_1, x_2)\phi_m(a)$ and observe that

$$\int f_m \, d\rho = \lim_{n \to \infty} \int f_m \, d\rho_n \leq \lim_{n \to \infty} \int f \, d\nu_n = \int f \, d(\mu \times \mu).$$

By letting $m \to \infty$ we obtain that

$$\int f \, d\nu = \sup_{m \geq 1} \int f_m \, d\rho \leq \int f \, d(\mu \times \mu).$$

As $f$ was arbitrary, this proves (1.9).

Since each of the probability measures $\rho_n$ is invariant under the skew-product action

$$\bar{T}_g^{(\bar{c})}(x_1, x_2, a) = (T_g x_1, T_g x_2, \bar{c}(g, (x_1, x_2)) + a)$$

of $G$ on $Y$, the same[1] is true for $\rho$, and hence the measure $\nu$ on $\bar{X}$ is invariant under $\bar{T}$. From (1.9) and the ergodicity of $\mu \times \mu$ it is clear that $\frac{1}{\nu(\bar{X})}\nu = \nu'$ is a $\bar{T}$-invariant probability measure on $\bar{X}$ which is absolutely continuous with respect to — and hence equal to — $\mu \times \mu$, and that the probability measure $\rho' = \frac{1}{\rho(Y)}\rho = \frac{1}{\nu(\bar{X})}\rho$ on $Y$ is invariant under $\bar{T}^{(\bar{c})}$ and satisfies that $\pi_*(\rho') = \mu \times \mu$.

By Lemma 1.4 there exists a compact subgroup $A_0 \subset A$ such that $\theta \circ \bar{c}$ is a coboundary, where $\theta: A \longrightarrow A' = A/A_0$ is the quotient map.

In order to simplify notation a little we set $\bar{c}' = \theta \circ \bar{c}: G \times \bar{X} \longrightarrow A'$ and $c' = \theta \circ c: G \times X \longrightarrow A'$. In the notation of [8, (6.1) and Theorem 6.2 (4)] we have proved that $j_*(c') = 0$, i.e. that $c' \in \Gamma_2(A')$ in the notation of [8, (7.5)]. As $\mu$ is weakly mixing, the triple diagonal action $T \times T \times T$ of $G$ on $(X \times X \times X, \mu \times \mu \times \mu)$ is ergodic, and [8, Corollary 7.2] shows that $\Gamma_2(A') = \Gamma_0(A')$ in the notation of [8, (7.3)–(7.5)]. Hence $c'$ is a homomorphism in the terminology of [8] or trivial in our terminology. $\square$

*Proof of Theorem* 1.2. We assume without loss in generality that $X$ is a compact metric space and that the $G$-action $T$ on $X$ is continuous (cf. [11]).

If (1.4) is violated, then there exist an $\varepsilon > 0$, a compact set $C \subset A$, an infinite subset $H \subset G$, and elements $\alpha_h \in A$, $h \in H$, with

$$\mu(\{x \in X : c(h, x) - \alpha_h \in C\}) \geq \varepsilon$$

for every $h \in H$. We can thus choose a sequence $(h_n)$ in $H$ with $\lim_{n \to \infty} h_n = \infty$ which satisfies the conditions of Proposition 1.3. Hence there exists a

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[1]In order to see this, consider the ring $\mathcal{R}$ of all Borel sets $B \subset Y$ with the property that $\bar{T}_g^{(\bar{c})}(B)$ has compact closure and boundary measure $\rho(\partial(\bar{T}_g^{(\bar{c})}B)) = 0$ for every $g \in G$. By assumption, $\rho(B) = \lim_{n \to \infty} \rho_n(B) = \lim_{n \to \infty} \rho_n(\bar{T}_g^{(\bar{c})}B) = \rho(\bar{T}_g^{(\bar{c})}B)$ for every $g \in G$ and $B \in \mathcal{R}$. Since $\mathcal{R}$ generates the Borel field of $Y$ this proves that $\rho$ is $\bar{T}^{(\bar{c})}$-invariant.

compact subgroup $A_0 \subset A$ such that the composition $\theta \circ c \colon G \times X \longrightarrow A/A_0$ of $c$ with the quotient map $\theta \colon A \longrightarrow A/A_0$ is trivial. This proves that $(1) \Rightarrow (2)$, and the reverse implication $(2) \Rightarrow (1)$ is obvious. $\qquad\square$

## 2. Mixing flows under functions

In order to apply Theorem 1.2 (or, more precisely, Proposition 1.3) to mixing properties of flows under functions we let $T$ be a measure-preserving automorphism of a standard probability space $(X, \mathcal{S}, \mu)$ and $f \colon X \longrightarrow \mathbb{R}$ a Borel map with $\int f\, d\mu = 1$ and $f(x) > 0$ for every $x \in X$. For every $n \in \mathbb{Z}$ and $x \in X$ we set

$$\mathbf{f}(n, x) = \begin{cases} \sum_{k=0}^{n-1} f(T^k x) & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -\mathbf{f}(-n, T^n x) & \text{if } n < 0. \end{cases} \tag{2.1}$$

The resulting map $\mathbf{f} \colon \mathbb{Z} \times X \longrightarrow \mathbb{R}$ is a cocycle for $T$ (or for the $\mathbb{Z}$-action $n \mapsto T^n$ on $(X, \mathcal{S}, \mu)$). We define an equivalence relation $R^f$ on $X \times \mathbb{R}$ by saying that

$$(x, t) \sim (T^n x, t - \mathbf{f}(n, x))$$

for every $(x, t) \in X \times \mathbb{T}$ and $n \in \mathbb{Z}$. The 'vertical' flow $S_t \colon (x, t') \mapsto (x, t + t')$, $(x, t') \in X \times \mathbb{R}$, $t \in \mathbb{R}$, preserves this equivalence relation and thus induces a flow $t \mapsto S_t^f$ on the space $(X \times \mathbb{R})_{R^f}$ of equivalence classes of the relation $R^f$. The set

$$X^f = \{(x, t) : x \in X, 0 \le t < f(x)\},$$

intersects each equivalence class of $R^f$ in exactly one point and may thus be identified with $(X \times \mathbb{R})_{R^f}$. We denote by $\mathcal{S}^f$ the Borel field of $X^f \subset X \times \mathbb{R}$, write $\lambda$ for the Lebesgue measure on $\mathbb{R}$, $\mu^f$ for the restriction to $X^f \subset X \times \mathbb{R}$ of the product measure $\mu \times \lambda$, and $T^f$ for the measure-preserving flow induced by $S^f$ on the probability space $(X^f, \mathcal{S}^f, \mu^f)$. This flow is usually called the *flow under the function $f$ with base $T$.*

Several authors have studied conditions on $f$ for a given ergodic base $T$ which determine whether the flow $T^f$ is mixing (cf. e.g. [3], [4], [2], [7] and the references listed there). In [7] the author proves the following result under the additional hypothesis that $T$ is an irrational rotation on $X = \mathbb{R}/\mathbb{Z}$, and asks whether Corollary 2.1 (under hypothesis (2)) holds for more general classes of ergodic automorphisms ([7, Remarque 2]).

**Corollary 2.1.** *Let $T$ be a measure-preserving automorphism of a standard probability space $(X, \mathcal{S}, \mu)$, $f \colon X \longrightarrow \mathbb{R}$ a Borel map with $\int f\, d\mu = 1$ and $f(x) > 0$ for every $x \in X$, and let $T^f$ be the flow under $f$ with base $T$ on the probability space $(X^f, \mathcal{S}^f, \mu^f)$. Suppose that either of the following conditions is satisfied.*

*(1) $T$ is mixing and $T^f$ is weakly mixing;*
*(2) $T^f$ is mixing.*

*Then the cocycle $\mathbf{f} \colon \mathbb{Z} \times X \longrightarrow \mathbb{R}$ in (2.1) disperses in the sense of (1.4).*

*Proof.* If $T$ is mixing, then Theorem 1.2 with $G = \mathbb{Z}$ and $A = \mathbb{R}$ shows that the cocycle $\mathbf{f}$ either disperses or is trivial, in which case the flow $T^f$ is not weakly mixing.

In order to prove dispersion of $\mathbf{f}$ under the hypothesis (2) we may replace $f$ by a cohomologous function $f' = f + b \circ T - b$ such that $b\colon X \longrightarrow \mathbb{R}$ is measurable and $f'$ is bounded above and below by positive constants; this will affect neither the hypotheses nor the conclusions of the corollary. We assume therefore without loss of generality that there exist positive constants $c_1 < c_2$ such that $c_1 \leq f(x) \leq c_2$ for every $x \in X$.

Define a map $F\colon X^f \longrightarrow \mathbb{R}$ by

$$F(x, s) = \frac{1}{f(x)}$$

for every $x \in X$ and $s \in [0, f(x))$ and consider the cocycle $c\colon \mathbb{R} \times X^f \longrightarrow \mathbb{R}$ given by

$$c(t, z) = \begin{cases} \int_0^t F(T_s^f z)\, ds & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -c(-t, T_t^f z) & \text{if } t < 0. \end{cases} \tag{2.2}$$

For every positive $t$ we denote by $d(t, z)$ the number of intersections of the set $\{(x, 0) : x \in X\} \subset X^f$ with the trajectory $\{T_s^f z : 0 \leq s < t\}$. Then

$$|c(t, z) - d(t, z)| \leq 2 \ \text{ and } \ |t - \mathbf{f}(d(t, (x, s)), x)| \leq 2c_2 \tag{2.3}$$

for every $t \geq 0$ and $z = (x, s) \in X^f$, and

$$\int c(t, z)\, d\mu^f(z) = t \tag{2.4}$$

for every $t \in \mathbb{R}$.

Suppose that (1.4) is not satisfied, i.e. that there exist an $\varepsilon > 0$, a constant $L > 0$, an increasing sequence $(m_n)$ of positive integers, and a sequence $(t_n)$ in $\mathbb{R}$ with

$$\mu(\{x \in X : |\mathbf{f}(m_n, x) - t_n| \leq L\}) \geq \varepsilon \tag{2.5}$$

for every $n \geq 1$. According to (2.3) and (2.5),

$$\mu^f(\{(x, s) \in X^f : |\mathbf{f}(d(t_n, (x, s)), x) - \mathbf{f}(m_n, x)| \leq L + 2c_2\}) \geq c_1 \varepsilon,$$

and hence

$$\mu^f\big(\big\{z \in X^f : |d(t_n, z) - m_n| \leq \tfrac{L + 2c_2}{c_1}\big\}\big) \geq c_1 \varepsilon,$$

and the first inequality in (2.3) implies that

$$\mu^f\big(\big\{z \in X^f : |c(t_n, z) - m_n| \leq \tfrac{L + 2c_2}{c_1} + 2\big\}\big) \geq c_1 \varepsilon \tag{2.6}$$

for every $n \geq 1$. Since $\mu^f$ is mixing and $\lim_{n \to \infty} t_n = \infty$ by (2.5), $\lim_{n \to \infty} \nu_{t_n} = \mu^f \times \mu^f$, where $\nu_{t_n}$ is the off-diagonal measure (1.5) with $\mu$ and $g$ replaced by $\mu^f$ and $t_n$.

We choose a countable dense subgroup $G \subset \mathbb{R}$ which contains the sequence $(t_n)$. Since $G$ is dense in $\mathbb{R}$ and $\mu^f \times \mu^f$ is ergodic under the diagonal $\mathbb{R}$-action $T^f \times T^f$ on $X^f \times X^f$, the measure $\mu^f \times \mu^f$ is also ergodic under the restriction of $T^f \times T^f$ to $G$, which implies that $\mu^f$ is weakly mixing under the restriction of $T^f$ to $G$. According to (2.6) and Proposition 1.3 there exist a homomorphism $\eta\colon G \longrightarrow \mathbb{R}$ and a Borel map $b\colon X^f \longrightarrow \mathbb{R}$ such that

$$c(t, z) = \eta(t) + b(T_t^f z) - b(z) \tag{2.7}$$

for every $t \in G$ and $\mu^f$-a.e. $z \in X^f$. From (2.4) we know that $\eta(t) = t$ for every $t \in G$, and the continuity of the map $t \mapsto c(t, \cdot)$ from $\mathbb{R}$ into $L^1(X^f, \mathcal{S}^f, \mu^f)$ in (2.2) guarantees that (2.7) holds for every $t \in \mathbb{R}$.

We re-trace our steps and conclude from (2.7) and (2.3) that the cocycle $\mathbf{f}' \colon \mathbb{Z} \times X \longrightarrow \mathbb{R}$, defined by

$$\mathbf{f}'(n, \cdot) = \mathbf{f}(n, \cdot) - n,\ n \in \mathbb{Z},$$

is bounded in $\mu$-measure. By [9, Theorem 11.8], [8, Theorem 5.2] or [10, Theorem 4.7] there exists a Borel map $b \colon X \longrightarrow \mathbb{R}$ with $f(x) = 1 + b(Tx) - b(x)$ for $\mu$-a.e. $x \in X$, which implies that $T^f$ is not even weakly mixing. This contradiction resulting from (2.5) proves the corollary. $\qquad\square$

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