ERGODICITY OF COCYCLES. 1: GENERAL THEORY

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Abstract. We prove several 'automatic' ergodicity results for cocycles on a discrete nonsingular ergodic equivalence relation on a probability space $(X, \mathcal{S}, \mu)$ with values in virtually nilpotent groups. The hypotheses required for automatic ergodicity are invariance or quasi-invariance of the cocycles under asymptotically central automorphisms of the equivalence relation. If the cocycles have certain recurrence properties then the condition of virtual nilpotency can be relaxed.

In a subsequent paper these ideas will be applied to ergodicity of noncompact and nonabelian covers of horocycle foliations on compact manifolds with nonconstant negative curvature.

1. Introduction

Let R be a discrete nonsingular equivalence relation on a standard probability space $(X, \mathcal{S}, \mu)$, H a Polish group, and $c\colon R \longrightarrow H$ a Borel cocycle (for the definitions we refer to Section 2). This paper is devoted to finding general conditions under which such a cocycle is *ergodic*, where we adopt a slightly more general definition of ergodicity of cocycles than usual which does not require the relation R to be ergodic or the group H to be locally compact (Definition 2.3). In a subsequent paper we apply the general ergodicity results obtained here to ergodicity of certain noncompact covers of horocyclic flows and foliations on compact connected negatively curved Riemannian manifolds with pinched sectional curvatures.

In order to describe conditions guaranteeing ergodicity of cocycles we denote by $[R]$ the full group of R and recall that a measure-preserving automorphism V of $(X, \mathcal{S}, \mu)$ is an *automorphism of* (R, μ) if $V^{-1}[R]V = [R]$. An automorphism V of (R, μ) is *weakly asymptotically central* if

$$\lim_{|n| to\infty} \mu(B \triangle V^n W V^{-n} B) = 0$$

for every $B \in \mathcal{S}$ and $W \in [R]$, and *strongly asymptotically central* if W and $V^n W' V^{-n}$ commute asymptotically as $|n| \to \infty$ for every $W, W' \in [R]$ (Definition 3.3).

2000 *Mathematics Subject Classification.* 37A20, 37D40, 60G09, 60G50.

Key words and phrases. Ergodic equivalence relations, asymptotically central automorphisms of equivalence relation, invariant cocycles, ergodicity of cocycles and skew products.

The authors would like to thank the following institutions for hospitality and support while this work was done:
VK: the Erwin Schrödinger Institute, Vienna, and the Newton Institute, Cambridge,
KS: the Mathematics Department, University of Manchester, the Royal Society, and the Newton Institute, Cambridge.

As was first observed in [17], every cocycle $c\colon R \longrightarrow H$ which is invariant under a weakly asymptotically central automorphism V of (R,μ) is automatically ergodic (cf. Theorem 4.1 in this paper and [12], [17], [18]).

Examples of cocycles invariant under asymptotically central automorphisms of equivalence relations are obtained by taking a two-sided shift-space $X \subset A^{\mathbb{Z}}$ with finite alphabet A, furnished with a shift-invariant probability measure μ on X, a μ-nonsingular subrelation R of the two-sided Gibbs (or tail) equivalence relation Δ_X of X, and an appropriate Borel map $f\colon X \longrightarrow H$, where H is an abelian Polish group. The *Gibbs cocycle* $a_f\colon R \longrightarrow H$, defined by

$$a_f(x,x') = \sum_{n\in\mathbb{Z}} f(\sigma^n x) - f(\sigma^n x') \tag{1.1}$$

for every $(x,x') \in R$, where σ is the shift on X, is obviously invariant under the weakly asymptotically central automorphism σ of (R,μ) and hence ergodic by Theorem 4.1 (cf. Section 6).

If the group H is nonabelian there is no analogue of the cocycle a_f. In order to obtain results about automatic ergodicity of cocycles with values in nonabelian groups one has to replace the notion of a V-invariant cocycle by that of a pair of cocycles which transforms under V in a certain way; in the setting of (1.1) this amounts to considering the cocycles

$$\begin{aligned} a_f^+(x,x') &= f(x)^{-1}f(\sigma x)^{-1}\cdots f(\sigma^n x)^{-1}\cdots f(\sigma^n x')\cdots f(\sigma x')f(x'),\\ a_f^-(x,x') &= f(\sigma^{-1}x)\cdots f(\sigma^{-n}x)\cdots f(\sigma^{-n}x')\cdots f(\sigma^{-2}x')f(\sigma^{-1}x')^{-1} \end{aligned} \tag{1.2}$$

on R, which have the property that

$$\begin{aligned} a_f^+(\sigma x,\sigma x') &= f(x)a_f^+(x,x')f(x')^{-1},\\ a_f^-(\sigma x,\sigma x') &= f(x)a_f^-(x,x')f(x')^{-1}, \end{aligned} \tag{1.3}$$

for all $(x,x') \in R$ (cf. Definition 3.1 and (6.6)–(6.8)). In particular, if we set $a_f^*(x,x') = a_f^-(x',x)a_f^+(x,x')$, then

$$a_f^*(\sigma x,\sigma x') = f(x')a_f^*(x,x')f(x')^{-1} \tag{1.4}$$

for all $(x,x') \in R$.

If H is discrete, then (1.4) implies that

$$\ker^*(a_f^-,a_f^+) = \{(x,x') \in R : a_f^*(x,x') = 1\}$$

is a V-invariant subrelation of R, and the 'noncommutative' analogue of Theorem 4.1 is that — under certain assumptions on the group H, such as virtual nilpotency — this equivalence relation $\ker^*(a_f^-,a_f^+)$ has the same ergodic components as R. One of the central problems of this paper is the search for conditions under which the ergodicity of $\ker^*(a_f^-,a_f^+)$ allows conclusions about the ergodicity of the kernels

$$\ker(a_f^{\pm}) = \{(x,x') \in R : a_f^{\pm}(x,x') = 1\}$$

of the individual cocycles $a_f^{\pm}$.

The definition of a *complementary quasi-invariant pair of cocycles* $(c_1, c_2)\colon R \longrightarrow H^2$ for a weakly asymptotically automorphism V of (R,μ) in Definition 3.1 and (3.15) is an abstraction of the properties (1.3)–(1.4) of the pair of cocycles (a_f^-, a_f^+), and the $*$-*ergodicity* of (c_1, c_2) expressed in the Theorems 4.2–4.3 is an abstract version of the ergodic properties of the equivalence relation $\ker^*(a_f^-, a_f^+)$.

Although $*$-ergodicity of a pair (c_1, c_2) of complementary quasi-invariant cocycles does not in general allow conclusions about ergodicity of its individual components c_i, it *does* imply that both components have identical ergodic behaviour in a very precise sense: they are *equally ergodic* in the terminology of Definition 5.1 (cf. Theorem 5.3).

For results about ergodicity of the individual components one has to impose much stronger conditions on the relation R, the measure μ and the pair of cocycles (c_1, c_2): we return to the setting of (1.1)–(1.4) and assume that $X \subset A^{\mathbb{Z}}$ is a mixing shift of finite type, $\mu = \mu_\phi$ the Gibbs measure arising from a Hölder continuous function $\phi\colon X \longrightarrow \mathbb{R}$, $f\colon X \longrightarrow H$ a cylinder map (i.e. a map which depends only on finitely many coordinates), $R = \Delta_X$ the Gibbs (or two-sided tail) equivalence relation on X, and $(c_1, c_2) = (a_f^-, a_f^+)\colon R \longrightarrow H^2$. If the group H is discrete and virtually nilpotent, or if the map f has certain recurrence properties, then there exists a cylinder map $b\colon X \longrightarrow H$ such that the cocycles $a_{f'}^{\pm}(x, x') = b(x)^{-1} a_f^{\pm}(x, x') b(x')$ are both ergodic; the precise statements under somewhat weaker hypotheses appear in Theorem 7.1 and Corollary 7.3.

In Section 8 we use Theorem 7.1 to prove ergodicity of the cocycles $a_f^{\pm}\colon R \longrightarrow H$ *without* any modification by a coboundary under a hypothesis on weights of periodic orbits originally introduced in [15], thereby extending the main result in [2] from abelian groups to virtually nilpotent groups.

Section 9 contains a number of examples. The first of these, 9.1, is taken from [17] and describes briefly the properties of the Radon-Nikodym derivatives of equivalence relations R admitting an asymptotically central automorphism. Example 9.2 shows that, in the case of a full shift X, a Gibbs measure $\mu = \mu_\phi$ on X arising from a Hölder continuous function $\phi\colon X \longrightarrow \mathbb{R}$, and a map $f\colon X \longrightarrow H$ taking values in a discrete virtually nilpotent group, which depends only on the zero coordinate, the cocycles $a_f^{\pm}\colon \Delta_X \longrightarrow H$ in (1.2) are individually ergodic.

If the random walk on H defined by the map f in Example 9.2 has certain recurrence properties then the hypotheses on H can be relaxed: this is illustrated in the case of the lamp-lighter group and other semi-direct products in Examples 9.3–9.4.

Examples 9.5–9.6 illustrate the break-down of individual ergodicity of the cocycles $a_f^{\pm}\colon X \longrightarrow H$ in (1.2) on general subshifts, even if the pair (a_f^-, a_f^+) is $*$-ergodic. In Example 9.6 (which was communicated to us by Dan Rudolph) we consider a subshift of the full 4-shift defined by the (T, T^{-1})-transformation (cf. [8]) and observe that the two-sided 'fine tail' (arising from the two-sided tail sigma-algebra and the symbol count) is ergodic (corresponding to the ergodicity of a cocycle a_f of the form (1.1) for an appropriate map f), but the one-sided fine tails are highly nonergodic (corresponding to the fact that the one-sided cocycles $a_f^{\pm}$ in (1.2) are nonergodic).

In Example 9.7 we present a refinement of an example in [18] about the difficulty of detecting local variations in (infinitely) long molecules based on the analysis of finite substrings of these molecules.

In the Examples 9.8–9.9 we apply our results on two-sided shift spaces to one-sided spaces. As an illustration we consider a discrete virtually nilpotent group H with a distinguished finite set of generators $A = \{h_1^{\pm 1}, \dots, h_m^{\pm 1}\}$ and denote by $X \subset A^{\mathbb{N}}$ the shift of finite type obtained by disallowing all words of length two of the form $h_i h_i^{-1}$ and $h_i^{-1} h_i$, $i = 1, \dots, m$. If $\mu = \mu_\phi$ is a Gibbs measure on X arising from a Hölder continuous function $\phi \colon X \longrightarrow \mathbb{R}$, and if R is the equivalence relation

$$R = \{(x, x') \in X \times X : x_0 \cdots x_n = x'_0 \cdots x'_n \text{ for all sufficiently large } n \geq 0\},$$

then μ is ergodic under R (Proposition 9.2).

In the final Example 9.10 we consider cocycles on the 'extended' Gibbs relation on a shift of finite type and explain how the earlier results about ergodicity of cocycles on the two- and one-sided Gibbs relations imply the ergodicity of cocycles on the two- and one-sided extended Gibbs relations.

2. Ergodicity of cocycles

Let $(X, \mathcal{S})$ be a standard Borel space and $\mathrm{Aut}(X, \mathcal{S})$ the group of Borel automorphisms of X. A Borel set $R \subset X \times X$ is a *discrete Borel equivalence relation* on X if R is an equivalence relation, and if the *equivalence class*

$$R(x) = \{y \in X : (x, y) \in R\} \tag{2.1}$$

of every $x \in X$ is countable.

Let R be a discrete Borel equivalence relation on X. The *full group* $[R]$ of R is the group of all $W \in \mathrm{Aut}(X, \mathcal{S})$ with $Wx \in R(x)$ for every $x \in X$. According to [3] there exists a countable subgroup $\Gamma \subset [R]$ with

$$R = R[\Gamma] = \{(\gamma x, x) : \gamma \in \Gamma, x \in X\}. \tag{2.2}$$

Conversely, if $\Gamma \subset \mathrm{Aut}(X, \mathcal{S})$ is a countable group, then (2.2) defines a discrete Borel equivalence relation $R[\Gamma]$ on X.

From (2.2) it follows that the *saturation*

$$R(B) = \bigcup_{x \in B} R(x) = \bigcup_{\gamma \in \Gamma} \gamma(B) \tag{2.3}$$

of every set $B \in \mathcal{S}$ lies in $\mathcal{S}$, and we write

$$\mathcal{S}^R = \{R(B) : B \in \mathcal{S}\} \subset \mathcal{S} \tag{2.4}$$

for the sigma-algebra of R-saturated Borel sets.

For every $C \in \mathcal{S}$ we denote by

$$R_C = R \cap (C \times C) \tag{2.5}$$

the equivalence relation *induced* by R on C.

A sigma-finite measure μ on $\mathcal{S}$ is *quasi-invariant* under R (or R is *μ-non-singular*) if $\mu(R(B)) = 0$ for every $B \in \mathcal{S}$ with $\mu(B) = 0$. The measure μ is *conservative* under R if there exists, for every $A \in \mathcal{S}$ with $\mu(A) > 0$, an element $(x, y) \in R_A$ with $x \neq y$, and μ is *ergodic* under R (or R is μ-ergodic) if either $\mu(B) = 0$ or $\mu(X \smallsetminus B) = 0$ for every $B \in \mathcal{S}^R$. The following proposition allows us to assume without much loss in generality that a discrete equivalence relation R is nonsingular with respect to a given measure μ on $\mathcal{S}$.

Proposition 2.1. *Let R be a discrete Borel equivalence relation on X and μ a sigma-finite measure on $\mathcal{S}$. Then there exists a set $N \in \mathcal{S}$ with the following properties.*

1. $\mu(N) = 0$;
2. *The measure μ is quasi-invariant under the discrete Borel equivalence relation*
$$R_\mu = R_{X \smallsetminus N} \cup \{(x, x) : x \in N\} \subset R; \tag{2.6}$$
3. $\mathcal{S}^R = \mathcal{S}^{R_\mu} \pmod{\mu}$.

Proof. This is [18, Lemma 2.3]. □

Let R be a discrete Borel equivalence relation on X and μ a sigma-finite measure on $\mathcal{S}$ which is quasi-invariant under R. A condition is satisfied $(\mathrm{mod}\ \mu)$ or μ-a.e. on R if there exists an μ-null set $N \in \mathcal{S}$ such that the condition holds for every $(x, y) \in R_{X \smallsetminus N}$ (we may obviously assume that N is R-saturated). Two discrete Borel equivalence relations R, R' on X are *equal* $(\mathrm{mod}\ \mu)$ if $R_{X \smallsetminus N} = R'_{X \smallsetminus N}$ for some μ-null set $N \in \mathcal{S}$.

Definition 2.2. Let H be a Polish (i.e. complete separable metric) group with identity element 1_H and Borel field $\mathcal{B}_H$. A Borel map $c\colon R \longrightarrow H$ is a *cocycle* on R if
$$c(x, x')c(x', x'') = c(x, x'') \tag{2.7}$$
for every $(x, x'), (x, x'') \in R$.

Two cocycles $c, c'\colon R \longrightarrow H$ are *cohomologous* if there exists a Borel map $b\colon X \longrightarrow H$ with
$$c(x, x') = b(x)^{-1}c'(x, x')b(x') \tag{2.8}$$
for every $(x, x') \in R$.

A cocycle $c\colon R \longrightarrow H$ is a *coboundary* if it is cohomologous to the cocycle $c' \equiv 1_H$.

For every cocycle $c\colon R \longrightarrow H$ we denote by
$$R^{(c)} = \{((x, c(x, y)h), (y, h)) : (x, y) \in R,\ h \in H\} \tag{2.9}$$
the *skew-product relation* on $X \times H$ defined by c.

Definition 2.3. Let R be a discrete Borel equivalence relation on a standard Borel space $(X, \mathcal{S})$, μ a sigma-finite measure on $\mathcal{S}$ which is quasi-invariant under R, and H a Polish group.

1. A Borel map $c\colon R \longrightarrow H$ is a *cocycle* $(\mathrm{mod}\ \mu)$ if there exists a set $N \in \mathcal{S}^R$ with $\mu(N) = 0$ such that the restriction of c to $R_{X \smallsetminus N}$ is a cocycle. Two cocycles $(\mathrm{mod}\ \mu)$ $c, c'\colon R \longrightarrow H$ are *cohomologous* $(\mathrm{mod}\ \mu)$ if there exists a set $N \in \mathcal{S}^R$ with $\mu(N) = 0$ such that the restrictions to $R_{X \smallsetminus N}$ of c and c' are cohomologous cocycles.
2. A cocycle $c\colon R \longrightarrow H$ is *μ-recurrent* (or simply *recurrent*) if there exists, for every $A \in \mathcal{S}$ with $\mu(A) > 0$ and every neighbourhood $N(1_H)$ of the identity in H, an element $(x, y) \in R$ with $x, y \in A$, $x \neq y$ and $c(x, y) \in N(1_H)$.

(3) A cocycle $c\colon R \longrightarrow H$ is *μ-ergodic* (or simply *ergodic*) if there exists, for every neighbourhood $N(1_H)$ of the identity in H and every pair $A, B \in \mathcal{S}$ with $\mu(R(A) \cap B) > 0$, an element $(x, y) \in R$ with

$$x \in A, \ \ y \in B \ \text{ and } \ c(x, y) \in N(1_H). \tag{2.10}$$

Remarks 2.4. (1) If $c\colon R \longrightarrow H$ is a cocycle (mod μ) we can choose a μ-null set $N \in \mathcal{S}^R$ such that the map

$$c'(x, y) = \begin{cases} c(x, y) & \text{if } (x, y) \in R_{X \smallsetminus N}, \\ 1_H & \text{otherwise} \end{cases}$$

is a cocycle on R. Similarly we can change cohomology (mod μ) into cohomology.

(2) Suppose that H is a discrete group and $c\colon R \longrightarrow H$ a cocycle. Then c is recurrent if and only if μ is conservative under the *kernel*

$$\ker(c) = \{(x, y) \in R : c(x, y) = 1_H\} \tag{2.11}$$

of c, and c is ergodic if and only if

$$\mathcal{S}^{\ker(c)} = \mathcal{S}^R \quad (\text{mod } \mu). \tag{2.12}$$

Note that the kernel of a cocycle is an equivalence relation.

(3) **Warning:** Our definition of ergodicity has the following features:
- The constant cocycle $c \equiv 1_H$ is ergodic.
- Unlike recurrence, ergodicity of cocycles is *not* a cohomology invariant: for example, if $c\colon R \longrightarrow H$ is a coboundary, then c is ergodic if and only if $c \equiv 1_H$.

The next propositions describe some basic properties of ergodic cocycles.

Proposition 2.5. *Let R be a discrete Borel equivalence relation on a standard Borel space $(X, \mathcal{S})$, μ a probability measure on $\mathcal{S}$ which is quasi-invariant and ergodic under R, H a Polish group, and $c\colon R \longrightarrow H$ an ergodic cocycle. Then there exist a unique smallest closed subgroup $H_0 \subset H$ and a cocycle $c'\colon R \longrightarrow H_0$ with $c = c'$ (mod μ) and $c'(x, y) \in H_0$ for every $(x, y) \in R$.*

Proof. For every $B \in \mathcal{S}$ with $\mu(B) > 0$ we set

$$\begin{aligned} H(B) = \{h \in H : &\text{ for every neighbourhood of the identity } N(1_H) \subset H \\ &\text{there exists an } (x, y) \in R_B \text{ with } c(x, y) \in hN(1_H)\}. \end{aligned}$$

Ergodicity of c and μ, together with the cocycle equation (2.7), imply that $H(B) = H(B')$ for all $B, B' \in \mathcal{S}$ with $\mu(B)\mu(B') > 0$, and that H_0 is a subgroup of H which is obviously closed and has the required properties. □

Proposition 2.6. *Let R be a discrete Borel equivalence relation on a standard Borel space $(X, \mathcal{S})$, μ a probability measure on $\mathcal{S}$ which is quasi-invariant under R, H a locally compact second countable group with left Haar measure λ_H, and $c\colon R \longrightarrow H$ a cocycle.*

(1) *The cocycle c is recurrent if and only if the product-measure $\mu \times \lambda_H$ is conservative under the skew-product relation $R^{(c)}$ on $X \times H$ defined in* (2.9)*;*

(2) *If R and c are ergodic, and if $H_0 \subset H$ is the subgroup appearing in Proposition* 2.5*, then the product measure $\mu \times \lambda_{H_0}$ is ergodic under $R^{(c)}$.*

Proof. The first assertion is obvious, and the second an immediate consequence of Proposition 2.5. □

Proposition 2.7. *Let R be a discrete Borel equivalence relation on a standard Borel space $(X, \mathcal{S})$, μ a probability measure on $\mathcal{S}$ which is quasi-invariant and ergodic under R, H a locally compact second countable group with left Haar measure λ_H, $\mathcal{T} = \mathcal{S} \otimes \mathcal{B}_H$ the product Borel field on $X \times H$, and $c\colon R \longrightarrow H$ a cocycle. We denote by $T\colon h \longrightarrow T^h$ the Borel action of H on $X \times H$ defined by*

$$T^h(x, h') = (x, h'h) \tag{2.13}$$

for every $h, h' \in H$ and $x \in X$. Then the following properties are equivalent.

(1) *The cocycle c is cohomologous* (mod μ) *to an ergodic cocycle $c'\colon R \longrightarrow H$;*
(2) *There exist a Borel set $C \in \mathcal{T}^{R^{(c)}}$ and a closed subgroup $H_0 \subset H$ with the following properties:*
   (a) *$\mu(\pi_1(C)) = 1$, where $\pi_1\colon X \times H \longrightarrow X$ is the first coordinate projection* (*note that $\pi_1(C)$ is μ-measurable by* [9]),
   (b) *for every $h \in H_0$, $(\mu \times \lambda_H)(T^h C \triangle C) = 0$,*
   (c) *for every $h \in H \smallsetminus H_0$, $(\mu \times \lambda_H)(T^h C \cap C) = 0$.*

Proof. Suppose that c is cohomologous (mod μ) to an ergodic cocycle c' with transfer function $b\colon X \longrightarrow H$. We denote by H_0 the closed subgroup of H described in Proposition 2.5 with c' replacing c. Fix a set $A \in \mathcal{S}^R$ with $\mu(A) = 1$ and $c(x, y) = b(x)^{-1} c'(x, y) b(y)$ for every $(x, y) \in R_A$ and put

$$C = \{(x, b(x)^{-1}h) : x \in A,\ h \in H_0\} \in \mathcal{T}^{R^{(c)}}.$$

Then C and H_0 satisfy (2).

Conversely, if $C \in \mathcal{T}^{R^{(c)}}$ and $H_0 \subset H$ satisfy (2), then we choose a set $A \in \mathcal{S}^R$ with $A \subset \pi_1(C)$ and $\mu(A) = 1$ and set $C' = C \cap (A \times H)$. Then C' and H_0 again satisfy (a)–(c). We denote by $\eta\colon H \longrightarrow H_0 \backslash H$ the quotient map from H onto the right coset space and conclude from (b)–(c) and [11, Lemma I.5.1] that there exists a Borel map $\psi\colon X \longrightarrow H$ with

$$C' = \{(x, \psi(x)h) : x \in A,\ h \in H_0\} \pmod{\mu \times \lambda_H}.$$

We put

$$b(x) = \begin{cases} \psi(x) & \text{if } x \in A, \\ 1_H & \text{otherwise,} \end{cases}$$

and obtain that the cocycle $c'(x, y) = b(x)^{-1} c(x, y) b(y)$ is ergodic. □

Remark 2.8. Cocycles satisfying the equivalent conditions (1) and (2) in Proposition 2.7 are called *regular* in [16]. Condition (2) can be interpreted as saying that the H-action $\bar{T}$ induced by T on $(X \times H, \mathcal{T}^{R^{(c)}}, \mu \times \lambda_H)$ is *transitive* (i.e. consists of a single orbit up to a null-set).

3. Invariance and quasi-invariance of cocycles

Definition 3.1. Let R be a discrete Borel equivalence relation on a standard Borel space $(X, \mathcal{S})$ and H a Polish group.

(1) An element $V \in \mathrm{Aut}(X, \mathcal{S})$ is an *automorphism of* R if $(V \times V)(R) = R$ or, equivalently, if $V(R(x)) = R(Vx)$ for every $x \in X$. We write $\mathrm{Aut}(R)$ for the group of all automorphisms of R and observe that $[R] \subset \mathrm{Aut}(R)$.

(2) If $V \in \mathrm{Aut}(R)$, then a cocycle $c\colon R \longrightarrow H$ is V*-invariant* if
$$c^V(x, x') := c(Vx, Vx') = c(x, x') \tag{3.1}$$
for every $(x, x') \in R$. The cocycle $c\colon R \longrightarrow H$ is V*-quasi-invariant* if the cocycles c and c^V are cohomologous, i.e. if there exists a Borel map $f\colon X \longrightarrow H$ with
$$c(x, x') = f(x)^{-1} c(Vx, Vx') f(x') \tag{3.2}$$
for every $(x, x') \in R$.

(3) Two cocycles $c_1, c_2\colon R \longrightarrow H$ form a V*-quasi-invariant pair*
$$(c_1, c_2)\colon R \longrightarrow H^2$$
if there exists a Borel map $f\colon X \longrightarrow H$ with
$$\begin{aligned} c_1(x, x') &= f(x)^{-1} c_1(Vx, Vx') f(x'), \\ c_2(x, x') &= f(x)^{-1} c_2(Vx, Vx') f(x') \end{aligned} \tag{3.3}$$
for every $(x, x') \in R$. If H is discrete and $(c_1, c_2)\colon R \longrightarrow H^2$ is a V-quasi-invariant pair we set
$$c^*(x, x') = c_1(x', x) c_2(x, x') \tag{3.4}$$
for every $(x, x') \in R$ and denote by
$$\begin{aligned} \ker^*(c_1, c_2) &= \{(x, x') \in R : c_1(x, x') = c_2(x, x')\} \\ &= \{(x, x') \in R : c^*(x, x') = 1_H\} \end{aligned} \tag{3.5}$$
the $*$*-kernel* of (c_1, c_2). The kernels of V-invariant cocycles and the $*$-kernels of V-quasi-invariant pairs are V-invariant subrelations of R.

(4) If $V \in \mathrm{Aut}(R)$ preserves a probability measure μ on $\mathcal{S}$ which is quasi-invariant under R, then a V-quasi-invariant pair of cocycles $(c_1, c_2)\colon R \longrightarrow H^2$ is $*$*-ergodic* if there exists, for every neighbourhood $N(1_H)$ of the identity in H and all $A, B \in \mathcal{S}$ with $\mu(R(A) \cap B) > 0$, an element $(x, y) \in R$ with
$$x \in A, \ \ y \in B \ \text{ and } \ c^*(x, y) \in N(1_H). \tag{3.6}$$
If H is discrete then (c_1, c_2) is $*$-ergodic if and only if
$$\mathcal{S}^R = \mathcal{S}^{\ker^*(c_1, c_2)} \pmod{\mu} \tag{3.7}$$
with $\ker^*(c_1, c_2)$ given by (3.5).

Remarks 3.2. (1) If the automorphism V of R in Definition 3.1 is fixed and no confusion is possible we shall simply speak of *invariant* or *quasi-invariant* cocycles and of *quasi-invariant pairs of cocycles.*

(2) The equations (3.2)–(3.4) imply that, for every quasi-invariant cocycle c and every quasi-invariant pair (c_1, c_2),

$$\begin{aligned} c(V^n x, V^n x') &= \mathbf{f}(n,x)c(x,x')\mathbf{f}(n,x')^{-1}, \\ c^*(V^n x, V^n x') &= \mathbf{f}(n,x')c^*(x,x')\mathbf{f}(n,x')^{-1} \end{aligned} \tag{3.8}$$

for every $n \in \mathbb{Z}$ and $(x,x') \in R$, where

$$\mathbf{f}(n,x) = \begin{cases} f(V^{n-1}x)\cdots f(x) & \text{if } n>0, \\ 1_H & \text{if } n=0, \\ f(V^n x)^{-1}\cdots f(V^{-1}x)^{-1} = \mathbf{f}(-n, V^n x)^{-1} & \text{if } n<0. \end{cases} \tag{3.9}$$

The map $f\colon \mathbb{Z} \times X \longrightarrow H$ in (3.9) has the property that

$$\mathbf{f}(m, V^n x)\mathbf{f}(n,x) = \mathbf{f}(m+n, x) \tag{3.10}$$

for every $m,n \in \mathbb{Z}$ and $x \in X$, and is called a *cocycle of* V.

(3) Let R be a discrete Borel equivalence relation on a standard Borel space $(X, \mathcal{S})$, $V \in \mathrm{Aut}(R)$ and μ a probability measure which is quasi-invariant both under R and V. We denote by

$$R^{(V)} = \{(V^n x, x') : (x,x') \in R, n \in \mathbb{Z}\} \tag{3.11}$$

the equivalence relation generated by R and V.

If H is a Polish group and $c\colon R^{(V)} \longrightarrow H$ a cocycle, then the restriction of c to $R \subset R^{(V)}$ is a quasi-invariant cocycle. For the converse one has to be a little more careful: put

$$N = \bigcup_{0 \neq n \in \mathbb{Z}} \{x \in X : V^n x = x\}, \qquad \bar{N} = R(N) = R^{(V)}(N).$$

Then there exists, for every $(x,x') \in R^{(V)}_{X \smallsetminus \bar{N}}$, a unique integer $p(x,x')$ with

$$(V^{p(x,x')}x, x') \in R. \tag{3.12}$$

We extend p to a cocycle $p\colon R^{(V)} \longrightarrow H$ by setting $p(x,x') = 0$ for every $(x,x') \in R^{(V)} \smallsetminus R^{(V)}_{X \smallsetminus \bar{N}}$. If $c\colon R \longrightarrow H$ is a quasi-invariant cocycle and $f\colon X \longrightarrow H$ a Borel map satisfying (3.2), then we can define a cocycle $\bar{c}\colon R^{(V)} \longrightarrow H$ by

$$\bar{c}(x,x') = \begin{cases} \mathbf{f}(p(x,x'),x)^{-1}c(V^{p(x,x')}x, x') & \text{if } (x,x') \in R^{(V)}_{X \smallsetminus \bar{N}}, \\ 1_H & \text{otherwise.} \end{cases} \tag{3.13}$$

In this somewhat technical sense quasi-invariant cocycles are precisely those cocycles on R which can be extended to $R^{(V)}$.

If V is an automorphism of a discrete Borel equivalence relation R on a standard Borel space $(X, \mathcal{S})$ and $c\colon R \longrightarrow H$ a quasi-invariant cocycle with values in a Polish group H, then the pair $(c,c)\colon R \longrightarrow H^2$ is again quasi-invariant, but obviously not very interesting. In order to avoid trivialities we assume that the components c_1, c_2 of a quasi-invariant pair satisfy a certain *independence condition* which is the subject of our next definition.

Definition 3.3 ([17])**.** Let R be a discrete Borel equivalence relation on a standard Borel space $(X, \mathcal{S})$ and μ a probability measure on $\mathcal{S}$ which is quasi-invariant under R. An element $V \in \mathrm{Aut}(R)$ is a *weakly asymptotically central automorphism of* (R, μ) if V preserves μ and

$$\lim_{|n|\to\infty} \mu(B \triangle V^n W V^{-n} B) = \lim_{|n|\to\infty} \mu(V^{-n} B \triangle W V^{-n} B) = 0 \tag{3.14}$$

for every $W \in [R]$ and $B \in \mathcal{S}$ (in [17]–[18] the term *asymptotically central* was used instead of *weakly asymptotically central*; for a brief discussion of weak and strong asymptotic centrality we refer to [6]).

Let V be an automorphism of R which preserves the measure μ. A V-quasi-invariant pair of cocycles $(c_1, c_2)\colon R \longrightarrow H^2$ is *complementary* if, for every $W \in [R]$,

$$\lim_{n\to\infty} c_1(V^{-n} W V^n x, x) = \lim_{n\to\infty} c_2(V^n W V^{-n} x, x) = 1_H \tag{3.15}$$

for μ-a.e. $x \in X$.

Remark 3.4. Examples of invariant cocycles and complementary quasi-invariant pairs of cocycles can be found in [17]–[18] (see also Section 6 and [6]). If $(c_1, c_2)\colon R \longrightarrow H^2$ is a quasi-invariant pair of cocycles for an automorphism $V \in \mathrm{Aut}(R)$, and if the group H is abelian, then the map $c^*\colon R \longrightarrow H$ in (3.4) is an invariant cocycle on R. There also exist cocycles with values in nonabelian groups which are invariant under weakly ergodic asymptotically central automorphisms ([6]). However, if H is nonabelian, and if V is is an ergodic strongly asymptotically central automorphism of (R, μ), where R is a discrete Borel equivalence relation and μ a probability measure which is quasi-invariant under R, then every invariant cocycle $c\colon R \longrightarrow H$ takes values in an abelian subgroup of H ([6]).

4. Automatic ergodicity of cocycles

Theorem 4.1. *Let R be discrete Borel equivalence relation on a standard Borel space $(X, \mathcal{S})$, μ a probability measure on $\mathcal{S}$ which is quasi-invariant under R, and V a weakly asymptotically central automorphism of (R, μ). If H is a Polish group and $c\colon R \longrightarrow H$ an invariant cocycle then c is ergodic.*

Proof. If μ is ergodic under R this is essentially Theorem 2.3 in [17].

Let $A, B \in \mathcal{S}$ be sets with $\mu(WA \cap B) > 0$ for some $W \in [R]$. We fix a metric δ on H, set $B_\delta(h, r) = \{h' \in H : \delta(h, h') < r\}$ for every $h \in H$ and $r > 0$, and choose an element $g \in H$ with $\mu(A(\varepsilon')) > 0$ for every $\varepsilon' > 0$, where

$$A(\varepsilon') = \{x \in A \cap W^{-1}(B) : c(Wx, x) \in B_\delta(g, \varepsilon')\}.$$

Fix $\varepsilon > 0$, choose ε' sufficiently small so that

$$\delta(ab^{-1}, 1_H) < \varepsilon$$

whenever $a, b \in B_\delta(g, \varepsilon')$, and set $A' = A(\varepsilon')$. The weak asymptotic centrality of V implies that

$$\lim_{n\to\infty} \mu(A' \cap V^{-n} W^{-1} V^n A') = \mu(A').$$

Since μ is V-invariant, the ergodic theorem and Hölder's inequality yield that

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{n-1}\mu(A'\cap V^{-n}A') &= \int 1_{A'}E_\mu(1_{A'}|\mathcal{S}^V)\,d\mu \\ &= \int E_\mu(1_{A'}|\mathcal{S}^V)^2\,d\mu \geq \Big(\int E_\mu(1_{A'}|\mathcal{S}^V)\,d\mu\Big)^2 = \mu(A')^2,\end{aligned} \tag{4.1}$$

where $\mathcal{S}^V = \{B\in\mathcal{S} : V^{-1}B = B\}$ and $1_{A'}$ is the indicator function of A'. Hence $\mu(A'\cap V^{-n}A')\geq\mu(A')^2/2$ for all n in an infinite subset $\mathbf{N}\subset\mathbb{N}$. For every sufficiently large $n\in\mathbf{N}$, the set

$$A'(n) = A'\cap V^{-n}A'\cap V^{-n}W^{-1}V^nA'$$

has positive measure, and every $x\in A'(n)$ satisfies that

$$x\in A',\ \ V^nx\in A',\ \ V^{-n}WV^nx\in A',\ \ Wx\in B,$$
$$c(x,V^{-n}WV^nx) = c(V^{-n}WV^nx,x)^{-1} = c(WV^nx,V^nx)^{-1}.$$

As $\delta(c(V^{-n}WV^nx,x,),g)<\varepsilon'$ and $\delta(c(Wx,x),g)<\varepsilon'$, we obtain that

$$\delta(c(Wx,V^{-n}WV^nx),1_H)<\varepsilon.$$

We have thus found an element $(y,y')\in R$ with $y' = V^{-n}WV^nx\in A$, $y = Wx\in B$ and $d(c(y,y'),1_H)<\varepsilon$. As A,B and ε were arbitrary we have proved the ergodicity of c. ☐

In order to prove $*$-ergodicity of quasi-invariant pairs of cocycles we have to make certain assumptions on the group H. Here we restrict ourselves to the cases where H is a compact extension of an abelian group or a discrete nilpotent group.

Theorem 4.2. *Let R be discrete Borel equivalence relation on a standard Borel space $(X,\mathcal{S})$, μ a probability measure on $\mathcal{S}$ which is quasi-invariant under R, and V a weakly asymptotically central automorphism of (R,μ). If H is a compact extension of a Polish abelian group then every complementary quasi-invariant pair of cocycles $(c_1,c_2)\colon R\longrightarrow H^2$ is $*$-ergodic.*

Proof. Let $f\colon X\longrightarrow H$ be a Borel map satisfying (3.3) and define $\mathbf{f}\colon\mathbb{Z}\times X\longrightarrow H$ by (3.9) – (3.10).

Since H is a compact extension of an abelian group, each conjugacy class $[g]=\{h^{-1}gh : h\in H\}$ of H carries a probability measure which is invariant under conjugation. We denote by $C(g)$ the centralizer of g and conclude that $H/C(g)$ carries a probability measure λ_g which is invariant under left translation and positive on open sets.

Let $N(1_H)$ be a symmetric neighbourhood of the identity in H which is invariant under inner automorphisms of H, and let $A,B\in\mathcal{S}$ have the property that $\mu(R(A)\cap B)>0$. By decreasing A and B, if necessary, we may assume in addition that there exist a $W\in[R]$ and a $g\in H$ with $WA=B$ and $c^*(Wx,x)\in gN(1_H)$ for every $x\in A$.

Consider the skew-product transformation S on $\bar{X}=X\times H/C(g)$, given by

$$S(x,g'C(g)) = (Vx,f(x)g'C(g))$$

for every $(x, g'C(g)) \in \bar{X}$, set $\nu = \mu \times \lambda_g$, write $\mathcal{T}$ for the product Borel field on $\bar{X}$, and put $Z = A \times N(1_H)C(g)$. As in (4.1) we observe that

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(A \cap V^{-n}A \cap \{x \in X : \mathbf{f}(n,x) \in N(1_H)^2 C(g)\}) \\ \geq \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \nu\big((A \times N(1_H)C(g)) \cap S^{-n}(A \times N(1_H)C(g))\big) \\ = \nu(A \times N(1_H)C(g))^2 = c, \end{aligned} \tag{4.2}$$

say. By using (3.14), (3.15) and (4.2) we find infinitely many $n \geq 0$ with $\mu(C(n)) > c/2$, where

$$\begin{aligned} C(n) = A \cap V^{-n}A \cap V^{-n}W^{-1}V^nA \cap \{x \in X : \mathbf{f}(n,x) \in N(1_H)^2C(g)\} \\ \cap \{x \in X : c_1(V^{-n}WV^nx, x) \in N(1_H)\}. \end{aligned}$$

For every $x \in C(n)$ the following conditions are satisfied:

(a) $V^nx \in A$, $V^{-n}WV^nx \in A$, $c^*(WV^nx, V^nx) \in gN(1_H)$, $\mathbf{f}(n,x) \in N(1_H)^2C(g)$, and hence $c^*(V^{-n}WV^nx, x) \in gN(1_H)^5$,

(b) $c_1(V^{-n}WV^nx, x) \in N(1_H)$ and hence $c_2(V^{-n}WV^nx, x) \in gN(1_H)^6$,

(c) $c^*(Wx, V^{-n}WV^nx) = c_1(V^{-n}WV^nx, x)c^*(Wx, x)c_2(x, V^{-n}WV^nx)$, and hence $c^*(Wx, V^{-n}WV^nx) \in N(1_H)^8$.

This shows that $D(n) = V^{-n}WV^nC(n)$ is a set of positive measure in A, $WV^{-n}W^{-1}V^nD(n) \subset B$, and $c^*(WV^{-n}W^{-1}V^ny, y) \in N(1_H)^8$ for every $y \in D(n)$. Since A, B and $N(1_H)$ were arbitrary, this proves the $*$-ergodicity of (c_1, c_2). □

Theorem 4.3. *Let R be discrete Borel equivalence relation on a standard Borel space $(X, \mathcal{S})$, μ a probability measure on $\mathcal{S}$ which is quasi-invariant under R, V a weakly asymptotically central automorphism of (R, μ), and H a discrete nilpotent group. Then every complementary quasi-invariant pair of cocycles $(c_1, c_2)\colon R \longrightarrow H^2$ is $*$-ergodic.*

Proof. For every group G we denote by $C(G)$ the centre of G. Since H is nilpotent there exists a filtration of subgroups

$$H = H_n \supset H_{n-1} \supset \cdots \supset H_0 = \{1_H\} \tag{4.3}$$

such that $H_m/H_{m-1} = C(H/H_{m-1})$ for $m = 1, \dots, n$. For every m we define $c^{(m)}\colon R \longrightarrow H/H_{m-1}$ and $c_i^{(m)}\colon R \longrightarrow H/H_{m-1}$ by

$$c^{(m)}(x,y) = c^*(x,y)H_{m-1}, \qquad c_i^{(m)}(x,y) = c_i(x,y)H_{m-1},\ i = 1,2,$$

where c^* is given by (3.4). Note that $c_i^{(m)}$ is a cocycle for $i = 1,2$ and $m = 1, \dots, n$.

Since H/H_{n-1} is abelian, $c^{(n)}$ is an invariant cocycle, and Theorem 4.1 and (2.12) show that

$$\mathcal{S}^R = \mathcal{S}^{R^{(n-1)}} \pmod{\mu}$$

with $R^{(n-1)} = \ker(c^{(n)}) \subset R$. Since $H_{n-1}/H_{n-2} = C(H/H_{n-2})$ and

$$c^{(n-1)}(x,y) \in H_{n-1}/H_{n-2},$$

(3.8) implies that $c^{(n-1)}(x,y) = c^{(n-1)}(Vx,Vy)$ for every $(x,y) \in R^{(n-1)}$. For all $W, W' \in [R^{(n-1)}]$ (3.15) shows that, for μ-a.e. $x \in X$,

$$\begin{aligned} c^{(n-1)}(Wx,x) &= c^{(n-1)}(V^kWx,V^kx) = c_2^{(n-1)}(V^kWx,V^kx),\\ c^{(n-1)}(W'Wx,x) &= c^{(n-1)}(V^kW'Wx,V^kx) = c_2^{(n-1)}(V^kW'Wx,V^kx),\\ c^{(n-1)}(W'Wx,Wx) &= c^{(n-1)}(V^kW'Wx,V^kWx)\\ &= c_2^{(n-1)}(V^kW'Wx,V^kWx), \end{aligned}$$

for infinitely many $k < 0$. We conclude that, for μ-a.e. $x \in X$, and for infinitely many $k < 0$,

$$\begin{aligned} c^{(n-1)}&(W'Wx,Wx)c^{(n-1)}(Wx,x)\\ &= c^{(n-1)}(V^kW'Wx,V^kWx)c^{(n-1)}(V^kWx,V^kx)\\ &= c_2^{(n-1)}(V^kW'Wx,V^kWx)c_2^{(n-1)}(V^kWx,V^kx)\\ &= c_2^{(n-1)}(V^kW'Wx,V^kx) = c^{(n-1)}(W'Wx,x). \end{aligned} \tag{4.4}$$

This shows that $c^{(n-1)}$ is a V-invariant cocycle on $R^{(n-1)}$ with values in the abelian group H_{n-1}/H_{n-2}, and Theorem 4.1 yields that

$$\mathcal{S}^R = \mathcal{S}^{R^{(n-2)}} \pmod{\mu}$$

with $R^{(n-2)} = \ker(c^{(n-1)}) \subset R^{(n-1)}$.

We proceed by induction and obtain equivalence relations

$$R = R^{(n)} \supset R^{(n-1)} \supset \dots \supset R^{(0)} = \{(x,y) : c^*(x,y) = 1_H\}$$

and, by (4.4), V-invariant cocycles $c^{(m)} \colon R^{(m)} \longrightarrow H_m/H_{m-1}$ such that

$$\mathcal{S}^R = \mathcal{S}^{R^{(m-1)}} \pmod{\mu}$$

with $R^{(m-1)} = \ker(c^{(m)})$ for every $m = 1, \dots, n$. By setting $m = 1$ we have proved the theorem. □

Corollary 4.4. *Let R be discrete Borel equivalence relation on a standard Borel space $(X, \mathcal{S})$, μ a probability measure on $\mathcal{S}$ which is quasi-invariant under R, V a weakly asymptotically central automorphism of (R,μ), and H a Polish group of the form $H = H_1 \times H_2$, where H_1 is a compact extension of an abelian group and H_2 is a discrete nilpotent group. Then every complementary quasi-invariant pair of cocycles $(c_1, c_2) \colon R \longrightarrow H^2$ is $*$-ergodic.*

Proof. Apply first Theorems 4.3 and then Theorem 4.2. □

We end this section with an elementary observation about complementary quasi-invariant pairs of cocycles taking values in arbitrary discrete groups.

Theorem 4.5. *Let R be discrete Borel equivalence relation on a standard Borel space $(X, \mathcal{S})$, μ a probability measure on $\mathcal{S}$ which is quasi-invariant and ergodic under R, and V an ergodic weakly asymptotically central automorphism of (R,μ). If H is a discrete group and $(c_1,c_2) \colon R \longrightarrow H^2$ a complementary quasi-invariant pair of cocycles, then the sigma-algebra $\mathcal{S}^{\ker^*(c_1,c_2)}$ is either nonatomic or trivial* (mod μ).

Proof. If $A \in \mathcal{S}^{\ker^*(c_1,c_2)}$ and $\mu(A) > 0$, then the V-invariance of $\ker^*(c_1, c_2)$ implies that $V^k A \in \mathcal{S}^{\ker^*(c_1,c_2)}$ for every $k \in \mathbb{Z}$. If A is an atom of $\mathcal{S}^{\ker^*(c_1,c_2)}$ then $V^k A$ is either equal to or disjoint from A. This shows that the orbit of A consists of finitely many atoms of equal size which are permuted cyclically by V. This is impossible by (3.14). □

5. Ergodicity of the components of a complementary pair

Let R be a discrete Borel equivalence relation on a standard Borel space $(X, \mathcal{S})$, μ a probability measure on $\mathcal{S}$ which is quasi-invariant under R, and V a weakly asymptotically central automorphism of (R, μ). If H is a Polish group of the form $H = H_1 \times H_2$, where H_1 is a compact extension of an abelian group and H_2 is a discrete nilpotent group, then Corollary 4.4 shows that any complementary quasi-invariant pair $(c_1, c_2)\colon R \longrightarrow H^2$ of cocycles is $*$-ergodic. In this section we investigate the ergodic behaviour of the individual components $c_i\colon R \longrightarrow H$ of a complementary quasi-invariant pair $(c_1, c_2)\colon R \longrightarrow H^2$.

In contrast to Corollary 4.4, the components of a complementary pair of cocycles are — in general — not ergodic (cf. Example 9.6), but Theorem 5.3 below shows that they have identical ergodic behaviour.

Definition 5.1. Let R be a discrete Borel equivalence relation on a standard Borel space $(X, \mathcal{S})$, H a Polish group and μ a probability measure on $\mathcal{S}$ which is quasi-invariant under R. Two cocycles $c, c'\colon R \longrightarrow H$ are *equally ergodic* if the following conditions are equivalent for every pair $A, B \in \mathcal{S}$ with $\mu(R(A) \cap B) > 0$:

(1) there exists an $(x, y) \in R$ with $x \in A$, $y \in B$ and $c(x, y) \in N(1_H)$,
(2) there exists an $(x, y) \in R$ with $x \in A$, $y \in B$ and $c'(x, y) \in N(1_H)$,
(3) there exists an $(x, y) \in R$ with $x \in A$, $y \in B$, $c(x, y) \in N(1_H)$ and $c'(x, y) \in N(1_H)$.

Remarks 5.2. (1) If the group H is discrete then two cocycles $c, c'\colon R \longrightarrow H$ are equally ergodic if and only if

$$\mathcal{S}^{\ker(c)} = \mathcal{S}^{\ker(c,c')} = \mathcal{S}^{\ker(c')} = \mathcal{S}^{\ker(c) \vee \ker(c')} \pmod{\mu}, \tag{5.1}$$

where $\ker(c, c') = \ker(c) \cap \ker(c')$ and $\ker(c') \vee \ker(c')$ is the smallest equivalence relation containing $\ker(c)$ and $\ker(c')$.

(2) Suppose that the group H is locally compact and second countable, and that λ_H is a left or right Haar measure of H. We denote by $R^{(c)}$ and $R^{(c')}$ the skew-product relations (2.9) on $X \times H$ with quasi-invariant sigma-finite measure $\mu \times \lambda_H$ and regard c and c' as cocycles $\mathbf{c}$ on $R^{(c')}$ and $\mathbf{c}'$ on $R^{(c)}$ by setting

$$\mathbf{c}((x, g), (x', g')) = c(x, x') \text{ for every } ((x, g), (x', g')) \in R^{(c')},$$
$$\mathbf{c}'((x, g), (x', g')) = c'(x, x') \text{ for every } ((x, g), (x', g')) \in R^{(c)}.$$

Then the following conditions are equivalent:

(a) c and c' are equally ergodic,
(b) $\mathbf{c}$ is ergodic on $R^{(c')}$ and vice versa.

Theorem 5.3. *Let R be a discrete Borel equivalence relation on a standard Borel space $(X, \mathcal{S})$, μ a probability measure on $\mathcal{S}$ which is quasi-invariant under R, and V a weakly asymptotically central automorphism of (R, μ). Assume furthermore that H is a Polish group of the form $H = H_1 \times H_2$, where H_1 a compact extension of an abelian group and H_2 is a discrete nilpotent group, and that $(c_1, c_2)\colon R \longrightarrow H^2$ is a complementary quasi-invariant pair of cocycles. Then c_1 and c_2 are equally ergodic.*

Proof. Since the proof of this result is similar to those of the Theorems 4.2–4.3 we restrict ourselves to describing the details in the case where $H = H_2$ is a discrete nilpotent group.

Let $A, B \in \mathcal{S}$ with $\mu(R(A)\cap B) \geq \mu(\ker(c_1)(A)\cap B) > 0$. Then we can find elements $g \in H$ and $W \in [\ker(c_1)]$ and a Borel set $A' \subset A$ with $\mu(A') > 0$, $B' = WA' \subset B$ and $c_2(Wx, x) = g$ for every $x \in A'$.

There exists a filtration of subgroups $H = H_n \supset H_{n-1} \supset \cdots \supset H_0 = \{1_H\}$ such that $H_m/H_{m-1} = C(H/H_{m-1})$ for $m = 1, \dots, n$. We define $c_i^{(n)}\colon R \longrightarrow H/H_{n-1}$ as in the proof of Theorem 4.3 and use (3.14)–(3.15) and (3.4) to ensure that

$$\lim_{k\to\infty} \mu(V^{-k}W^{-1}V^kB' \triangle B') = 0,$$
$$\lim_{k\to\infty} \mu(\{x \in X : c_1(V^{-k}W^{-1}V^kx, x) = 1_H\}) = 1.$$

For every $k \geq 0$ we put

$$B_k = B' \cap V^{-k}WV^kB' \cap \{x \in X : c_1(V^{-k}WV^kx, x) = 1_H\}$$

and obtain that $\lim_{k\to\infty} \mu(B' \smallsetminus B_k) = 0$, and that there exist infinitely many $k \geq 0$ with $\mu(B_k \cap V^{-k}B_k) > \mu(B')^2/2$. If $A_k = W^{-1}(B_k \cap V^{-k}B_k)$, $W_k = V^{-k}W^{-1}V^kW$, $x \in A_k$, $y = Wx \in B_k$, then

$$x \in A',\ W_kx \in B',\ V^ky \in B',\ c_1(W^{-1}y, y) = 1_H,\ c_2(W^{-1}y, y) = g^{-1},$$
$$\begin{aligned} c_2(V^{-k}W^{-1}V^ky, y) &= c_1(y, V^{-k}W^{-1}V^ky)c_2(V^{-k}W^{-1}V^ky, y) \\ &= \mathbf{f}(k, y)^{-1}c_1(V^ky, W^{-1}V^ky)c_2(W^{-1}V^ky, V^ky)\mathbf{f}(k, y) \\ &= \mathbf{f}(k, y)^{-1}c_2(W^{-1}V^ky, V^ky)\mathbf{f}(k, y) \\ &= \mathbf{f}(k, y)^{-1}g^{-1}\mathbf{f}(k, y), \end{aligned}$$

with $\mathbf{f}\colon \mathbb{Z} \times X \longrightarrow H$ defined by (3.9), and

$$c_1(W_kx, x) = 1_H,$$
$$c_2(W_kx, x) = c_2(W_kx, Wx)c_2(Wx, x) = \mathbf{f}(k, y)^{-1}g^{-1}\mathbf{f}(k, y)g \in H_{n-1}.$$

Choose a subset $A^{(n)} \subset A$ with positive measure and an element $W^{(n)} \in [\ker(c_1)]$ such that $W^{(n)}A^{(n)} \subset B$ and $c_2(W^{(n)}x, x) = g_{n-1}$ for some $g_{n-1} \in H_{n-1}$ and every $x \in A'_n$, and repeat the above argument to find a Borel $A^{(n-1)} \subset A^{(n)}$ with $\mu(A^{(n-1)}) > 0$ and an element $W^{(n-1)} \in [\ker(c_1)]$ with

$$W^{(n-1)}A^{(n-1)} \subset B,\ \ c_2(W^{(n-1)}x, x) \in H_{n-2},$$

for every $x \in A^{(n-1)}$.

By repeating this argument we eventually obtain a Borel set $A^{(1)} \subset A$ with $\mu(A^{(1)}) > 0$ and an element $W^{(1)} \in [\ker(c_1)]$ with

$$W^{(1)}A^{(1)} \subset B, \;\; c_2(W^{(1)}x, x) = 1_H,$$

for every $x \in A^{(1)}$. This shows that

$$\mathcal{S}^{\ker(c_1)} \supset \mathcal{S}^{\ker(c_1,c_2)} = \mathcal{S}^{\ker(c_1)\cap\ker(c_2)} \pmod{\mu}.$$

By symmetry,

$$\mathcal{S}^{\ker(c_2)} = \mathcal{S}^{\ker(c_1,c_2)} \pmod{\mu}.$$

We choose countable groups $\Gamma_i \subset [\ker(c_i)]$ satisfying (2.2) and obtain that every $B \in \mathcal{S}^{\ker(c_1,c_2)}$ differs by a null-set from a set B' which is both Γ_1- and Γ_2-invariant. Hence $\mathcal{S}^{\ker(c_1,c_2)} = \mathcal{S}^{\ker(c_1)\vee\ker(c_2)} \pmod{\mu}$, which completes the proof of the theorem. □

6. Gibbs cocycles on shift spaces

Let A be a finite set and $A^{\mathbb{Z}}$ the compact space of all two-sided sequences $x = (x_n)$ with $x_n \in A$ for every $n \in \mathbb{Z}$. We write $\mathcal{S} = \mathcal{B}_{A^{\mathbb{Z}}}$ for the Borel field of $A^{\mathbb{Z}}$ and $\sigma\colon A^{\mathbb{Z}} \longrightarrow A^{\mathbb{Z}}$ for the *shift*

$$(\sigma x)_n = x_{n+1} \tag{6.1}$$

on $A^{\mathbb{Z}}$. A closed, shift-invariant subset $X \subset A^{\mathbb{Z}}$ is a *subshift*, and the restriction of σ to a subshift X is again denoted by σ.

We fix a subshift $X \subset A^{\mathbb{Z}}$. The *Gibbs relation*

$$\Delta_X = \{(x, x') \in X \times X : x_n \neq x'_n \text{ for only finitely many } n \in \mathbb{Z}\} \tag{6.2}$$

is a discrete Borel equivalence relations on X, and $\sigma \in \mathrm{Aut}(\Delta_X)$.

Let H be a Polish group. A map $f\colon X \longrightarrow H$ is a *cylinder map* if it is continuous when viewed as a map into the discrete group H, i.e. if there exists an integer $N \geq 0$ such that f is constant on each cylinder set

$$C_N(x) = \{y \in X : x_n = y_n \text{ for } n = -N, \dots, N\}, \;\; x \in X. \tag{6.3}$$

If H has a distinguished bi-invariant metric δ (i.e. $\delta(hh_1, hh_2) = \delta(h_1, h_2) = \delta(h_1h, h_2h)$ for all $h, h_1, h_2 \in H$), then a map $f\colon X \longrightarrow H$ has *summable variation* if

$$\sum_{n\geq 0} \omega_n(f) < \infty, \tag{6.4}$$

where

$$\omega_n(f) = \sup_{\substack{x,y\in X\\ x_k=y_k \text{ for } k=-n,\dots,n}} \delta(f(x), f(y)) \tag{6.5}$$

for every $n \geq 0$. For $H = \mathbb{R}^d$, $d \geq 1$, the metric δ is assumed to be the Euclidean metric. If H is discrete, a map $f\colon X \longrightarrow H$ has summable variation if and only if it is continuous.

We fix a cylinder map (or, if H admits a bi-invariant metric, a map with summable variation) $f\colon X \longrightarrow H$, define the map $\mathbf{f}\colon \mathbb{Z} \times X \longrightarrow H$ by (3.9)–(3.10) with σ replacing V, and set, for every $(x, x') \in \Delta_X$ and $L \geq 0$,

$$\begin{aligned} a_f^+(x, x')^{(L)} &= \mathbf{f}(L, x)^{-1}\mathbf{f}(L, x'), \\ a_f^-(x, x')^{(L)} &= \mathbf{f}(L, \sigma^{-L}x)\mathbf{f}(L, \sigma^{-L}x')^{-1} = \mathbf{f}(-L, x)^{-1}\mathbf{f}(-L, x'). \end{aligned} \tag{6.6}$$

Put

$$a_f^+(x,x') = \lim_{L\to\infty} a_f^+(x,x')^{(L)}, \qquad a_f^-(x,x') = \lim_{L\to\infty} a_f^-(x,x')^{(L)},$$
$$a_f^*(x,x') = a_f^-(x',x)a_f^+(x,x'). \tag{6.7}$$

Then the maps $a_f^\pm \colon \Delta_X \longrightarrow H$ are well defined cocycles, i.e.

$$a_f^+(x,x')a_f^+(x',x'') = a_f^+(x,x''), \qquad a_f^-(x,x')a_f^-(x',x'') = a_f^-(x,x''),$$

and

$$\begin{aligned} a_f^+(\sigma x,\sigma x') &= f(x)a_f^+(x,x')f(x')^{-1}, \\ a_f^-(\sigma x,\sigma x') &= f(x)a_f^+(x,x')f(x')^{-1}, \\ a_f^*(\sigma x,\sigma x') &= f(x')a_f^*(x,x')f(x')^{-1} \end{aligned} \tag{6.8}$$

for all $(x,x'),(x,x'') \in \Delta_X$. The cocycles $a_f^-, a_f^+ \colon \Delta_X \longrightarrow H$ will be called the *Gibbs cocycles* defined by f.

The following observation allows us to apply Corollary 4.4 to the pair of Gibbs cocycles (a_f^-, a_f^+).

Theorem 6.1. *Let A be a finite set, $X \subset A^{\mathbb{Z}}$ a subshift, μ a nonatomic shift-invariant probability measure on X, and $R \subset \Delta_X$ a μ-nonsingular shift-invariant subrelation. Then the shift σ is a weakly asymptotically central automorphism of (R,μ).*

Suppose furthermore that H is a Polish group and $f \colon X \longrightarrow H$ a cylinder map (if H admits a bi-invariant metric it suffices to assume that f has summable variation). Then the pair of Gibbs cocycles $(a_f^-, a_f^+) \colon R \longrightarrow H^2$ in (6.6)–(6.7) *is quasi-invariant and complementary.*

Corollary 6.2. *Let A be a finite set, $X \subset A^{\mathbb{Z}}$ a subshift, μ a nonatomic shift-invariant probability measure on X, and $R \subset \Delta_X$ a μ-nonsingular shift-invariant subrelation.*

Suppose furthermore that H is a Polish group of the form $H = H_1 \times H_2$, where H_1 is a compact extension of an abelian group and H_2 is a discrete nilpotent group, and that $f = (f_1, f_2) \colon X \longrightarrow H$ is a map with the property that $f_1 \colon X \longrightarrow H_1$ has summable variation (with respect to some fixed bi-invariant metric δ on H_1), and $f_2 \colon X \longrightarrow H_2$ is a cylinder map. Then the pair of Gibbs cocycles $(a_f^-, a_f^+) \colon R \longrightarrow H^2$ in (6.6)–(6.7) *is $*$-ergodic and equally ergodic.*

Proof. Theorem 6.1, Corollary 4.4 and Theorem 5.3. □

Remark 6.3. If the group H in Corollary 6.2 is discrete then the shift-invariant subrelation $\ker^*(a_f^-, a_f^+) \subset \Delta_X$ satisfies that $(x,x') \in \ker^*(a_f^-, a_f^+)$ if and only if

$$f(\sigma^n x) \cdots f(\sigma^{-n} x) = f(\sigma^n x') \cdots f(\sigma^{-n} x') \tag{6.9}$$

for all sufficiently large $n \geq 0$ (cf. [17] and [18]).

Although the assumptions on the group H in Corollary 6.2 may be unnecessarily strong, it is easy to see that (6.9) cannot hold without any restrictions on H: if H is the free group on two generators a, b, $A = \{a, b\}$, $X = A^{\mathbb{Z}}$ and $f(x) = x_0$ for every $x \in X$, then (6.9) shows that $(x,x') \in \ker^*(a_f^-, a_f^+)$ if and only if $x = x'$, i.e. that (a_f^-, a_f^+) cannot be ergodic with respect to any nonatomic shift-invariant measure on X. In order to get beyond the hypotheses of Corollary 6.2 we have to impose a recurrence condition on the map $f\colon X \longrightarrow H$ in (6.6)–(6.8).

Definition 6.4. Let $(X, \mathcal{S})$ be a standard Borel space, $V \in \mathrm{Aut}(X, \mathcal{S})$, μ a V-nonsingular probability measure on $\mathcal{S}$, H a Polish group, $f\colon X \longrightarrow H$ a Borel map, and $\mathbf{f}\colon \mathbb{Z} \times X \longrightarrow H$ the cocycle defined in (3.9)–(3.10). The map f and the cocycle $\mathbf{f}$ are *recurrent* if there exist, for every $A \in \mathcal{S}$ with $\mu(A) > 0$ and every neighbourhood $N(1_H)$ of the identity in H, an $x \in A$ and an $n \in \mathbb{Z}$ with $V^n x \neq x$ and $\mathbf{f}(n, x) \in N(1_H)$. The map f and the cocycle $\mathbf{f}$ are *inner recurrent* if there exist, for every $A \in \mathcal{S}$ with $\mu(A) > 0$, every $g \in H$ and every neighbourhood $N(1_H)$ of the identity in H, an $x \in A$ and an $n \in \mathbb{Z}$ with $V^n x \neq x$ and $\mathbf{f}(n, x) g \mathbf{f}(n, x)^{-1} \in N(1_H) g$.

Examples 6.5. (1) Every recurrent map is inner recurrent; however, recurrence is obviously a much stronger condition than inner recurrence. Whereas every Borel map with values in a Polish group H with compact conjugacy classes is inner recurrent, the results on recurrence of random walks in [19] raise the following question (cf. [7]): suppose that $X \subset A^{\mathbb{Z}}$ is a mixing shift of finite type (cf. (7.1)) and $\mu = \mu_\phi$ the Gibbs measure arising from a map $\phi\colon X \longrightarrow \mathbb{R}$ with summable variation (cf. (7.2)); is every continuous recurrent map $f\colon X \longrightarrow H$ with values in a discrete group H cohomologous to a map $f'\colon X \longrightarrow H' \subset H$, where H' is a finite extension of $\mathbb{Z}^d$ with $d \leq 2$?

(2) Let H' be a discrete group, H'' a Polish group, and let $\alpha\colon H' \longrightarrow \mathrm{Aut}(H'')$ be a homomorphism from H' into the group of continuous automorphisms of H''. We denote by $H' \ltimes_\alpha H''$ (or simply by $H' \ltimes H''$) the semi-direct product of H', H'', defined as

$$H = H' \times H'' \tag{6.10}$$

with group operation

$$(h_1', h_1'') \cdot (h_2', h_2'') = (h_1' h_2', h_1'' \alpha^{h_1'}(h_2'')). \tag{6.11}$$

We identify the groups H', H'' with the subgroups

$$\begin{aligned} \bar{H}' &= \{(h', 1_{H''}) : h' \in H'\} \cong H', \\ \bar{H}'' &= \{(1_{H'}, h'') : h'' \in H''\} \cong H'' \end{aligned} \tag{6.12}$$

of H. If X is a set and $f\colon X \longrightarrow H = H' \ltimes H''$ a map then we write f as

$$f = (f', f'') \tag{6.13}$$

with components $f'\colon X \longrightarrow H'$ and $f''\colon X \longrightarrow H''$.

If one of the two groups H', H'' is abelian and the other one is compact, then the conjugacy class of every $h \in H = H' \ltimes H''$ has compact closure, and every Borel map $f\colon X \longrightarrow H$ is inner recurrent.

(3) Let H be a Polish group with centre $C(H)$, and let $f\colon X \longrightarrow H$ be a Borel map such that $\pi \circ f\colon X \longrightarrow H/C(H)$ is recurrent, where $\pi\colon H \longrightarrow H/C(H)$ is the quotient map. Then f is inner recurrent.

Theorem 6.6. *Let A be a finite set, $X \subset A^{\mathbb{Z}}$ a subshift, H a locally compact second countable group, and $f\colon X \longrightarrow H$ an inner recurrent cylinder map* (*if*

H admits a bi-invariant metric it suffices to assume that f is inner recurrent and has summable variation).

Let furthermore μ be a nonatomic shift-invariant probability measure on X and $R \subset \Delta_X$ a μ-nonsingular shift-invariant subrelation. Then the pair of Gibbs cocycles (a_f^-, a_f^+) is $*$-ergodic and equally ergodic on R.

Proof. We prove the theorem in the special case where H is discrete and f is continuous; the modifications necessary for the general case will be indicated at the end of the proof.

For convenience we go to a higher block representation of X and assume that f depends only on the coordinate x_0 of every $x \in X$.

Denote by $\Gamma' \subset [R]$ a countable set of maps of the following form:

(a) $R = \{(\gamma x, x) : \gamma \in \Gamma',\, x \in X\}$,

(b) for every $\gamma \in \Gamma'$ there exist nonempty cylinder sets

$$\begin{aligned} A(\gamma) &= [i_{-N(\gamma)}, \dots, i_{N(\gamma)}] \\ &\quad = \{x = (x_n) \in X : x_k = i_k \text{ for } k = -N(\gamma), \dots, N(\gamma)\}, \\ B(\gamma) &= [j_{-N(\gamma)}, \dots, j_{N(\gamma)}] \\ &\quad = \{x = (x_n) \in X : x_k = j_k \text{ for } k = -N(\gamma), \dots, N(\gamma)\} \end{aligned}$$

with $N(\gamma) \geq 1$, $i_{-N(\gamma)} = j_{-N(\gamma)}$, $i_{N(\gamma)} = j_{N(\gamma)}$, and Borel sets $A'(\gamma) \subset A(\gamma)$, $B'(\gamma) \subset B(\gamma)$, such that $\mu(A'(\gamma)) > 0$, $\gamma A'(\gamma) = B'(\gamma)$ and

$$(\gamma x)_k = \begin{cases} x_k & \text{if } x \notin A'(\gamma) \cup B'(\gamma) \text{ and } k \in \mathbb{Z}, \\ & \qquad \text{or if } x \in A'(\gamma) \cup B'(\gamma) \text{ and } |k| \geq N(\gamma), \\ j_k & \text{if } x \in A'(\gamma) \text{ and } k = -N(\gamma), \dots, N(\gamma) \\ i_k & \text{if } x \in B'(\gamma) \text{ and } k = -N(\gamma), \dots, N(\gamma). \end{cases}$$

Then the complementary pair (a_f^-, a_f^+) satisfies that

$$a_f^-(\sigma^{-n}\gamma x, \sigma^{-n} x) = a_f^+(\sigma^n \gamma x, \sigma^n x) = 1_H \tag{6.14}$$

whenever $\gamma \in \Gamma'$, $n \geq N(\gamma)$ and $x \in X$.

Let $A, B \in \mathcal{S}$ satisfy that $\mu(R(A) \cap B) > 0$. We choose a Borel set $A' \subset A \cap A'(\gamma)$ and elements $\gamma \in \Gamma'$, $g \in H$, with $\mu(A') > 0$, $B' = \gamma A' \subset B \cap B'(\gamma)$ and $a_f^*(\gamma x, x) = g$ for every $x \in A'$ (cf. (6.7)).

We claim that there exist a $y \in A'$ and an integer $n > 2N(\gamma)$ with

$$\mathbf{f}(n, y) g = g \mathbf{f}(n, y), \quad \sigma^n y \in A' \quad \text{and} \quad \sigma^{-n} \gamma \sigma^n y \in A'. \tag{6.15}$$

In order to prove (6.15) we denote by λ_H the Haar (= counting) measure on H and set $\nu = \mu \times \lambda_H$. Since f is inner recurrent, the ν-preserving skew-product transformation

$$S(x, h) = (\sigma x, f(x) h f(x)^{-1}) \tag{6.16}$$

on $X \times H$ is conservative. We induce S on the set $\bar{X} = X \times \{g\}$ and view the induced transformation as a μ-preserving automorphism of X: for every

$x \in X$ we set

$$r(x) = \begin{cases} \min\{k \geq 1 : \mathbf{f}(k,x)g = g\mathbf{f}(k,x)\} & \text{if the sets} \\ & \{k \geq 1 : \mathbf{f}(k,x)g = g\mathbf{f}(k,x)\} \text{ and} \\ & \{k \geq 1 : \mathbf{f}(-k,x)g = g\mathbf{f}(-k,x)\} \\ & \text{are both infinite,} \\ 0 & \text{otherwise,} \end{cases} \tag{6.17}$$

and put $Tx = S^{r(x)}x$. Then $T \in [R[\sigma]]$, where

$$R[\sigma] = \{(\sigma^k x, x) : k \in \mathbb{Z},\, x \in X\}$$

is the equivalence relation generated by σ.

For every $n \in \mathbb{Z}$ and $x \in X$, $T^n x = \sigma^{\mathbf{r}(n,x)}x$, where

$$\mathbf{r}(n,x) = \begin{cases} \sum_{k=0}^{n-1} r(T^k x) & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -\mathbf{r}(-n, T^n x) & \text{if } n < 0. \end{cases}$$

is the n-th return time of (x, g) to $\bar{X}$ under S.

Put $C = \sigma^{-N(\gamma)}(A(\gamma) \cup B(\gamma))$ and denote by T_C the automorphism of C induced by T. Then there exists a Borel map $\mathbf{r}' \colon \mathbb{Z} \times C \longrightarrow \mathbb{Z}$ with $T_C^n x = \sigma^{\mathbf{r}'(n,x)}x$ for every $n \in \mathbb{Z}$ and $x \in C$.

For $n \geq 0$, $\mathbf{r}'(-n, x)$ depends only on the coordinates x_k, $k < 0$, which are unaffected by $\sigma^{-N(\gamma)}\gamma\sigma^{N(\gamma)}$. Hence

$$\mathbf{r}'(-n, \sigma^{-N(\gamma)}\gamma\sigma^{N(\gamma)}x) = \mathbf{r}'(-n, x)$$

for every $x \in C$ and $n \geq 0$, and the map

$$T_n' x = \begin{cases} T_C^{-n}\sigma^{-N(\gamma)}\gamma\sigma^{N(\gamma)}T_C^n x & \text{if } x \in C, \\ x & \text{otherwise,} \end{cases}$$

lies in $[R]$ for every $n \geq 0$. For every cylinder set $D \subset C$, $T_n' D = D$ for every sufficiently large $n \geq 1$, and by approximating arbitrary Borel sets in C by closed and open subsets of C we obtain that

$$\lim_{n \to \infty} \mu(T_n' D \triangle D) = 0 \tag{6.18}$$

for every Borel set $D \subset C$.

In order to complete the proof of (6.15) we set

$$D = \sigma^{-N(\gamma)}A' \subset \sigma^{-N(\gamma)}A(\gamma)$$

and use (6.18) and the conservativity of T_C to conclude that there exists a point $x \in D$ with $\sigma^{\mathbf{r}'(n,x)}x \in D$,

$$\sigma^{-\mathbf{r}'(n,x)}\sigma^{-N(\gamma)}\gamma\sigma^{N(\gamma)}\sigma^{\mathbf{r}'(n,x)}x \in D,$$

and $\mathbf{f}(\mathbf{r}'(n,x), x)g = g\mathbf{f}(\mathbf{r}'(n,x), x)$ for every n in an infinite subset $\mathbf{N} \subset \mathbb{N}$.

The point $y = \sigma^{N(\gamma)}x \in A'$ satisfies that $\sigma^{\mathbf{r}'(n,x)}y \in A'$, $\mathbf{f}(\mathbf{r}'(n,x), y)g = g\mathbf{f}(\mathbf{r}'(n,x), y)$ and $\sigma^{-\mathbf{r}'(n,x)}\gamma\sigma^{\mathbf{r}'(n,x)}y \in A'$ for every $n \in \mathbf{N}$. This proves (6.15).

We fix an integer $n > 2N(\gamma)$ in $\mathbf{N}$ and set

$$a = f(y_{-1}) \cdots f(y_{-N(\gamma)}), \;\; b = f(y_{N(\gamma)}) \cdots f(y_0),$$

$$a' = f((\gamma y)_{-1}) \cdots f((\gamma y)_{-N(\gamma)}), \;\; b' = f((\gamma y)_{N(\gamma)}) \cdots f((\gamma y)_0), \\ c = f(y_{n-N(\gamma)-1}) \cdots f(y_{N(\gamma)+1}).$$

Then $\mathbf{f}(n,y)$ commutes with g, $a_f^*(\gamma y, y) = g$, $a_f^-(\sigma^{-n}\gamma\sigma^n y, y) = 1_H$, and

$$\begin{aligned} a_f^*(\gamma y, \sigma^{-n}\gamma\sigma^n y) &= a_f^-(\sigma^{-n}\gamma\sigma^n y, \gamma y) a_f^+(\gamma y, \sigma^{-n}\gamma\sigma^n y) \\ &= a_f^-(y, \gamma y) a_f^+(\gamma y, y) a_f^+(y, \sigma^{-n}\gamma\sigma^n y) \\ &= a_f^*(\gamma y, y) a_f^+(\sigma^{-n}\gamma\sigma^n y, y)^{-1} \\ &= a_f^*(\gamma y, y) a_f^+(\sigma^{-n}\gamma\sigma^n y, y)^{-1} a_f^-(y, \sigma^{-n}\gamma\sigma^n y)^{-1} \\ &= a_f^*(\gamma y, y) a_f^*(\sigma^{-n}\gamma\sigma^n y, y)^{-1} \\ &= a_f^*(\gamma y, y) \mathbf{f}(n,y)^{-1} a_f^*(\gamma\sigma^n y, \sigma^n y)^{-1} \mathbf{f}(n,y) \\ &= gg^{-1} = 1_H. \end{aligned}$$

Since $y' = \sigma^{-n}\gamma\sigma^n y \in A$, $\gamma y \in B$ and $(\gamma y, y') \in R$, we have proved the $*$-ergodicity of (a_f^-, a_f^+).

If one assumes at the beginning of this proof that $a_f^-(\gamma x, x) = 1_H$ and $a_f^*(\gamma x, x) = a_f^+(\gamma x, x) = g$, then the same proof shows that the element $(\gamma y, y') \in R$ satisfies that $y' \in A$, $\gamma y \in B$, $(\gamma y, y') \in R$ and $a_f^-(\gamma y, y') = a_f^+(\gamma y, y') = 1_H$. As in the proof of Theorem 5.3, this implies that a_f^- and a_f^+ are equally ergodic.

Finally we mention the changes necessary if H is an arbitrary locally compact second countable group with left Haar measure λ_H. In this case the map S in (6.16) has to be replaced by the map $\bar{S}\colon X \times H \times \mathbb{R}$ of the form

$$S(x,h,t) = (\sigma x, f(x) h f(x)^{-1}, \Lambda_H(f(x)) t),$$

where Λ_H is the modular function of H. Then S preserves the measure $\nu = \mu \times \lambda_H \times \lambda_{\mathbb{R}}$ and is conservative. We induce S on $\bar{X} = X \times N(1_H) g \times [1-\varepsilon, 1+\varepsilon]$ for some compact neighbourhood $N(1_H)$ of the identity in H and some $\varepsilon > 0$. The map r in (6.17) is now a map from $X \times N(1_H) g \times [1-\varepsilon, 1+\varepsilon]$ to $\mathbb{Z}$, and the rest of the proof goes through with obvious changes. ☐

By combining the proofs of the Theorems 4.3 and 6.6 we obtain the following result.

Theorem 6.7. *Let A be a finite set, $X \subset A^{\mathbb{Z}}$ a subshift, μ a nonatomic shift-invariant probability measure on X and $R \subset \Delta_X$ a μ-nonsingular shift-invariant subrelation.*

Let furthermore $H = H' \ltimes H''$ be a semi-direct product of two discrete groups, where H'' is nilpotent, and let $f = (f', f'')\colon X \longrightarrow H$ be a cylinder map (cf. (6.13)). If f' is recurrent, then the pair of Gibbs cocycles (a_f^-, a_f^+) is $*$-ergodic and equally ergodic on R.

Proof. We use the same notation as in the proof of Theorem 6.6.

Let $A, B \in \mathcal{S}$ satisfy that $\mu(R(A) \cap B) > 0$. We choose a Borel set $A' \subset A$ and elements $\gamma \in \Gamma'$, $g \in H$, with $\mu(A') > 0$, $B' = \gamma A' \subset B$ and $a_f^*(\gamma x, x) = g$ for every $x \in A'$. As in the proof of Theorem 6.6 we show that there exists, for μ-a.e. $x \in A'$, an infinite subset $\mathbf{N}_x \subset \mathbb{N}$ with $\mathbf{f}(n,x) \in \bar{H}''$, $\sigma^n x \in A'$,

$a_f^-(\sigma^{-n}\gamma\sigma^n x, x) = 1_H$, and $\sigma^{-n}\gamma\sigma^n x \in A'$ for every $n \in \mathbf{N}_x$, where $\bar{H}''$ is defined in (6.12). Then

$$\begin{aligned} a_f^*(\gamma x, \sigma^{-n}\gamma\sigma^n x) &= a_f^-(\sigma^{-n}\gamma\sigma^n x, \gamma x) a_f^+(\gamma x, \sigma^{-n}\gamma\sigma^n x) \\ &= a_f^-(x, \gamma x) a_f^+(\gamma x, x) a_f^+(x, \sigma^{-n}\gamma\sigma^n x) \\ &= a_f^*(\gamma x, x) a_f^+(\sigma^{-n}\gamma\sigma^n x, x)^{-1} \\ &= a_f^*(\gamma x, x) a_f^+(\sigma^{-n}\gamma\sigma^n x, x)^{-1} a_f^-(x, \sigma^{-n}\gamma\sigma^n x)^{-1} \\ &= a_f^*(\gamma x, x) a_f^*(\sigma^{-n}\gamma\sigma^n x, x)^{-1} \\ &= a_f^*(\gamma x, x)\mathbf{f}(n, x)^{-1} a_f^*(\gamma\sigma^n x, \sigma^n x)^{-1}\mathbf{f}(n, x) \in \bar{H}''. \end{aligned} \tag{6.19}$$

The same induction argument as in the proof of Theorem 4.3 allows us to find an element $W \in [R]$ and a subset $A'' \subset A$ with $\mu(A'') > 0$, $WA'' \subset B$, and $a_f^*(Wx, x) = 1_H$ for every $x \in A''$.

The proof that a_f^- and a_f^+ are equally ergodic is analogous. □

Corollary 6.8. *Let A be a finite set, $X \subset A^{\mathbb{Z}}$ a subshift, μ a nonatomic shift-invariant probability measure on X and $R \subset \Delta_X$ a μ-nonsingular shift-invariant subrelation.*

Let furthermore H be a Polish group of the form $H = H_1 \times H_2$, where H_1 is a compact extension of an abelian group and $H_2 = H_2' \ltimes H_2''$ is a semi-direct product of two discrete groups, where H_2'' is nilpotent.

Suppose that $f = (f_1, f_2)\colon X \longrightarrow H$ is a map with the following properties:

(a) *the map $f_1\colon X \longrightarrow H_1$ has summable variation with respect to some fixed bi-invariant metric δ on H_1,*
(b) *the map $f_2'\colon X \longrightarrow H_2'$ is recurrent, where $f_2 = (f_2', f_2'')\colon X \longrightarrow H_2$ is written in the form* (6.13).

Then the pair of Gibbs cocycles (a_f^-, a_f^+) is $*$-ergodic and equally ergodic on R.

Proof. By assumption, H_1 contains a normal abelian subgroup H_1'' such that $H_1' = H_1/H_1''$ is compact. We denote by $\pi_1\colon H_1 \longrightarrow H_1'$ the quotient map. Then the map $f' = (\pi_1 \circ f_1, f_2')\colon X \longrightarrow H_1' \times H_2'$ is recurrent.

We fix an invariant symmetric neighbourhood $N(1_{H_1})$ of the identity in H_1, set $N(1_{H_1'}) = \pi_1(N(1_{H_1})) \subset H_1'$, and observe that $N(1_{H_1'})$ and $N(1_H) = N(1_{H_1}) \times \{1_{H_2}\}$ are invariant neighbourhoods of the identity in H_1' and H, respectively.

Choose a Borel set $A' \subset A$ and elements $\gamma \in \Gamma'$, $g \in H$, with $\mu(A') > 0$, $B' = \gamma A' \subset B$, and $a_f^*(\gamma x, x) \in gN(1_H)$. For μ-a.e. $x \in X$ there exists an infinite set $\mathbf{N}_x \subset \mathbb{N}$ with $\mathbf{f}'(n, x) \in N(1_{H_1'}) \times \{1_{H_2'}\}$, $\sigma^n x \in A'$, $a_f^-(\sigma^{-n}\gamma\sigma^n x, x) \in N(1_H)$, and $\sigma^{-n}\gamma\sigma^n x \in A'$ for every $n \in \mathbf{N}_x$.

The calculation (6.19) allows us to find an element $W \in [R]$ and a subset $A'' \subset A$ with $\mu(A'') > 0$, $WA'' \subset B$ and $a_f^*(Wx, x) \in N(1_{H_1})^2 \times \bar{H}_2''$, and the same induction argument as in the proof of Theorem 6.7 yields an element $W' \in [R]$ and a set $A''' \subset A$ with $\mu(A''') > 0$, $W'A''' \subset B$ and $a_f^*(Wx, x) \in N(1_{H_1})^m \times \{1_{H_2}\}$ for some m depending only on the length of the filtration (4.3) of H_2''. Since $N(1_{H_1})$ was arbitrary, we conclude that (a_f^-, a_f^+) is $*$-ergodic. The proof of equal ergodicity is completely analogous. □

Corollary 6.9. *Let A be a finite set, $X \subset A^{\mathbb{Z}}$ a subshift, μ a nonatomic shift-invariant probability measure on X and $R \subset \Delta_X$ a μ-nonsingular shift-invariant subrelation.*

Let furthermore H be a Polish group of the form $H = H' \ltimes_\alpha H''$, where H' and H'' are abelian groups and α is an action of H' by automorphisms of H'', and let $f = (f', f'')\colon X \longrightarrow H$ be a cylinder map with the property that $\alpha \circ f'\colon X \longrightarrow \mathrm{Aut}(H'')$ is recurrent with respect to the natural Polish topology on $\mathrm{Aut}(H'')$. Then the pair of Gibbs cocycles (a_f^-, a_f^+) is $*$-ergodic and equally ergodic on R.

Proof. The proof is essentially identical to that of Theorem 6.7. We denote by G the countable subgroup of H' generated by the values of f', furnished with the discrete topology, and consider f as a map from X into $\bar{H} = G \ltimes H''$ with the property that the map $\alpha \circ f\colon X \longrightarrow \mathrm{Aut}(H'')$ is recurrent. Since it suffices to prove the corollary with $\bar{H}$ replacing H we assume for simplicity that H' is countable and discrete.

The same calculation as in (6.19) shows that we can find, for every neighbourhood $N(1_{H''})$ of the identity in H'', and for all $A, B \in \mathcal{S}$ with $\mu(R(A) \cap B) > 0$, a pair of points $(x, x') \in R$ with $x \in A$, $x' \in B$ and $a_f^*(x, x') \in \{1_{H'}\} \times N(1_{H''})$. Similarly one proves equal ergodicity of (a_f^-, a_f^+). □

7. Ergodicity of cocycles on shifts of finite type

In this section we investigate the individual ergodicity of the Gibbs cocycles a_f^-, a_f^+ on a shift space after modification by a suitable coboundary.

Let $X \subset A^{\mathbb{Z}}$ be a *shift of finite type*, i.e. assume that there exists an integer $k \geq 1$ and a subset $P \subset A^k$ such that

$$X = \{(x_n) \in A^{\mathbb{Z}} : (x_n, \dots, x_{n+k-1}) \in P \text{ for every } n \in \mathbb{Z}\}. \tag{7.1}$$

If $\phi\colon X \longrightarrow \mathbb{R}$ is a map with summable variation, then there exists a unique probability measure μ_ϕ on the Borel field $\mathcal{S}$ of X which is quasi-invariant and ergodic under the Gibbs relation Δ_X of X, and whose Radon-Nikodym derivative (9.1) under Δ_X satisfies that

$$\rho_{\mu_\phi}(x, x') = a_\phi^*(x, x') = \sum_{k \in \mathbb{Z}} (\phi(\sigma^k x) - \phi(\sigma^k x')) \tag{7.2}$$

μ-a.e. on Δ_X (cf. [1], [10],[14], [20]).

Theorem 7.1. *Let A be a finite set, $X \subset A^{\mathbb{Z}}$ a mixing shift of finite type, and let μ_ϕ be the Gibbs measure arising from a map $\phi\colon X \longrightarrow \mathbb{R}$ with summable variation* (*cf.* (6.4)–(7.2)). *Let furthermore H be a Polish group and $f\colon X \longrightarrow H$ a cylinder map such that the pair of Gibbs cocycles $(a_f^-, a_f^+)\colon \Delta_X \longrightarrow H^2$ in* (6.6)–(6.8) *is $*$-ergodic and equally ergodic* (*cf. Corollary* 6.2 *and Theorems* 6.6–6.7). *Then there exists a cylinder map $b\colon X \longrightarrow H$ such that the cocycles*

$$a_{f'}^{\pm}(x, x') = b(x)^{-1} a_f^{\pm}(x, x') b(x') \tag{7.3}$$

on Δ_X are ergodic, where

$$f' = (b \circ \sigma)^{-1} f b. \tag{7.4}$$

If the group H admits a bi-invariant metric and $f\colon X \longrightarrow H$ has summable variation, and if the Gibbs cocycles $(a_f^-, a_f^+)\colon \Delta_X \longrightarrow H^2$ are $*$-ergodic and equally ergodic, then there exists a continuous *map $b\colon X \longrightarrow H$ such that the cocycles $a_{f'}^{\pm}$ in* (7.3)–(7.4) *are ergodic.*

Proof. We begin by assuming that f is a cylinder map and assume without loss in generality (by going over to a higher block representation of X, if necessary) that X satisfies (7.1) with $k = 2$, and that f depends only on the coordinate x_0 of each $x \in X$. For simplicity we also assume that H is discrete; the general case is proved by an approximate argument similar to (but simpler than) the second part of this proof.

From (6.6)–(6.7) we know that

$$\ker(a_f^-) \supset \Delta_X^+, \;\; \ker(a_f^+) \supset \Delta_X^-,$$

where

$$\begin{aligned} \Delta_X^+ &= \{(x,x') \in \Delta_X : x_n = x_n' \;\text{ for all }\; n < 0\}, \\ \Delta_X^- &= \{(x,x') \in \Delta_X : x_n = x_n' \;\text{ for all }\; n \geq 0\}. \end{aligned} \tag{7.5}$$

Put

$$\Delta_X^* = \Delta_X^- \vee \Delta_X^+ \tag{7.6}$$

and apply Theorem 5.3 and (5.1) to see that

$$\mathcal{S}^{\ker(a_f^+)\cap\ker(a_f^-)} \subset \mathcal{S}^{\Delta_X^*} \pmod{\mu_\phi}. \tag{7.7}$$

Since X is mixing, there exists an integer $N \geq 0$ such that the relation $\Delta^*_{C_N(x)}$ induced by Δ_X^* on each cylinder set $C_N(x)$ in (6.3) is μ_ϕ-ergodic for every $x \in X$ (cf. (7.7)), and hence that the relation $(\ker(a_f^+)\cap\ker(a_f^-))_{C_N(x)}$ is μ_ϕ-ergodic on $C_N(x)$ for every $x \in X$.

The collection $\mathcal{P} = \{C_N(x) : x \in X\}$ is a finite partition of X. We fix a $P \in \mathcal{P}$ and an element $x \in P$ and use $*$-ergodicity to find, for every $Q \neq P$ in $\mathcal{P}$, Borel sets $A_Q \subset P$, $B_Q \subset Q$, and elements $g_Q \in H$, $V_Q \in [\ker^*(a_f^-, a_f^+)]$ with

$$\mu_\phi(A_Q \cap \{x \in X : a_f^+(V_Q x, x) \in gN(1_H),\, a_f^-(V_Q x, x) \in gN(1_H)\}) > 0$$

for every neighbourhood $N(1_H)$ of the identity in H. Put

$$b(x) = \begin{cases} 1_H & \text{if } \; x \in P, \\ g_Q & \text{if } \; x \in Q \neq P. \end{cases} \tag{7.8}$$

Then b is a cylinder map, and (6.6)–(6.7) show that

$$a_{f'}^{\pm}(x,x') = b(x)^{-1} a_f^{\pm}(x,x') b(x'), \tag{7.9}$$

where f' is defined in (7.4). Our choice of b implies that the cocycles $a_{f'}^{\pm}$ are μ_ϕ-ergodic.

If H admits a bi-invariant metric and $f\colon X \longrightarrow H$ has summable variation, then we set, for every $N \geq 0$,

$$\Delta_X^{(N)} = \{(x,x') \in \Delta_X : x_k = x_k' \;\text{ for }\; k = -N, \dots, N\},$$
$$\Delta_N^{\pm} = \Delta_X^{(N)} \cap \Delta_X^{\pm},$$

and conclude that there exists, for every $\varepsilon > 0$, an integer $N_0 \geq 0$ with $\delta(a_f^-(x,x'), 1_H) < \varepsilon$ for every $(x,x') \in \Delta_{N_0}^+$ and $\delta(a_f^+(x,x'), 1_H) < \varepsilon$ for every $(x,x') \in \Delta_{N_0}^-$. If N_0 is sufficiently large then the argument in Theorem 5.3 allows us to find, for every $x \in X$ and every pair of Borel sets $A, B \subset C_{N_0}(x)$ with $\mu_\phi(A)\mu_\phi(B) > 0$, an element $W \in [\Delta_X]$ and a Borel set $A' \subset A$ with $\mu_\phi(A') > 0$, $WA' \subset B$ and $\delta(a_f^\pm(Wx,x), 1_H) < 2\varepsilon$ for every $x \in A'$. We write $\mathcal{P}_0$ for the partition of X into cylinders of the form $C_{N_0}(x)$, $x \in X$, and construct a map $b_0 \colon X \longrightarrow H$ as in (7.8) with the following properties:

(a:0) b_0 is constant on every $P \in \mathcal{P}_0$,

(b:0) if $f_1 = (b_0 \circ \sigma)^{-1} f b_0$, and if $a_{f_1}^\pm \colon \Delta_X \longrightarrow H$ is defined as in (6.6)–(6.7), then there exist, for every pair of sets $A, B \in \mathcal{S}$ with $\mu_\phi(A) > 0$ and $\mu_\phi(B) > 0$, an element $W \in [\Delta_X]$ and a Borel set $A' \subset A$ with $\mu_\phi(A') > 0$, $WA' \subset B$ and $\delta(a_{f_1}^\pm(Wx,x), 1_H) < 3\varepsilon$ for every $x \in A'$.

Choose $N_1 > N_0$ such that $\delta(a_{f_1}^-(x,x'), 1_H) < \varepsilon_2$ for all $(x,x') \in \Delta_X^{(N_1)}$ and repeat this argument to find a map $b_1 \colon X \longrightarrow H$ with the following properties:

(a:1) b_1 is constant on each cylinder set $C_{N_1}(x)$, $x \in X$,

(b:1) if $f_2 = (b_1 \circ \sigma)^{-1} f b_1$, then there exist, for every pair of sets $A, B \in \mathcal{S}$ with $\mu_\phi(A) > 0$ and $\mu_\phi(B) > 0$, an element $W \in [\Delta_X]$ and a Borel set $A' \subset A$ with $\mu_\phi(A') > 0$, $WA' \subset B$ and $\delta(a_{f_2}^\pm(Wx,x), 1_H) < 3\varepsilon/2$ for every $x \in A'$,

(c:1) $\delta(b_1(x), 1_H) < 3\varepsilon$ for every $x \in X$.

By repeating this construction we obtain recursively an increasing sequence of integers $(N_k,\ k \ge 0)$ and sequences of continuous maps $b_k, f_k \colon X \longrightarrow H$ with the following properties for every $k \ge 1$:

(a:k) b_k is constant on each cylinder set $C_{N_k}(x)$, $x \in X$,

(b:k) if $f_{k+1} = (b_k \circ \sigma) f_k b_k$, then there exist, for every pair of sets $A, B \in \mathcal{S}$ with $\mu_\phi(A) > 0$ and $\mu_\phi(B) > 0$, an element $W \in [\Delta_X]$ and a Borel set $A' \subset A$ with $\mu_\phi(A') > 0$, $WA' \subset B$ and $\delta(a_{f_{k+1}}^\pm(Wx,x), 1_H) < 3\varepsilon/2^k$ for every $x \in A'$,

(c:k) $\delta(b_k(x), 1_H) < 3\varepsilon/2^{k-1}$ for every $x \in X$.

The map $b = \lim_{n\to\infty} b_0 \cdots b_n$ is continuous, and we set $f' = (b \circ \sigma)^{-1} f b$ and obtain that $a_{f'}^\pm(x,x') = b(x)^{-1} a_f^\pm(x,x') b(x')$ for every $(x,x') \in \Delta_X$, and that $a_{f'}^\pm$ is ergodic. □

Remark 7.2. As we observed in the proof of Theorem 7.1, the existence of a cylinder map $b \colon X \longrightarrow H$ such that the cocycles $a_{f'}^\pm \colon \Delta_X \longrightarrow H$ in (7.3)–(7.4) are ergodic is equivalent to the existence of a finite algebra $\mathcal{A}$ of closed and open subsets of X such that $\mathcal{S}^{\ker(a_f^\pm)} \subset \mathcal{A} \pmod{\mu_\phi}$.

Corollary 7.3. *Let A be a finite set, $X \subset A^{\mathbb{Z}}$ a mixing shift of finite type, μ_ϕ the Gibbs measure arising from a map $\phi \colon X \longrightarrow \mathbb{R}$ with summable variation, H a Polish group, and $f \colon X \longrightarrow H$ a cylinder map.*

Suppose that one of the following conditions (1) *or* (2) *is satisfied.*

(1) (a) *The group H is of the form $H = H_1 \times H_2$, where H_1 is a compact extension of an abelian group and $H_2 = H_2' \ltimes H_2''$ a semi-direct product of two discrete groups with H_2 nilpotent,*

(b) *The map $f = (f_1, f_2) \colon X \longrightarrow H$ has the property that $f_2' \colon X \longrightarrow H_2'$ is recurrent* (*in the notation of* (6.13))*;*

(2) *The group H is locally compact and f is an inner recurrent cylinder map.*

Then there exists a cylinder map $b\colon X \longrightarrow H$ such that the cocycles $a^{\pm}_{f'}\colon \Delta_X \longrightarrow H$ in (7.3)–(7.4) *are ergodic.*

Proof. Apply Corollary 6.8 and the Theorems 6.6 and 7.1. □

Remark 7.4. If the map $f_1\colon X \longrightarrow H_1$ in Corollary 7.3 (1) has summable variation, then the map $b = (b_1, b_2)\colon X \longrightarrow H = H_1 \times H_2$ in the conclusion of the corollary satisfies that $b_1\colon X \longrightarrow H_1$ is continuous, $b_2\colon X \longrightarrow H_2$ has summable variation, and the cocycles $a^{\pm}_{f'}\colon \Delta_X \longrightarrow H$ in (7.3)–(7.4) are ergodic.

8. Ergodicity of cocycles and weights of periodic points

This section is motivated by the papers [15] and [2]. The hypothesis (3) in Theorem 8.1 is introduced in [15], and in [2] Theorem 8.1 is proved under the additional hypothesis that the group H is discrete and abelian.

Suppose that H is a Polish group and $f\colon X \longrightarrow H$ a map which is either a cylinder map or has summable variation (the latter hypothesis requires the existence of a bi-invariant metric on H). For every periodic point $x \in X$ with minimal period $p \geq 1$ (i.e. with $\sigma^p x = x$, but $\sigma^k x \neq x$ for $0 < k < p$) we define the *f-weight* of x by

$$w_f(x) = f(\sigma^{p-1}x) \cdots f(x). \tag{8.1}$$

Denote by $\Delta^{(\sigma)}$ the equivalence relation (3.11) generated by Δ_X and σ, and write $\bar{a}^{\pm}_f\colon \Delta^{(\sigma)}_X \longrightarrow H$ for the extensions (3.13) to $\Delta^{(\sigma)}_X$ of the Gibbs cocycles $a^{\pm}_f\colon \Delta_X \longrightarrow H$.

Theorem 8.1. *Let A be a finite set, $X \subset A^{\mathbb{Z}}$ a mixing shift of finite type, μ_ϕ the Gibbs measure arising from a map $\phi\colon X \longrightarrow \mathbb{R}$ with summable variation, H a compact extension of a nilpotent locally compact second countable group of the form $H = H_1 \times H_2$, where H_1 is abelian, and $f = (f_1, f_2)\colon X \longrightarrow H$ a map with the following properties:*

(1) *the map $f_1\colon X \longrightarrow H_1$ has summable variation,*
(2) *the map $f_2\colon X \longrightarrow H_2$ is a cylinder map,*
(3) *the collection of f-weights*

$$W_f = \{w_f(x)^{\pm 1} : x \in X \text{ is periodic}\} \tag{8.2}$$

is dense in H (cf. (8.1)*).*

Suppose furthermore that the pair of cocycles (a^-_f, a^+_f) is $*$-ergodic and equally ergodic (cf. Corollary 6.2 *and Theorems* 6.6–6.7*). Then the cocycles $\bar{a}^-_f$ and $\bar{a}^+_f$ are ergodic on $\Delta^{(\sigma)}_X$.*

Proof. According to Theorem 7.1 and Remark 7.4 there exists a map $b = (b_1, b_2)\colon X \longrightarrow H = H_1 \times H_2$ such that b_1 is continuous, b_2 is a cylinder map, and the cocycles $a^{\pm}_{f'}\colon \Delta_X \longrightarrow H$ in (7.3)–(7.4) are μ_ϕ-ergodic. If $\mathcal{P}$ is the partition of X into the closed and open sets of constancy of b_2, then $a^{\pm}_{f_2}$ — and hence $\bar{a}^{\pm}_{f_2}$ — is ergodic on each $P \in \mathcal{P}$ (cf. Remark 7.2).

We denote by $H_0 \subset H$ the closed subgroup of H associated with the ergodic restriction of $\bar{a}^{\pm}_{f'}$ to $(\Delta^{(\sigma)}_X)_P$ by Proposition 2.5 (since $(\bar{a}^{-}_{f'}, \bar{a}^{+}_{f'})$ is $*$-ergodic, it is easy to see that the subgroups arising from $\bar{a}^{+}_{f'}$ and $\bar{a}^{-}_{f'}$ are identical for every $P \in \mathcal{P}$).

For every $P \in \mathcal{P}$ we denote by $G_P \subset H$ the closed subgroup of H associated with the ergodic restriction of $\bar{a}^{\pm}_{f'}$ to $(\Delta^{(\sigma)}_X)_P$ by Proposition 2.5. Clearly, the closure of the set

$$\{(w_{f'_1}(x), w_{f'_2}(x))^{\pm 1} : x \in P \text{ is periodic}\}$$

is contained in G_P, where $f' = (f'_1, f'_2)$.

We fix $P \in \mathcal{P}$, set $G = G_P$, and conclude that there exists, for every $P' \in \mathcal{P}$, an element $h_{P'} \in H_2$ such that

$$\begin{aligned}\{(w_{f'_1}(x), w_{f'_2}(x))^{\pm 1} &: x \in P' \text{ is periodic}\} \\ &= \{(w_{f_1}(x), h_{P'}^{-1} w_{f_2}(x) h_{P'})^{\pm 1} : x \in P \text{ is periodic}\}\end{aligned}$$

is contained in $G_{P'}$. Here we are using that H_1 is abelian and b_2 is constant on each $P' \in \mathcal{P}$.

Since every periodic point of X lies in some $P' \in \mathcal{P}$ and W_f is dense in H, H is the union of finitely many conjugacy classes of G. By Baire's category theorem G is open, and we claim that $G = H$.

Since H is a compact extension of a nilpotent group, there exists a left-invariant mean $\mathfrak{m}\colon \mathrm{UCB}(H) \longrightarrow \mathbb{C}$ on H, where $\mathrm{UCB}(H)$ is the space of uniformly continuous bounded complex valued functions on H (cf. e.g. [13]). Since finitely many conjugacy classes of G cover H, the indicator function $1_{gGg^{-1}}$ of at least one of these conjugacy classes satisfies that $\mathfrak{m}(1_{gGg^{-1}}) > 0$ (since gGg^{-1} is open, its indicator function lies in $\mathrm{UCB}(H)$). The left-invariance and finite additivity of $\mathfrak{m}$ guarantee that gGg^{-1} — and hence G — has finite index in H. Hence G contains a subgroup $K \subset H$ which is normal and has finite index in H.

I am grateful to G. Kowol and H. Rindler for the following argument which completes the proof of the theorem.

Suppose that $H' \supset G$ is a maximal proper subgroup of H such that H is the union of finitely many conjugacy classes of H'. We denote by $N(H') \supseteq H'$ the normalizer of H' in H. Our assumption on H' implies that $N(H') = H'$, and hence that $h^{-1}H'h \neq H'$ for every $h \in H \smallsetminus H'$. Hence

$$\begin{aligned}|H/K| = |H/H'| \cdot |H'/K| &= \sum_{h \in C} |(h^{-1}H'h)/K| \\ &> \Bigl|\bigcup_{h \in C} (h^{-1}H'h)/K\Bigr| = |H/K|,\end{aligned}$$

where $C \subset H$ is a set which intersects each coset of H' in H in exactly one point, and where $|\cdot|$ denotes cardinality. This contradiction implies that $H' = G = H$. □

9. Examples

9.1. **The Radon-Nikodym cocycle.** Let R be a discrete Borel equivalence relation on a standard Borel space $(X, \mathcal{S})$ and μ a sigma-finite measure on $\mathcal{S}$ which is quasi-invariant under R. Then μ is quasi-invariant under every

$W \in [R]$. By choosing a countable subgroup $\Gamma \subset [R]$ satisfying (2.2) and selecting a suitable version of the Radon-Nikodym derivative of $\frac{d\mu\gamma}{d\mu}$ for every $\gamma \in \Gamma$ we obtain a cocycle $\rho_\mu \colon R \longrightarrow \mathbb{R}$ with

$$\rho_\mu(\gamma x, x) = \log \frac{d\mu\gamma}{d\mu}(x) \text{ for every } \gamma \in \Gamma \text{ and } \mu\text{-a.e. } x \in X. \tag{9.1}$$

The cocycle ρ_μ is uniquely determined (mod μ) and is called the *Radon-Nikodym cocycle* of (R, μ).

If $V \in \mathrm{Aut}(R)$ preserves μ then the cocycles $(x, y) \mapsto \rho_\mu(x, y)$ and $(x, y) \mapsto \rho_\mu(Vx, Vy)$ coincide (mod μ). Hence we may assume without loss in generality that ρ_μ is V-invariant.

In the special case where μ is a probability measure on $\mathcal{S}$ which is ergodic under V and V is weakly asymptotically central on (R, μ), Theorem 4.1 implies that μ is either invariant (and hence of type II_1) or of type III_λ under R with $0 < \lambda \leq 1$ (cf. [17]).

If A is a finite set, $X \subset A^{\mathbb{Z}}$ a shift of finite type and μ_ϕ the Gibbs measure arising from a map $\phi \colon X \longrightarrow \mathbb{R}$ with summable variation (cf. (7.2)), then we can combine the resulting cocycles $a_\phi^\pm \colon \Delta_X \longrightarrow \mathbb{R}$ with the cocycles $a_f^\pm \colon \Delta_X \longrightarrow H$ in Theorem 7.1 or Corollary 7.3. For example, if $H = H_1 \times H_2$ is a Polish group and $f = (f_1, f_2) \colon X \longrightarrow H$ is a map satisfying the hypotheses of Corollary 7.3 (1) and Remark 7.4, then we can replace f_1 by $(\phi, f_1) \colon X \longrightarrow \mathbb{R} \times H_1$ and obtain a continuous map $b = (b_0, b_1, b_2) \colon X \longrightarrow \mathbb{R} \times H_1 \times H_2$ such that $b_2 \colon X \longrightarrow H_2$ is a cylinder map and the cocycles $(a_{\phi'}^\pm, a_{f'}^\pm) \colon \Delta_X \longrightarrow \mathbb{R} \times H$ are ergodic, where $\phi'(x) = (b_0 \circ \sigma)^{-1} + \phi(x) - b_0(x)$, $a_{\phi'}^\pm(x, x') = a_\phi^\pm(x, x') - b_0(x) + b_0(x')$, and where $a_{f'}^\pm$ is defined by (7.3)–(7.4) with b_1 replacing b.

9.2. **Ergodicity of cocycles on full shifts.** Let A be a finite set, $X = A^{\mathbb{Z}}$, $\mu = \mu_\phi$ the Gibbs measure arising from a map $\phi \colon X \longrightarrow \mathbb{R}$ with summable variation, H a discrete group, and $f \colon X \longrightarrow H$ a map which depends only on the zero coordinate, i.e. $f(x) = f(x')$ for all $x, x' \in X$ with $x_0 = x'_0$. Then μ is is quasi-invariant and ergodic under Δ_X.

If H and f satisfy the conditions of Corollary 7.3, then there exists a subgroup $H_0 \subset H$ with the following properties:

(a) $a_f^+(x, x') \in H_0$ and $a_f^-(x, x') \in H_0$ for every $(x, x') \in \Delta_X$,

(b) if λ is the counting measure on $H_0 \times H_0$, then the measure $\mu \times \lambda$ on $X \times H_0^2$ is ergodic under the skew-product relation $\Delta_X^{(a_f^-, a_f^+)}$ defined in (2.9).

Condition (b) implies that the cocycle $(a_f^-, a_f^+) \colon \Delta_X \longrightarrow H^2$ is ergodic.

Indeed, Theorem 5.3 and (5.1) imply that

$$\mathcal{S}^{\ker(a_f^-)} = \mathcal{S}^{\ker(a_f^-, a_f^+)} = \mathcal{S}^{\ker(a_f^+)} = \mathcal{S}^{\ker(a_f^-) \vee \ker(a_f^+)} \pmod{\mu}.$$

Furthermore, $\ker(a_f^-) \supset \Delta_X^+$, $\ker(a_f^+) \supset \Delta_X^-$ (cf. (7.5)), and

$$\Delta_X = \Delta_X^- \vee \Delta_X^+ = \ker(a_f^-) \vee \ker(a_f^+). \tag{9.2}$$

Hence

$$\mathcal{S}^{\ker(a_f^-)} = \mathcal{S}^{\ker(a_f^+)} = \{X, \varnothing\} \pmod{\mu} \tag{9.3}$$

by (5.1), which proves the ergodicity of (a_f^-, a_f^+).

The ergodicity of the relation $\Delta_X^{(a_f^-, a_f^+)}$ on $X \times H_0^2$ is equivalent the condition that there exists, for all $A, B \in \mathcal{S}$ with $\mu(\Delta_X(A) \cap B) > 0$ and all $(g, h) \in H_0^2$, a point $(x, x') \in \Delta_X$ with $x \in A, x' \in B$, $a_f^-(x, x') = g$ and $a_f^-(x, x') = h$.

9.3. **The lamp-lighter process.** Let $d \geq 1$, $A = \{\pm \mathbf{e}^{(i)} : i = 1, \dots, d\} \subset \mathbb{Z}^d$, where $\mathbf{e}^{(i)}$ is the i-th unit vector in $\mathbb{Z}^d$, $X = A^{\mathbb{Z}}$, and let μ be the uniform Bernoulli measure on X.

Let $G = \sum_{\mathbb{Z}^d} \mathbb{Z}$ be the direct sum of copies of $\mathbb{Z}$, indexed by $\mathbb{Z}^d$. Every element $g \in G$ is viewed as a map $g \colon \mathbb{Z}^d \longrightarrow \mathbb{Z}$ with $g(\mathbf{n}) \neq 0$ for only finitely many $\mathbf{n} \in \mathbb{Z}^d$, and the shift-action α of $\mathbb{Z}^d$ on G is defined by

$$\alpha^{\mathbf{m}}(g)(\mathbf{n}) = g(\mathbf{m} + \mathbf{n})$$

for every $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$ and $g \in G$. The semi-direct product $H_1 = \mathbb{Z}^d \ltimes_\alpha G$ consists of all pairs $(\mathbf{n}, g) \in \mathbb{Z}^d \times G$ with group operation $(\mathbf{n}, g) \cdot (\mathbf{n}', g') = (\mathbf{n} + \mathbf{n}', g + \alpha^{\mathbf{n}} g')$.

Consider the cylinder map $f \colon X \longrightarrow H_1$ with $f(x) = (x_0, 1_{\{x_0\}})$ for every $x = (x_n) \in X$, where

$$1_{\{x_0\}}(\mathbf{n}) = \begin{cases} 1 & \text{if } \mathbf{n} = x_0, \\ 0 & \text{otherwise.} \end{cases}$$

For $d \leq 2$, the first component $f' \colon X \longrightarrow \mathbb{Z}^d$ of f is recurrent, and Theorem 6.7 shows that the pair of Gibbs cocycles $(a_f^-, a_f^+) \colon \Delta_X \longrightarrow H_1^2$ in (6.6)–(6.7) is $*$-ergodic and equally ergodic. By Corollary 7.3 and Example 9.2, the individual cocycles $a_f^{\pm} \colon \Delta_X \longrightarrow H_1$ are, in fact, ergodic.

If $d \geq 3$, then f_1 is transient (i.e. not recurrent), and Theorem 6.7 cannot be applied.

For $1 \leq d \leq 2$ we can interpret the ergodicity of a_f^+ geometrically as follows: define the one-sided space X_+ and the one-sided Gibbs relation Δ_{X_+} by (9.5)–(9.6) below and identify every $x \in X_+$ in the obvious manner with a path in $\mathbb{Z}^d$ starting at $\mathbf{0}$. If we call two such paths *equivalent* if they differ in only finitely many steps and visit every lattice site equally often, then the resulting equivalence relation is ergodic with respect to the uniform Bernoulli measure on X_+.

We modify this example by replacing the group G with the direct sum $G_{/2} = \sum_{\mathbb{Z}^d} \mathbb{Z}/2\mathbb{Z}$. Every element of $G_{/2}$ is identified with a finite subset of $\mathbb{Z}^d$, with symmetric difference $\triangle$ as group operation. The shift-action α of $\mathbb{Z}^d$ on $G_{/2}$ is defined as above by

$$\alpha^{\mathbf{m}} F = F - \mathbf{m}$$

for all $\mathbf{m} \in \mathbb{Z}^d$ and $F \subset \mathbb{Z}^d$, and the semi-direct product $H_2 = \mathbb{Z}^d \ltimes_\alpha G_{/2}$ is called the d-dimensional *lamp-lighter group.*

If $f \colon X \longrightarrow H_2$ is defined by $f(x) = (x_0, \{x_0\})$ for every $x \in X$, then the same argument as above shows that the cocycles $a_f^{\pm} \colon \Delta_X \longrightarrow H_2$ are ergodic if $1 \leq d \leq 2$.

The same results hold if we replace the uniform Bernoulli measure on X by any Gibbs measure μ_ϕ arising from a map $\phi\colon X \longrightarrow \mathbb{R}$ with summable variation such that $f'\colon X \longrightarrow \mathbb{Z}^d$ is μ_ϕ-recurrent (and therefore $d \le 2$).

9.4. **An application of inner recurrence.** Let A be a finite set, $X \subset A^{\mathbb{Z}}$ a subshift, and μ a shift-invariant probability measure on X. Let furthermore $d \ge 1$, $n \ge 2$, and let $\alpha\colon \mathbb{Z}^d \longrightarrow \mathrm{GL}(n, \mathbb{Z})$ be a group homomorphism. Then α defines $\mathbb{Z}^d$-actions by automorphisms of $\mathbb{Z}^n$ and of $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, and we denote by $H_1 = \mathbb{Z}^d \ltimes_\alpha \mathbb{T}^n$ and $H_2 = \mathbb{Z}^d \ltimes_\alpha \mathbb{Z}^n$ the resulting semi-direct products.

Every cylinder map $f\colon X \longrightarrow H_1$ is inner recurrent by Example 6.5 (2). According to Theorem 6.6, the Gibbs cocycles $(a_f^-, a_f^+)\colon \Delta_X \longrightarrow H_1^2$ are $*$-ergodic and equally ergodic on any shift-invariant subrelation $R \subset \Delta_X$.

If X is a shift of finite type and $\mu = \mu_\phi$ is the Gibbs measure arising from a map $\phi\colon X \longrightarrow \mathbb{R}$ with summable variation, then Theorem 7.1 shows that there exists a cylinder map $b\colon X \longrightarrow H_1$ such that the cocycles $a_{f'}^{\pm}\colon \Delta_X \longrightarrow H$ in (7.3)–(7.4) are ergodic.

If $f\colon X \longrightarrow H_2$ is a cylinder map, then Theorem 6.7 shows that the Gibbs cocycles $(a_f^-, a_f^+)\colon \Delta_X \longrightarrow H_2^2$ are $*$-ergodic and equally ergodic on any shift-invariant subrelation $R \subset \Delta_X$, *provided that — in the notation of* (6.13) *— the map $f'\colon X \longrightarrow \mathbb{Z}^d$ is recurrent* (which essentially implies that $d \le 2$). In this case Theorem 7.1 again yields a cylinder map $b\colon X \longrightarrow H_2$ such that the cocycles $a_{f'}^{\pm}\colon \Delta_X \longrightarrow H_2$ in (7.3)–(7.4) are ergodic.

If the $\mathbb{Z}^d$-action α is *irreducible*[1] we can weaken the recurrence condition on the map $f'\colon X \longrightarrow \mathbb{Z}^d$.

Proposition 9.1. *Let X be a shift of finite type, $\mu = \mu_\phi$ the Gibbs measure arising from a map $\phi\colon X \longrightarrow \mathbb{R}$ with summable variation, and let $\alpha\colon \mathbb{Z}^d \longrightarrow \mathrm{GL}(n, \mathbb{Z})$ be an irreducible linear $\mathbb{Z}^d$-action on $\mathbb{R}^n$. Suppose furthermore that $f = (f', f'')\colon X \longrightarrow H = \mathbb{Z}^d \ltimes_\alpha \mathbb{R}^n$ is a cylinder map (where $f'\colon X \longrightarrow \mathbb{Z}^d$ and $f''\colon X \longrightarrow \mathbb{R}^n$) such that $\eta \circ f'\colon X \longrightarrow \mathbb{R}$ is recurrent[2] for every linear map $\eta\colon \mathbb{R}^d \longrightarrow \mathbb{R}$. Then the pair of Gibbs cocycles $(a_f^-, a_f^+)\colon \Delta_X \longrightarrow H^2$ is $*$-ergodic and equally ergodic, and there exists a cylinder map $b\colon X \longrightarrow H$ such that the cocycles $a_{f'}^{\pm}\colon \Delta_X \longrightarrow H$ in (7.3)–(7.4) are both ergodic.*

Proof. We view α as a linear $\mathbb{Z}^d$-action on $\mathbb{C}^n$ and set $\bar{H} = \mathbb{Z}^d \ltimes_\alpha \mathbb{C}^n$. Since α is irreducible, we can diagonalize the commuting matrices $\alpha^{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^d$, simultaneously and find α-invariant subspaces V_i, $i = 1, \dots, k$, of $\mathbb{C}^n$ and continuous group homomorphisms $\beta_i\colon \mathbb{R}^d \longrightarrow \mathbb{C}^\times$ (where $\mathbb{C}^\times$ is the multiplicative group of nonzero complex numbers) such that $\mathbb{C}^n = V_1 \oplus \dots \oplus V_k$ and the restriction of $\alpha^{\mathbf{n}}$ to V_i is multiplication by $\beta_i(\mathbf{n})$ for every $i = 1, \dots, k$ and $\mathbf{n} \in \mathbb{Z}^d$.

Put $G_i = \mathbb{C}^\times \ltimes V_i$, where $\mathbb{C}^\times$ acts on V_i by multiplication, $G = \bigoplus_{i=1}^k G_k$, and define an injective group homomorphism $\theta\colon \bar{H} \longrightarrow G$ by

$$\theta(\mathbf{n}, (v_1, \dots, v_k)) = ((\beta_1(\mathbf{n}), v_1), \dots, (\beta_k(\mathbf{n}), v_k))$$

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[1]Irreducibility means that every nonzero α-invariant subspace $V \subset \mathbb{R}^d$ is equal to $\mathbb{R}^d$.

[2]This is equivalent to assuming that $\int \eta \circ f \, d\mu = 0$ for every linear map $\eta\colon \mathbb{R}^d \longrightarrow \mathbb{R}$.

for every $\mathbf{n} \in \mathbb{Z}^d$ and $v = (v_1, \dots, v_k) \in \bigoplus_{i=1}^k V_k = \mathbb{C}^n$. If we regard f as a cylinder map from X into $\bar{H}$, then $\theta \circ f \colon X \longrightarrow G$ is of the form

$$\theta \circ f = (f_1, \dots, f_k),$$

where $f_i = (f_i', f_i'') \colon X \longrightarrow G_i = \mathbb{C}^\times \ltimes V_i$ is a cylinder map for $i = 1, \dots, k$.

Our hypothesis on f' is easily seen to imply that $f_i' \colon X \longrightarrow \mathbb{C}^\times$ is recurrent for $i = 1, \dots, k$. Corollary 6.9 shows that the pair of Gibbs cocycles $(a_{f_i}^-, a_{f_i}^+) \colon \Delta_X \longrightarrow G_i^2$ is $*$-ergodic and equally ergodic, and Theorem 7.1 and Remark 7.2 show that there exists a finite algebra $\mathcal{A}_i$ of closed and open subsets of X such that the cocycles $a_{f_i}^\pm$ are ergodic on each atom of $\mathcal{A}_i$. By combining these subalgebras we obtain a finite algebra $\mathcal{A}$ of closed and open subsets of X such that the cocycles $a_{f_i}^\pm$ are all ergodic on each atom of $\mathcal{A}$. Hence the cocycles $a_{\theta\circ f}^\pm \colon \Delta_X \longrightarrow G^2$ are both ergodic on each atom of $\mathcal{A}$, which implies the ergodicity of $a_f^\pm \colon \Delta_X \longrightarrow H$ on each atom of $\mathcal{A}$. Remark 7.2 completes the proof of the proposition. □

9.5. **Ergodicity of cocycles on subshifts.** Let A be a finite set, $X \subset A^\mathbb{Z}$ a subshift, μ a nonatomic shift-invariant measure on X, H a finite extension of a discrete nilpotent group and $f \colon X \longrightarrow H$ a cylinder map. Choose a shift-invariant subrelation $\Delta_\mu \subset \Delta_X$ such that μ is quasi-invariant under Δ_μ and $\Delta_X = \Delta_\mu \pmod{\mu}$ (cf. Proposition 2.1), and let

$$\begin{aligned}\Delta_\mu^+ &= \{(x, x') \in \Delta_\mu : x_n = x_n' \text{ for all } n < 0\},\\ \Delta_\mu^- &= \{(x, x') \in \Delta_X : x_n = x_n' \text{ for all } n \geq 0\}.\end{aligned}$$

As we saw in Example 9.2 above, the cocycles $a_f^\pm \colon \Delta_\mu \longrightarrow H$ satisfy that

$$\ker(a_f^-) \supset \Delta_\mu^+, \ \ker(a_f^+) \supset \Delta_\mu^-,$$

and hence that

$$\mathcal{S}^{\ker(a_f^-)} = \mathcal{S}^{\ker(a_f^+)} \subset \mathcal{S}^{\Delta_\mu^- \vee \Delta_\mu^+} \pmod{\mu}.$$

The size of the relation $\Delta_\mu^* = \Delta_\mu^- \vee \Delta_\mu^+ \subset \Delta_\mu$ depends, of course, on the nature of the shift space X and the measure μ.

If X is a shift of finite type and $\mu = \mu_\phi$ the Gibbs measure corresponding to a map $\phi \colon X \longrightarrow \mathbb{R}$ with summable variation we set $\Delta_\mu = \Delta_X$ and observe that

$$\Delta_X^* = \Delta_X^- \vee \Delta_X^+ \subset \Delta_X$$

has only finitely many ergodic components (cf. [12]). If $\Delta_X^* = \Delta_X$, and if the map $f \colon X \longrightarrow H$ depends on the single coordinate x_0 of every $x \in X$, then (7.7) shows that the Gibbs cocycles $a_f^\pm \colon \Delta_X \longrightarrow H$ are ergodic without any modification.

Here is an elementary example where $\Delta_X^* \neq \Delta_X$: in the notation of (7.1) we assume that $k = 2$, $A = \{1, 2, 3\}$, and that the set of allowed transitions $P \subset A^2$ is given by

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

in the sense that the transition (i, j) is allowed if and only if $P(i, j) = 1$, $1 \leq i, j \leq 3$. The shift of finite type $X \subset A^\mathbb{Z}$ determined by these data has the property that 1 has to be followed by 2 and 2 has to be preceded by 1. Hence the cylinder set

$$[1,2] = \{x \in X : x_{-1} = 1, x_0 = 2\}$$

is an atom (mod μ_ϕ) of $\mathcal{S}^{\Delta^*_X}$ for any Gibbs measure μ_ϕ arising from a map $\phi\colon X \longrightarrow \mathbb{R}$ with summable variation.

If we define $f\colon X \longrightarrow \mathbb{Z}/2\mathbb{Z}$ by

$$f(x) = \begin{cases} 1 & \text{if } x_0 \in \{1,2\}, \\ 0 & \text{otherwise,} \end{cases}$$

then the cocycles $a^\pm_f\colon \Delta_X \longrightarrow \mathbb{Z}/2\mathbb{Z}$ are nonergodic with respect to μ_ϕ.

For general subshifts the situation can be much more complicated. The following example was pointed out by D. Rudolph.

9.6. **Tail-fields of** (T, T^{-1})**.** Let

$$Y = Z = \{1, -1\}^{\mathbb{Z}},$$

and let S, T be the shifts (6.1) on Y and Z, respectively. We write $\mathcal{B}_Y, \mathcal{B}_Z$ the Borel fields and μ_Y, μ_Z for the equidistributed Bernoulli measures on Y and Z, set $X = Y \times Z$, denote by $\mathcal{S}$ and $\mu = \mu_Y \times \mu_Z$ the product Borel field and the product measure on X, and write $\pi_Y\colon X \longrightarrow Y$, $\pi_Z\colon X \longrightarrow Z$ for the coordinate projections.

Let $V\colon X \longrightarrow X$ be the map

$$V(y,z) = (Sy, T^{y_0}z)$$

for every $y = (y_n) \in Y$ and $z = (z_n) \in Z$. Then V is a Borel automorphism of $(X, \mathcal{S})$ which preserves μ and is called the (T, T^{-1})-*transformation* on $(X, \mathcal{S}, \mu)$ (cf. [8]).

We define a continuous map $\phi\colon X \longrightarrow X$ by setting $\phi(x) = (y', z')$ for every $x = (y,z) \in X$, where

$$y' = y, \;\; z'_n = z_{s(n,y)},$$

where $s(y) = y_0$ for every $y \in Y$ and $s\colon \mathbb{Z} \times Y \longrightarrow \mathbb{Z}$ is defined by (3.9). Then $\phi \circ V = \sigma \circ \phi$, where $\sigma = S \times T$ is the shift on X, and ϕ is injective μ-a.e., i.e. there exists a V-invariant μ-null set $N \in \mathcal{S}$ with $\phi(x) = \phi(x')$ whenever $x \neq x'$ and $x, x' \in X \smallsetminus N$. The map ϕ allows us to identify the automorphism V on $(X, \mathcal{S}, \mu)$ with the shift σ on $(X, \mathcal{S}, \nu)$, where $\nu = \mu\phi^{-1}$.

Denote by Δ_X the Gibbs equivalence relation (6.2) on X and choose a σ-invariant subrelation $\Delta_\nu \subset \Delta_X$ according to Proposition 2.1 such that ν is quasi-invariant under Δ_ν and $\Delta_X = \Delta_\nu$ (mod ν). Clearly, σ is a weakly asymptotically central automorphism of (Δ_ν, ν), and we claim that ν is ergodic under Δ_ν.

In order to prove this we write Δ_Y for the Gibbs relation on Y and note that μ_Y is quasi-invariant and ergodic under Δ_Y. We define the cocycle $(a^-_s, a^+_s)\colon \Delta_Y \longrightarrow (2\mathbb{Z})^2$ by (6.6)–(6.7), denote by λ the counting measure on $\mathbb{Z}^2$, and obtain from Example 9.2 that the product measure $\mu_Y \times \lambda$ on $Y \times (2\mathbb{Z})^2$ is ergodic under the skew-product relation $\Delta_Y^{(a^-_s, a^+_s)}$. Hence we can find, for all $A, B \in \mathcal{B}_Y$ with $\mu_Y(\Delta_Y(A) \cap B) > 0$ and all $n \in \mathbb{Z}$, a point

$(x,x') \in \Delta_Y$ with $x \in A, x' \in B$, $a_s^-(x,x') = a_s^+(x,x') = n$. In particular, $(x,x') \in \ker^*(a_s^-, a_s^+)$.

A moment's reflection on Δ_X shows that there exists a ν-null set $N \in \mathcal{S}$ with ν quasi-invariant under $\Delta_\nu = \Delta_{X\smallsetminus N} \cup \{(x,x) : x \in X\}$ and $(x,x') \in \Delta_{X\smallsetminus N}$ if and only if $x = (y,z)$, $x' = (y',z')$ with

$$(y,y') \in \ker^*(a_s^-, a_s^+),\ \ T^{a_s^+(y,y')}z = z'. \tag{9.4}$$

Above we saw that we can — in an appropriate sense — prescribe the value of a_s^+ arbitrarily on $\ker^*(a_s^-, a_s^+)$, and by taking into account the ergodicity of T we conclude that ν is indeed ergodic under Δ_ν.

We define $f\colon X \longrightarrow \mathbb{Z}$ by $f(x) = s(y)$ for every $x = (y,z) \in X$ and obtain from Theorem 4.1 (or from the discussion above) that ν is ergodic under $\ker^*(a_f^-, a_f^+)$ (cf. (6.6)–(6.7)). However, ν is seriously nonergodic under the equivalence relation $\ker(a_f^+) \subset \Delta_\nu$: for every $(x,x') \in \ker(a_f^+)$, the points x,x' are of the form $x = (y,z)$, $x' = (y',z')$ with $(y,y') \in \Delta_Y$ and $a_f^+(x,x') = a_s^+(y,y') = 0$, and by comparing this with (9.4) we see that $z = z'$. Hence $\mathcal{S}^{\ker(a_f^+)} \supset \pi_Z^{-1}(\mathcal{B}_Z)$ (mod ν), and the two sigma-algebras are, in fact, equal (mod ν).

The next example is a slight refinement of an example discussed in [18].

9.7. **Local variations in long molecules.** Let A be a finite set, $X \subset A^{\mathbb{Z}}$ a subshift, σ the shift (6.1) on X, and μ a shift-invariant probability measure on the Borel field $\mathcal{S}$ of X such that $\mathcal{S}^{\Delta_X} = \mathcal{N} = \{\varnothing, X\}$ (mod μ) (i.e. Δ_X is μ-ergodic). Choose a shift-invariant μ-nonsingular subrelation $\Delta_\mu \subset \Delta_X$ with $\mathcal{S}^{\Delta_X} = \mathcal{S}^{\Delta_\mu} = \mathcal{N}$ (mod μ) according to Proposition 2.1. If X is a mixing shift of finite type, then every Markov measure and, more generally, every Gibbs measure on X arising from a function $\phi\colon X \longmapsto \mathbb{R}$ with summable variation has this property with $\Delta_\mu = \Delta_X$ — cf. e.g. [12].

We fix $n \geq 1$, denote by $G^{(n)}$ the free abelian group generated by A^n, and define a continuous map $f_1\colon X \longmapsto G$ by $f_1(x) = (x_0, \dots, x_{n-1}) \in A^n \subset G^{(n)}$ for every $x \in X$. Then the equivalence relation $\ker^*(a_{f_1}^-, a_{f_2}^+) \subset \Delta_\mu$ in (6.9) consists of all pairs $(x,y) \in \Delta_\mu$ which differ in only finitely many coordinates, and for which the n-blocks $((x_k, \dots x_{k+n-1}),\ k \in \mathbb{Z})$ and $((y_k, \dots y_{k+n-1}),\ k \in \mathbb{Z})$ occurring in x and y differ only by a finite permutation. Theorem 4.1 shows that $\ker^*(a_{f_1}^-, a_{f_1}^+)$ is ergodic; this fact can be expressed by saying that, for a typical point $x \in X$, a $\ker^*(a_{f_1}^-, a_{f_2}^+)$-equivalent point y could lie anywhere in the space X.

Now consider the d-dimensional Euclidean group $E(d) = SO(d) \ltimes \mathbb{R}^d$, furnished with the group operation

$$(B,v) \cdot (B',v') = (BB', v + Bv')$$

for all $v,v' \in \mathbb{R}^d$ and $B,B' \in SO(d)$. If $f_2\colon X \longmapsto G$ is a cylinder map, then Corollary 6.8 implies that the pair of cocycles $(a_f^-, a_f^+)\colon \Delta_\mu \longrightarrow G^2$ is $*$-ergodic, where $f = (f_1, f_2)\colon X \longrightarrow G = G^{(n)} \times E(d)$. This observation has the following geometrical interpretation.

We define the cocycle $\mathbf{f}_2\colon \mathbb{Z} \times X \longrightarrow E(g)$ by (3.9) and write $\mathbf{f}_2(n,x) = (B_n x, v_n(x))$ with $B_n(x) \in SO(d)$ and $v_n(x) \in \mathbb{R}^d$. If $v(x) = v_1(x)$ and

$A(x) = A_1(x)$ then

$$v_n(x) = \begin{cases} v(x) + v(Tx)A(x) + \cdots + v(T^{n-1}x)A(T^{n-1}x)\cdots A(x) \\ \qquad\qquad\qquad\qquad \text{if } n > 0, \\ 0 \qquad\qquad\qquad\qquad \text{if } n = 0, \\ -v(T^{-n}x)A(T^{-n}x)^{-1}\cdots A(x)^{-1} - \cdots - v(T^{-1}x)A(x)^{-1} \\ \qquad\qquad\qquad\qquad \text{if } n < 0. \end{cases}$$

By connecting successive points in the sequence $(v_n(x),\, n \in \mathbb{Z})$ by straight line segments we obtain an infinite polygonal curve in $\mathbb{R}^d$ which may, of course, have self-intersections. If we call two such polygonal curves associated with $x, y \in X$ *ε-equivalent* if $(x, y) \in \ker^*(a_{f_1}^-, a_{f_1}^+)$ and $\|v_n(x) - v_n(y)\| > \varepsilon$ for only finitely many $n \in \mathbb{Z}$ (where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$), then Corollary 6.8 implies that there exist, for every $\varepsilon > 0$ and all sets $B, B' \in \mathcal{S}$ with $\mu(B) \cdot \mu(B') > 0$, ε-equivalent points (x, y) with $x \in B$ and $y \in B'$. In other words, ε-equivalent curves may have very significant local differences.

If one were to interpret A as a finite set of molecules and X as a collection of two-sided infinite concatenations of these molecules, then the $*$-ergodicity of (a_f^-, a_f^+) would imply the unreliability of any chemical analysis of the structure of such a concatenation based on an investigation of substrings of a given length. The sequence of coordinates $(v_n(x),\, n \in \mathbb{Z})$ would correspond to a spatial arrangement of the chain x determined by its molecular structure, and the $*$-ergodicity assertion of Corollary 6.8 could be interpreted as a statement about quite dissimilar chains having identical substrings (up to permutation) and spatial arrangements with only local — but significant — differences.

9.8. **Cocycles on one-sided shift spaces.** Let A be a finite set, $X \subset A^{\mathbb{Z}}$ a subshift, write $\pi_+ \colon A^{\mathbb{Z}} \longrightarrow A^{\mathbb{N}}$ for the projection of each point onto onto its nonnegative coordinates, and consider the one-sided shift-space

$$X_+ = \pi_+(X) \subset A^{\mathbb{N}}. \tag{9.5}$$

Denote by $\mathcal{T}_+$ the Borel field of X_+, and set $\mathcal{S}_+ = \pi_+^{-1}(\mathcal{T}_+)$ and $\mu_\phi^+ = \mu_\phi \pi_+^{-1}$. The shift σ_+ on X_+ is defined as in (6.1), and the Gibbs relations on X and X_+ are denoted by Δ_X and

$$\Delta_{X_+} = (\pi_+ \times \pi_+)(\Delta_X). \tag{9.6}$$

Assume that H is a Polish group and $f \colon X_+ \longrightarrow H$ a cylinder map. We view f as an $\mathcal{S}_+$-measurable map on X. If the pair of Gibbs cocycles $(a_f^-, a_f^+) \colon \Delta_X \longrightarrow H^2$ is $*$-ergodic and equally ergodic, then a minor modification of the proof of Theorem 7.1 shows that there exists a map $b \colon X_+ \longrightarrow H$ such that the cocycle $a_{f'}^+ \colon \Delta_{X_+} \longrightarrow H$ in (7.3)–(7.4) is μ_ϕ^+-ergodic. If H admits a bi-invariant metric we can again weaken the hypotheses on f along the lines of Remark 7.4.

9.9. **Words in groups.** Let H be a discrete finitely generated group with a distinguished set of generators $A = \{h_1^{\pm 1}, \dots, h_m^{\pm 1}\}$, and let $X \subset A^{\mathbb{Z}}$ be the shift of finite type obtained by disallowing all words of length two of the form $h_i h_i^{-1}$ and $h_i^{-1} h_i$, $i = 1, \dots, m$. We fix a map $\phi \colon X \longrightarrow \mathbb{R}$ with summable variation, denote by μ_ϕ the corresponding Gibbs measure, and define $f\colon X \longrightarrow H$ by $f(x) = x_0$ for every $x = (x_n) \in H$. We define the Gibbs cocycles $(a_f^-, a_f^+)\colon \Delta_X \longrightarrow H^2$ by (6.6)–(6.7) and observe that $\ker^*(a_f^-, a_f^+)$ is the set of all pairs $(x, x') \in \Delta_X$ satisfying

$$x_k \cdots x_{-k} = x'_k \cdots x'_{-k}$$

in the notation of (6.9).

If H is a finite extension of a nilpotent group (or, equivalently, if H has polynomial growth — cf. [4], [5]), then Corollary 4.4, Theorems 5.3 and 7.1, and Remark 7.2 imply that (a_f^-, a_f^+) and (a_f^-, a_f^+) are $*$-ergodic, and that there exists a finite algebra $\mathcal{A}$ of closed and open subsets of X such that $\ker(a_f^\pm)$ — and hence $\ker(a_f^\pm)$ — is ergodic on every atom P of $\mathcal{A}$. Since $\Delta_X = \Delta_X^- \vee \Delta_X^+$ in (7.6), (7.7) implies as in Example 9.2 that the cocycles $a_f^\pm$ are, in fact, ergodic on Δ_X.

The ergodicity of the cocycle a_f^+ can be translated into a statement about the one-sided shift-space X_+ (cf. Example 9.8): since a_f^+ depends only on nonnegative coordinates, it is a well-defined map from Δ_{X_+} to H which is ergodic with respect to the one-sided Gibbs measure μ_ϕ^+. We summarize this in a proposition.

Proposition 9.2. *Let H be a finite extension of a discrete nilpotent group with a distinguished set of generators $A = \{h_1^{\pm 1}, \dots, h_m^{\pm 1}\}$, and let $X \subset A^{\mathbb{Z}}$ be the shift of finite type obtained by disallowing all words of length two of the form $h_i h_i^{-1}$ and $h_i^{-1} h_i$, $i = 1, \dots, m$. We fix a map $\phi\colon X_+ \longrightarrow \mathbb{R}$ with summable variation and denote by μ_ϕ^+ the corresponding one-sided Gibbs measure. If we call two elements x, x' in X_+ equivalent if $\sigma^n x = \sigma^n x'$ and*

$$x_0 \cdots x_n = x'_0 \cdots x'_n \tag{9.7}$$

for some $n \geq 0$, then this equivalence relation is μ_ϕ-ergodic.

Furthermore we can find, for every $\varepsilon > 0$ and all Borel sets A, B in X_+ with positive measure, equivalent points $x, x' \in X_+$ with $x \in A$, $x' \in B$, and $|\rho_{\mu_\phi^+}(x, x')| < \varepsilon$ (cf. (9.1)).

Proof. The only point worth noting is that we have reversed the order of the products in (9.7) (which amounts to interchanging h_i and h_i^{-1} for every $i = 1, \dots, m$). □

9.10. **Gibbs cocycles on the extended Gibbs relation.** Let A be a finite set, $X \subset A^{\mathbb{Z}}$ a shift of finite type, $\phi\colon X \longrightarrow \mathbb{R}$ a map with summable variation, and μ_ϕ the corresponding Gibbs measure. We denote by $N \subset X$ the shift-invariant Borel set consisting of all points whose positive or negative coordinates are eventually periodic (i.e. for which there exist positive integers p, N such that $x_{k+p} = x_p$ either for all $k \geq N$ or for all $k \leq -N$). Since $\mu_\phi(N) = 0$ we shall ignore this set in the following discussion.

The *extended Gibbs relation* $\bar{\Delta}_X$ on X is defined as

$$\begin{aligned} \bar{\Delta}_X = \{(x, x') \in X \times X : &\text{ there exist } l, r \in \mathbb{Z} \text{ and } K \in \mathbb{N} \\ &\text{with } x_{-k+l} = x'_{-k} \text{ and } x_{k+r} = x'_k \text{ for all } k \geq K\}. \end{aligned} \tag{9.8}$$

It is not difficult to check that $\bar{\Delta}_X$ is μ_ϕ-nonsingular, and that $\Delta_X \subsetneq \Delta_X^{(\sigma)} \subsetneq \bar{\Delta}_X$ (cf. (3.11)).

If H is a Polish group and $f\colon X \longrightarrow H$ a cylinder map, then the Gibbs cocycles $(a_f^-, a_f^+)\colon \Delta_X \longrightarrow H^2$ can be extended to $\bar{\Delta}_X$ by setting

$$\begin{aligned} \bar{a}_f^-(x,x') &= \mathbf{f}(l,x)^{-1} a_f^-(\sigma^l x, x'), \\ \bar{a}_f^+(x,x') &= \mathbf{f}(r,x)^{-1} a_f^+(\sigma^r x, x'), \end{aligned} \tag{9.9}$$

for every $(x,x') \in \bar{\Delta}_X$ of the form (9.8) (cf. (3.13)). The pair of *extended Gibbs cocycles* $(\bar{a}_f^-, \bar{a}_f^+)\colon \bar{\Delta}_X \longrightarrow H^2$ is again quasi-invariant, but no longer complementary. As in (3.4) and (6.7) we put

$$\bar{a}^*(x,x') = \bar{a}_f^-(x',x)\bar{a}_f^+(x,x') \tag{9.10}$$

for every $(x,x') \in \bar{\Delta}_X$ and obtain the same relations as in (6.8) with $\bar{a}$ replacing a in each case. The $*$-ergodicity of (a_f^-, a_f^+) obviously implies that of $(\bar{a}_f^-, \bar{a}_f^+)$. Furthermore, if H is discrete, then the $*$-ergodicity is equivalent to the ergodicity of

$$\ker^*(\bar{a}_f^-, \bar{a}_f^+) = \{(x,x') \in \bar{\Delta}_X : \bar{a}_f^*(x,x') = 1_H\},$$

and $(x,x') \in \ker^*(\bar{a}_f^-, \bar{a}_f^+)$ if and only if $(x,x') \in \bar{\Delta}_X$ is of the form (9.8) and

$$f(\sigma^{k+r}x)\cdots f(\sigma^{-k+l}x) = f(\sigma^k x')\cdots f(\sigma^{-k}x'). \tag{9.11}$$

for all sufficiently large $k \geq 0$. Finally, if there exists a continuous map $b\colon X \longrightarrow H$ such that the cocycles $a_{f'}^{\pm}$ in (7.3)–(7.4) are ergodic (as in Section 7), then the same is true for the cocycles

$$\bar{a}_{f'}^{\pm}(x,x') = b(x)^{-1}\bar{a}_f^{\pm}(x,x')b(x'), \tag{9.12}$$

and if H is discrete and $\ker(a_f^{\pm})$ has only finitely many ergodic components, then the same is true for the extended Gibbs cocycles $\bar{a}_f^{\pm}$.

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