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INVARIANT COCYCLES, RANDOM TILINGS AND THE SUPER-K AND STRONG MARKOV PROPERTIES

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Abstract. We consider 1-cocycles with values in locally compact, second countable abelian groups on discrete, nonsingular, ergodic equivalence relations. If such a cocycle is invariant under certain automorphisms of these relations we show that the skew product extension defined by the cocycle is ergodic. As an application we obtain an extension of many of the results in [9] to higher-dimensional shifts of finite type, and prove a transitivity result concerning rearrangements of certain random tilings.

1. Introduction

Let $\mathbf{R}$ be a discrete, nonsingular, ergodic equivalence relation of a standard probability space $(X, \mathfrak{B}_X, \mu)$, and let V be a measure preserving automorphism of $(X, \mathfrak{B}_X, \mu)$ which *normalises* $\mathbf{R}$, i.e. which sends $\mathbf{R}$-equivalence classes to $\mathbf{R}$-equivalence classes. We consider 1-cocycles $c\colon \mathbf{R} \longmapsto G$ on $\mathbf{R}$ taking values in a locally compact, second countable, abelian group G which are *invariant* under V, i.e. which satisfy that $c(Vx, Vx') = c(x, x')$ μ-*a.e.* on $\mathbf{R}$ (for the definitions we refer to Section 2). If the automorphism V is *asymptotically central* (Definition 2.2), then every V-invariant cocycle has the following property: there exists a null set $N \in \mathfrak{B}_X$ such the closure H in G of the set $\{c(x, x') : (x, x') \in \mathbf{R} \cap ((X \smallsetminus N) \times (X \smallsetminus N))\}$ is a subgroup of G, and that c defines an ergodic skew product extension of $\mathbf{R}$ by H (Theorem 2.3). This theorem is applied in Section 3 to the Gibbs equivalence relation Δ_X of a d-dimensional shift of finite type (*SFT*) X, where $d \geq 1$, and to a shift-invariant probability measure μ which is quasi-invariant and ergodic under the Gibbs relation of X (such as Gibbs measures of a sufficiently rapidly decaying continuous function $\phi\colon X \longmapsto \mathbb{R}$). For such a measure μ the shifts $\sigma_{\mathbf{m}}$, $\mathbf{0} \neq \mathbf{m} \in \mathbb{Z}^d$, are asymptotically central automorphisms of the Gibbs relation Δ_X, and the resulting application of Theorem 2.3 yields some surprising properties of the *SFT* X which go well beyond the *super-K property* discussed in [9]. In [9] it was shown that, if X is a one-dimensional *SFT*, then any Gibbs measure μ of a function with summable variation on X is ergodic under a large family of subrelations of the Gibbs relation, and in particular under the relation in which two points $x, x' \in X$ are equivalent if their coordinates are finite permutations of each other. Here we extend this result to dimensions $d \geq 1$, and to any subrelation of the Gibbs relation arising as the kernel of a shift-invariant cocycle $c\colon \Delta_X \longmapsto G$ with values in a locally compact, second countable group G (Theorem 3.1 and Corollary 3.3).

As an illustration we consider the two-dimensional 2-shift $X = \{0, 1\}^2$ with uniform (i.e. equidistributed) Bernoulli measure μ and regard every $x \in X$ as a colouring of the lattice $\mathbb{Z}^2$ with the colours white and black, corresponding to the possible values 0 and 1 of each coordinate of x. The connected monochromatic subsets of $\mathbb{Z}^2$ of such a colouring are known to be finite for μ-*a.e.* $x \in X$, and

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constitute a random tiling of $\mathbb{Z}^2$ by irregularly shaped black and white tiles of finite size. As a consequence of Corollary 3.3 one obtains that, for a typical point $x \in X$, the tiles occurring in this tiling can be rearranged by a finite permutation so that they still fit together exactly, that no tile touches another tile of the same colour, and that the resulting point lies in any given subset $B \subset X$ of positive measure (Example 3.4). This represents a considerable strengthening of the super-K property, since every rearrangement of these tiles corresponds, in particular, to a finite permutation of the coordinates of x.

Another illustration, related to the strong Markov property of shifts of finite type, appears in Example 3.5: let X be a one-dimensional *SFT*, furnished with the Gibbs measure μ of a function with summable variation, and let $B \subset X$ be a Borel set of positive measure. Then μ-*a.e.* $x \in X$ visits B infinitely often both in the past and the future, and we write $S(B,x) = (n_k,\, k \in \mathbb{Z}) \subset \mathbb{Z}$ for the sequence of successive times at which $x = (x_m)$ visits B. For each visit at time n_k, $k \in \mathbb{Z}$, we consider the string $C(x,k) = (x_{n_k}, \dots, x_{n_{k+1}-1})$ consisting of the coordinates of x until the next visit to B. If we call two points $x, x' \in X$ equivalent if their sequences of strings $(C(x,k),\, k \in \mathbb{Z})$ and $(C(x',k),\, k \in \mathbb{Z})$ differ only by a finite permutation, then this equivalence relation turns out to be ergodic whenever B can be approximated sufficiently quickly by closed and open sets. Again we conclude that, for typical points $x, x' \in X$, the sequence of strings arising from x can be finitely permuted to resemble locally the sequence of strings coming from x'.

This work arose out of discussions with R. Burton and J. Steif during their visit to the Erwin Schrödinger Institute in September 1995. I am particularly grateful to them for introducing me to the setting of Example 3.4, and to J. Steif for the reference [5]. This paper is a direct continuation of [9]; as it turns out, the results presented here require little more than stripping down some of the methods in [9] to their bare essentials.

2. Invariant cocycles

We begin by recalling a few definitions and results from [4]. Let X be a standard Borel space with Borel sigma-algebra $\mathfrak{B}_X$. A *discrete Borel equivalence relation* $\mathbf{R} \subset X \times X$ is an equivalence relation which is a Borel set, and for which each equivalence class $\mathbf{R}(x) = \{x' \in X : (x,x') \in \mathbf{R}\}$, $x \in X$, is countable. Since we only deal with discrete Borel equivalence relations we shall use the term *equivalence relation* to denote a discrete Borel equivalence relation.

The *full group* $[\mathbf{R}]$ of an equivalence relation $\mathbf{R} \subset X \times X$ is the group of all Borel automorphisms W of X with $Wx \in \mathbf{R}(x)$ for every $x \in X$. Under our assumption that $\mathbf{R}$ is discrete there exists a countable subgroup $\Gamma \subset [\mathbf{R}]$ with $\Gamma x = \{\gamma x : \gamma \in \Gamma\} = \mathbf{R}(x)$ for every $x \in X$. It follows that the *saturation* $\mathbf{R}(B) = \bigcup_{x\in B} \mathbf{R}(x)$ of every $B \in \mathfrak{B}_X$ lies in $\mathfrak{B}_X$. A sigma-finite measure μ on $\mathfrak{B}_X$ is *quasi-invariant* under $\mathbf{R}$ if $\mu(\mathbf{R}(B)) = 0$ for every $B \in \mathfrak{B}_X$ with $\mu(B) = 0$, and *ergodic* if it is quasi-invariant and either $\mu(\mathbf{R}(B)) = 0$ or $\mu(X \smallsetminus \mathbf{R}(B)) = 0$ for every $B \in \mathfrak{B}_X$. Every μ which is quasi-invariant under $\mathbf{R}$ is also quasi-invariant under every $W \in [\mathbf{R}]$, and by piecing together the Radon-Nikodym derivatives $d\mu\gamma/d\mu$, $\gamma \in \Gamma$, we can define a Borel map $\rho_\mu \colon \mathbf{R} \longmapsto \mathbb{R}$ such that

(1) $\rho_\mu(Wx,x) = (d\mu W/d\mu)(x)$ μ-*a.e.*, for every $W \in [\mathbf{R}]$,
(2) $\rho_\mu(x,x')\rho_\mu(x',x'') = \rho_\mu(x,x'')$ for every $(x,x'),(x,x'') \in \mathbf{R}$.

The map ρ_μ is the *Radon-Nikodym derivative* of μ under $\mathbf{R}$, and μ is ($\mathbf{R}$-)invariant if there exists a μ-null set $N \in \mathfrak{B}_X$ with $\rho_\mu(x,x') = 1$ for every $(x,x') \in \mathbf{R} \cap ((X \smallsetminus N) \times (X \smallsetminus N))$. In order to simplify terminology we say that a property holds (mod μ), or μ-*a.e.* on $\mathbf{R}$, or for μ-*a.e.* $(x,x') \in \mathbf{R}$, if there exists a μ-null set $N \in \mathfrak{B}_X$ such that the property holds everywhere on $\mathbf{R} \cap ((X \smallsetminus N) \times (X \smallsetminus N))$.

Furthermore, if $\mathbf{R} \subset X \times X$ is an equivalence relation and μ a sigma-finite measure on $\mathfrak{B}_X$ which is quasi-invariant (resp. ergodic) under $\mathbf{R}$, then $\mathbf{R}$ is said to be a *nonsingular* (resp. *ergodic*) equivalence relation on $(X, \mathfrak{B}_X, \mu)$. A nonsingular equivalence relation $\mathbf{R}$ on $(X, \mathfrak{B}_X, \mu)$ is *hyperfinite* if there exists a (nonsingular) automorphism V of $(X, \mathfrak{B}_X, \mu)$ such that $\mathbf{R} = \{(V^k x, x) : k \in \mathbb{Z}\} \pmod{\mu}$.

Fix an equivalence relation $\mathbf{R} \subset X \times X$ on a standard Borel space X and a nonatomic, sigma-finite measure μ on $\mathfrak{B}_X$ which is quasi-invariant and ergodic under $\mathbf{R}$. A Borel automorphism V of X is a (μ-nonsingular) *automorphism* of $(\mathbf{R}, \mu)$ if μ is quasi-invariant under V and $V(\mathbf{R}(x)) = \mathbf{R}(V(x))$ for μ-a.e. $x \in X$. Write $\mathrm{Aut}(\mathbf{R}, \mu)$ for the group of all nonsingular automorphisms of $\mathbf{R}$ and observe that $[\mathbf{R}] \subset \mathrm{Aut}(\mathbf{R})$. Let G be a locally compact, second countable, abelian group. A Borel map $c\colon \mathbf{R} \longmapsto G$ is a (1-)*cocycle* of $\mathbf{R}$ if

$$c(x, x') + c(x', x'') = c(x, x'')$$

for every $(x, x'), (x, x'') \in \mathbf{R}$. A cocycle $c\colon \mathbf{R} \longmapsto G$ is a (1-)*coboundary* if there exists a Borel map $b\colon X \longmapsto G$ with

$$c(x, x') = b(x) - b(x')$$

for μ-a.e. $(x, x') \in \mathbf{R}$, and two cocycles $c, c'\colon \mathbf{R} \longmapsto G$ are *cohomologous* if they differ by a coboundary. If $V \in \mathrm{Aut}(\mathbf{R}, \mu)$, then a cocycle $c\colon \mathbf{R} \longmapsto G$ is *invariant* under V if

$$c(x, x') = c(Vx, Vx')$$

for μ-a.e. $(x, x') \in \mathbf{R}$. Under pointwise addition, the set of G-valued cocycles on $\mathbf{R}$ forms a group $Z^1(\mathbf{R}, G)$, the sets $B^1(\mathbf{R}, G) \subset Z^1(\mathbf{R}, G)$ of coboundaries and $Z^1(\mathbf{R}, G)^V \subset Z^1(\mathbf{R}, G)$ of V-invariant cocycles are subgroups, and the quotient group $H^1(\mathbf{R}, G) = Z^1(\mathbf{R}, G)/B^1(\mathbf{R}, G)$ is called the (first) cohomology group of $\mathbf{R}$ with coefficients in G.

An element $g \in G$ is an *essential value* of a cocycle $c \in Z^1(\mathbf{R}, G)$ if

$$\{(x, x') \in \mathbf{R} \cap (B \times B) : c(x, x') \in N(g)\} \neq \varnothing$$

for every neighbourhood $N(g)$ of g in G and every $B \in \mathfrak{B}_X$ with $\mu(B) > 0$ (cf. [10]). The set $E(c)$ of essential values of c is a closed subgroup of G. If G is noncompact we say that ∞ is an essential value of c if

$$\{(x, x') \in \mathbf{R} \cap (B \times B) : c(x, x') \notin K\} \neq \varnothing$$

for every compact set $K \subset G$ and every $B \in \mathfrak{B}_X$ with $\mu(B) > 0$, and set

$$\bar{E}(c) = \begin{cases} E(c) \cup \{\infty\} & \text{if } \infty \text{ is an essential value of } c, \\ E(c) & \text{otherwise.} \end{cases}$$

An important feature of $\bar{E}(c)$ is that $\bar{E}(c_1) = \bar{E}(c_2)$ whenever $c_1, c_2 \in Z^1(\mathbf{R}, G)$ are cohomologous, and that $c \in B^1(\mathbf{R}, G)$ if and only if $\bar{E}(c) = \{0\}$. For later use we also define the *range* $R(c)$ of a cocycle $c \in Z^1(\mathbf{R}, \mu)$: an element $g \in G$ lies in $R(c)$ if and only if there exists, for every neighbourhood $N(g)$ of g in G and every μ-null set $N \in \mathfrak{B}_X$, an element $(x, x') \in \mathbf{R} \cap ((X \smallsetminus N) \times (X \smallsetminus N))$ with $c(x, x') \in N(g)$.

Proposition 2.1. *Assume that $\mathbf{R} \subset X \times X$ is a nonsingular and ergodic equivalence relation on a standard probability space $(X, \mathfrak{B}_X, \mu)$, and that $V \in \mathrm{Aut}(\mathbf{R}, \mu)$ is weakly mixing, i.e. that V has no nonconstant eigenfunction in $L^\infty(X, \mathfrak{B}_X, \mu)$. Then $B^1(\mathbf{R}, G) \cap Z^1(\mathbf{R}, G)^V = \{0\}$.*

Proof. If $0 \neq c \in B^1(\mathbf{R}, G) \cap Z^1(\mathbf{R}, G)^V$, then there exists a Borel map $b\colon X \longmapsto G$ with $c(x, x') = b(x) - b(x') = b(Vx) - b(Vx')$ for μ-a.e. $(x, x') \in \mathbf{R}$, and the ergodicity of μ under $\mathbf{R}$ implies that the map $b \cdot V - b$ is μ-a.e. equal to a constant $g \in G$. For every character $\chi \in \hat{G}$ the map $\chi \cdot b\colon X \longmapsto \mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$ is an eigenfunction of V with eigenvalue $\chi(g)$. As V is mixing, $g = 0$, b is constant, and $c = 0$. □

For the remainder of this section we assume that $\mathbf{R} \subset X \times X$ is an equivalence relation, and that μ is a probability measure on $\mathfrak{B}_X$ which is quasi-invariant and ergodic under $\mathbf{R}$.

Definition 2.2. A measure preserving automorphism $V \in \mathrm{Aut}(\mathbf{R}, \mu)$ is *asymptotically central* if

$$\lim_{n\to\infty} \mu(B \triangle V^n W V^{-n} B) = \lim_{n\to\infty} \mu(V^{-n} B \triangle W V^{-n} B) = 0 \tag{2.1}$$

for every $W \in [\mathbf{R}]$ and $B \in \mathfrak{B}_X$.

Theorem 2.3. *Let $\mathbf{R} \subset X \times X$ be a nonsingular and ergodic equivalence relation on a standard probability space $(X, \mathfrak{B}_X, \mu)$, and let $V \in \mathrm{Aut}(\mathbf{R}, \mu)$ be an asymptotically central automorphism. Then V is strongly mixing. Furthermore, if G is a locally compact, second countable, abelian group, then every cocycle $c \in Z^1(\mathbf{R}, G)^V$ satisfies that $E(c) = R(c)$. Finally, if $\lambda_{E(c)}$ is the Haar measure of the closed subgroup $E(c) = R(c) \subset G$, then $\mu \times \lambda_{E(c)}$ is nonsingular and ergodic under the skew product equivalence relation*

$$\begin{aligned} \mathbf{R}^{(c)} &= \{((x, g + c(x, x')), (x', g)) : (x, x') \in \mathbf{R},\ g \in G\} \\ &\subset ((X \times E(c)) \times (X \times E(c))). \end{aligned} \tag{2.2}$$

Proof. In the terminology of [11] and [6], Definition 2.2 can be expressed as saying that a measure preserving automorphism V of $(\mathbf{R}, \mu)$ is asymptotically central if and only if, for every $B \in \mathfrak{B}_X$, the sequence $(V^{-n}B,\ n \geq 0)$ is asymptotically invariant under $\mathbf{R}$. Every weak limit of the sequence $(1_{V^{-n}B},\ n \geq 0)$ of indicator functions must therefore be invariant under every $W \in [\mathbf{R}]$ and hence, by ergodicity, constant. Since V preserves μ, this constant is equal to $\mu(B)$. Hence $\lim_{n\to\infty} \mu(B_1 \cap V^{-n}B_2) = \mu(B_1)\mu(B_2)$ for all $B_1, B_2 \in \mathfrak{B}_X$, so that every asymptotically central automorphism of $(\mathbf{R}, \mu)$ is mixing.

If $c \in Z^1(\mathbf{R}, G)$ then it is obvious from the definitions that $E(c) \subset R(c)$. Conversely, if $g \in R(c)$, then there exists, for every neighbourhood $N(g)$ of g in G, a set $C \in \mathfrak{B}_X$ with $\mu(C) > 0$ and an element $W \in [\mathbf{R}]$ such that $c(Wx, x) \in N(g)$ for every $x \in C$. Let $B \in \mathfrak{B}_X$ with $\mu(B) > 0$. Since V is mixing,

$$\lim_{n\to\infty} \mu(B \cap V^{-n}C) = \mu(B)\mu(C),$$

and (2.1) guarantees that $\lim_{n\to\infty} \mu(B \triangle V^n W^{-1} V^{-n} B) = 0$. It follows that

$$\lim_{n\to\infty} \mu(B \cap V^n W^{-1} V^{-n} B \cap V^{-n} C) = \mu(B)\mu(C) > 0,$$

and every $x \in B \cap V^n W^{-1} V^{-n} B \cap V^{-n} C$ satisfies that $x \in B$, $V^n W V^{-n} x \in B$, $V^n x \in C$, and $c(V^n W V^{-n} x, x) = c(W V^{-n} x, V^{-n} x) \in N(g)$. This proves that $g \in E(c)$.

Since $E(c) = R(c)$ we may assume (after modifying c on $\mathbf{R} \cap ((X \smallsetminus N) \times (X \smallsetminus N))$ for some null set $N \in \mathfrak{B}_X$, if necessary) that $c(x, x') \in E(c)$ for every $(x, x') \in \mathbf{R}$, so that the skew product relation $\mathbf{R}^{(c)}$ in (2.2) is well defined. The ergodicity of $\mu \times \lambda_{E(c)}$ under $\mathbf{R}^{(c)}$ is a well known and easy consequence of the definition of $E(c)$ (cf. e.g. [10]). □

Corollary 2.4. *Let $\mathbf{R} \subset X \times X$ be a nonsingular and ergodic equivalence relation on a standard probability space $(X, \mathfrak{B}_X, \mu)$. If there exists an asymptotically central automorphism of $(\mathbf{R}, \mu)$, then μ is either $\mathbf{R}$-invariant, or $(\mathbf{R}, \mu)$ is of type III_λ for some $\lambda \in (0, 1]$ (i.e. $E(\log \rho_\mu) \neq \{0\}$ whenever μ is not invariant under $\mathbf{R}$).*

Proof. By assumption, the cocycle $c = \log \rho_\mu \colon \mathbf{R} \longmapsto \mathbb{R}$ lies in $Z^1(\mathbf{R}, \mathbb{R})^V$, and Theorem 2.3 guarantees that either $\bar{E}(c) = \{0\}$ (in which case μ is $\mathbf{R}$-invariant), or that $E(c) = R(c)$ is a closed, nonzero subgroup of $\mathbb{R}$. It follows that, if μ is not invariant, then $(\mathbf{R}, \mu)$ can neither be of type II_∞ nor of type III_0. $\square$

Corollary 2.5. *Let $\mathbf{R} \subset X \times X$ be a nonsingular and ergodic equivalence relation on a standard probability space $(X, \mathfrak{B}_X, \mu)$, $V \in \mathrm{Aut}(\mathbf{R}, \mu)$ an asymptotically central automorphism, (G, δ) a locally compact, metric, abelian group with identity element 0_G, and $c \in Z^1(\mathbf{R}, G)^V$.*

Then there exists a μ-null set $N \in \mathfrak{B}_X$ with the following property: for every $x \in X \smallsetminus N$, every $\varepsilon > 0$, and every $B \in \mathfrak{B}_X$ with $\mu(B) > 0$ there exists an $x' \in \mathbf{R}(x) \cap B$ with $\delta(c(x, x'), 0_G) < \varepsilon$. In particular, if $E(c)$ is a discrete subgroup of G then μ is ergodic under the equivalence relation

$$\mathbf{R}^{(c,0)} = \{(x, x') \in \mathbf{R} : c(x, x') = 0\} \subset \mathbf{R}. \tag{2.3}$$

Proof. The assertion of this corollary is equivalent to the ergodicity of μ under $\mathbf{R}^{(c)}$. $\square$

The equivalence relation $\mathbf{R}^{(c,0)}$ in (2.3) is called the *kernel* of the cocycle $c \colon \mathbf{R} \longmapsto G$. If the automorphism V of $(\mathbf{R}, \mu)$ is not asymptotically central the conclusion of Theorem 2.3 and its corollaries cannot be expected to hold, as the following example shows.

Example 2.6. Let $(X, \mathfrak{B}_X, \mu)$ be a standard probability space, and let S, T be commuting ergodic, measure preserving automorphisms of $(X, \mathfrak{B}_X, \mu)$ such that the $\mathbb{Z}^2$-action $(n_1, n_2) \mapsto S^{n_1} T^{n_2}$ is *free* in the sense that $\mu(\{x \in X : S^{n_1} T^{n_2} x = x\}) = 0$ whenever $(0, 0) \neq (n_1, n_2) \in \mathbb{Z}^2$. If $\mathbf{R} = \{(x, S^n x) : x \in X, n \in \mathbb{Z}\}$, then $T \in \mathrm{Aut}(\mathbf{R}, \mu)$ has *infinite outer period* (i.e. $\{T^n x : n \in \mathbb{Z}\} \cap \mathbf{R}(x) = \varnothing$ for μ-*a.e.* $x \in X$—cf. [2]), but $T^{-n} S T^n = S$ for every $n \geq 0$. In particular, (2.1) cannot hold for any $B \in \mathfrak{B}_X$ with $0 < \mu(B) < 1$.

The cocycle $c \in Z^1(\mathbf{R}, \mathbb{Z})^T$ given by

$$c(S^n x, x) = n$$

for every $x \in X$ and $n \in \mathbb{Z}$ satisfies that $\bar{E}(c) = \{0, \infty\}$. In fact, if G is a locally compact, second countable group, then every cocycle $c' \in Z^1(\mathbf{R}, G)^T$ looks essentially like this: the function $x \mapsto c'(Sx, x)$ is T-invariant, since

$$c'(Sx, x) = c'(TSx, Tx) = c'(STx, x),$$

and therefore μ-*a.e.* equal to a constant $g_0 \in G$. This shows that $c'(S^n x, x) = n g_0$ for every $n \in \mathbb{Z}$ and $x \in X$. If G is compact and S is weakly mixing, the cocycle c' *does* have the property that $E(c') = R(c')$, but if $R(c') = \overline{\{n g_0 : n \in \mathbb{Z}\}} \subset G$ in noncompact, then $E(c') \neq R(c')$.

We end this section with another example which shows that the inverse of an asymptotically central automorphism of an ergodic equivalence relation need not be asymptotically central.

Example 2.7. *Unstable and homoclinic relations of ergodic toral automorphisms.* Let $n \geq 2$, and let $A \in GL(n, \mathbb{Z})$ be an ergodic automorphism of $X = \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$. We regard A as a linear map on $\mathbb{R}^n$, write $\| \cdot \|$ for the Euclidean norm on $\mathbb{R}^n$, and denote by

$$F^- = \{v \in \mathbb{R}^n : \lim_{k \to \infty} \|A^k v\| = 0\}, \qquad F^+ = \{v \in \mathbb{R}^n : \lim_{k \to \infty} \|A^{-k} v\| = 0\}$$

the contracting and expanding subspaces of A. Under the factor map $\mathbb{R}^n \longmapsto \mathbb{T}^n$ the spaces F^- and F^+ are sent to the *unstable group* E^- and *stable group* E^+ of A, which are A-invariant, dense subgroups of X. In order to remain within the framework of discrete equivalence relations we fix a countable, dense, A-invariant subgroup $\Gamma \subset E^-$ and denote by $\mathbf{R} = \mathbf{R}_\Gamma = \{(x+\gamma, x) : x \in X, \gamma \in \Gamma\}$ the equivalence relation on X generated by the orbits of Γ.

Let μ be an A-invariant probability measure on $\mathfrak{B}_X$ which is quasi-invariant and ergodic under translation by Γ. Then we claim that A is an asymptotically central automorphism of $(\mathbf{R}, \mu)$.

Indeed, denote by $T_\gamma x = x + \gamma$ the translation by an element $\gamma \in \Gamma$, and let $B \in \mathfrak{B}_X$ with $\mu(B) > 0$. As μ is A-invariant, the sequence of Radon-Nikodym derivatives $(d\mu T_{A^k\gamma}/d\mu,\ k \geq 0)$ is uniformly integrable. If d is a translation invariant metric on X we set, for every $\delta > 0$,

$$B_d(B, \delta) = \{x \in X : \inf_{y \in B} d(x, y) < \delta\},$$

and conclude that there exists, for every $\varepsilon > 0$, a $\delta > 0$ with

$$\mu(T_{A^k\gamma}(B_d(B, \delta) \smallsetminus B)) < \varepsilon$$

for every $k \geq 0$. As $\lim_{k\to\infty} A^k\gamma = 0$ there exists an integer $K \geq 0$ with $T_{\pm A^k\gamma}(B) = B \pm A^k\gamma \subset B_d(B, \delta)$ for every $k \geq K$. It follows that, for $k \geq K$,

$$\mu(T_{A^k\gamma}B \smallsetminus B) \leq \mu(B_d(B, \delta) \smallsetminus B) < \varepsilon,$$
$$\mu(B \smallsetminus T_{A^k\gamma}B) \leq \mu(T_{A^k\gamma}(B_d(B, \delta)) \smallsetminus T_{A^k\gamma}B) < \varepsilon,$$

so that

$$\mu(B \triangle T_{A^k} B) = \mu(B \triangle A^k T_\gamma A^{-k} B) < 2\varepsilon.$$

This proves that A is asymptotically central on $(\mathbf{R}, \mu)$.

Now assume that $\mu = \lambda_X$ is the normalised Lebesgue measure on X. If A^{-1} is an asymptotically central automorphism of $(\mathbf{R}, \mu)$, then Definition 2.2 implies that

$$\lim_{n\to\infty} \mu(B \triangle (B + A^{-n}\gamma)) = 0$$

for every $B \in \mathfrak{B}_X$ and $\gamma \in \Gamma$. As the set B is arbitrary, we conclude that, for every $\gamma \in \Gamma$, 0 is the only limit point of the sequence $(A^{-n}\gamma,\ n \geq 0)$, so that every $\gamma \in \Gamma$ is *homoclinic to* 0 (i.e. $\gamma \in F^- \cap F^+$). However, if $A \in GL(n, \mathbb{Z})$ is irreducible and ergodic, but not expansive (= hyperbolic), then A has no nonzero points which are homoclinic to 0, so that A^{-1} cannot be an asymptotically central automorphism of $(\mathbf{R}, \mu)$.

Finally assume that the toral automorphism $A \in GL(n, \mathbb{R})$ is irreducible and expansive. In this case the group $\Delta = E^- \cap E^+ \subset X$ of points which are homoclinic to 0 is dense in X, and we choose a countable, dense, A-invariant subgroup $\Gamma \subset \Delta$ and define $\mathbf{R} = \mathbf{R}_\Gamma$ as before. The above proof shows that both A and A^{-1} are asymptotically central automorphisms of $(\mathbf{R}, \mu)$ for every probability measure μ on $\mathfrak{B}_X$ which is quasi-invariant and ergodic under translation by Γ.

In this case it is easy to construct A-invariant cocycles $c\colon \mathbf{R} \longmapsto \mathbb{R}$: take a Hölder continuous map $\phi\colon X \longmapsto \mathbb{R}$, and put, for every $x \in X$, $\gamma \in \Gamma$,

$$c(x+\gamma, x) = \sum_{k\in\mathbb{Z}} \phi(A^k(\gamma + x)) - \phi(A^k x). \tag{2.4}$$

Since $\lim_{|k|\to\infty} d(A^k\gamma, 0) = 0$ exponentially fast as $|k| \to \infty$ we see that the cocycle c in (2.4) is well defined, and obviously A-invariant. From [8] we know that $c = 0$ if and only if there exists a constant $a \in \mathbb{R}$ and a Hölder continuous function $b\colon X \longmapsto \mathbb{R}$ with $\phi = a + b \cdot A - b$ (cf. also [12]). Via Markov partitions this example is closely related to the examples discussed in the next section.

3. Shifts of finite type, random tilings and the super–K and strong Markov properties

This section is devoted to a particular class of examples of asymptotically central automorphisms obtained by considering the shift-action of $\mathbb{Z}^d$ on the Gibbs equivalence relation of a d-dimensional shift of finite type X.

Let A be a nonempty finite set, $d \geq 1$, and let $A^{\mathbb{Z}^d}$ be the space of all maps $x\colon \mathbb{Z}^d \longmapsto A$. For every set $F \subset \mathbb{Z}^d$ we denote by $\pi_F\colon A^{\mathbb{Z}^d} \longmapsto A^F$ the projection map which restricts every $x \in A^{\mathbb{Z}^d}$ to the set F. The space $A^{\mathbb{Z}^d}$ is compact in the product topology. We write a typical point $x \in A^{\mathbb{Z}^d}$ as $x = (x_{\mathbf{m}}) = (x_{\mathbf{m}}, \mathbf{m} \in \mathbb{Z}^d)$, where $x_{\mathbf{m}}$ denotes the value of x at $\mathbf{m}$, and define, for every $\mathbf{n} \in \mathbb{Z}^d$, a homeomorphism $\sigma_{\mathbf{n}}$ of $A^{\mathbb{Z}^d}$ by

$$(\sigma_{\mathbf{n}}(x))_{\mathbf{m}} = x_{\mathbf{m}+\mathbf{n}} \tag{3.1}$$

for every $x = (x_{\mathbf{m}}) \in A^{\mathbb{Z}^d}$. The map $\sigma\colon \mathbf{n} \longmapsto \sigma_{\mathbf{n}}$ is the *shift-action* of $\mathbb{Z}^d$ on $A^{\mathbb{Z}^d}$, and a subset $X \subset A^{\mathbb{Z}^d}$ is *shift-invariant* if $\sigma_{\mathbf{n}}(X) = X$ for every $\mathbf{n} \in \mathbb{Z}^d$. A closed, shift-invariant subset $X \subset A^{\mathbb{Z}^d}$ is a *subshift*, and X is a *shift of finite type* (*SFT*) if there exists a non-empty, finite set $F \subset \mathbb{Z}^d$ with

$$X = \{x \in A^{\mathbb{Z}^d} : \pi_F \cdot \sigma_{\mathbf{n}}(x) \in \pi_F(X) \text{ for every } \mathbf{n} \in \mathbb{Z}^d\}. \tag{3.2}$$

The restriction of the shift-action σ to X will again be denoted by σ.

If $X \subset A^{\mathbb{Z}^d}$ is a *SFT* then there exists an integer $k \geq 1$ such that the set F in (3.2) is contained in $E = \{-k, \dots, k\}^d \subset \mathbb{Z}^d$. Put $E' = \{-k, \dots, k-1\}^{\mathbb{Z}^d}$, $A' = \pi_{E'}(X) \subset A^{E'}$, define a continuous, injective, shift-commuting map $\eta\colon X \longmapsto A'^{\mathbb{Z}^d}$ by setting

$$(\eta(x))_{\mathbf{n}} = \pi_{E'}(\sigma_{\mathbf{n}}(x))$$

for every $x \in X$ and $\mathbf{n} \in \mathbb{Z}^d$, and observe that

$$Y = \eta(X) = \{y \in A'^{\mathbb{Z}^d} : \pi_{\{0,1\}^d}(\sigma_{\mathbf{n}}(y)) \in \pi_{\{0,1\}^d}(X) \text{ for every } \mathbf{n} \in \mathbb{Z}^d\}.$$

This shows that we may change the 'alphabet' A, if necessary, and assume without loss in generality that

$$F = \{0,1\}^d \subset \mathbb{Z}^d \tag{3.3}$$

in (3.2).

For the remainder of this section we fix a finite alphabet A and a *SFT* $X \subset A^{\mathbb{Z}^d}$ of the form (3.2)–(3.3). The *Gibbs* (*or homoclinic*) *equivalence relation* $\Delta_X \subset X \times X$ is defined by

$$\Delta_X = \{(x, x') \in X \times X : x_{\mathbf{m}} \neq x'_{\mathbf{m}} \text{ for only finitely many } \mathbf{m} \in \mathbb{Z}^d\}. \tag{3.4}$$

We shall be interested in shift-invariant probability measures on $\mathfrak{B}_X$ which are quasi-invariant and ergodic under Δ_X. The standard examples of such measures are the *Gibbs measures* of suitable functions $\phi\colon X \longmapsto \mathbb{R}$. In order to describe these measures and some related ideas we assume that G is a locally compact, second countable group with a distinguished translation invariant metric δ. For every continuous map $\phi\colon X \longmapsto G$ and every $k \geq 0$ we set

$$\omega_k(\phi) = \max_{\{(x,x') \in X \times X : \pi_{B(k)}(x) = \pi_{B(k)}(x')\}} \delta(\phi(x), \phi(x')),$$

where

$$B(k) = \{\mathbf{n} = (n_1, \dots, n_d) : |n_i| \leq k \text{ for } i = 1, \dots, d\}.$$

The map ϕ has *l-summable variation*, $l \geq 0$, if

$$\omega(\phi) = \sum_{n \geq 0} n^l \omega_n(\phi) < \infty.$$

Maps with 0-summable variation are simply said to have *summable variation*. If $G = \mathbb{R}$, $\delta(a, a') = |a - a'|$, and if $\phi\colon X \longmapsto \mathbb{R}$ is a map with $(d-1)$-summable variation we call a probability measure μ on $\mathfrak{B}_X$ a *Gibbs measure* of ϕ if μ is quasi-invariant under Δ_X with Radon-Nikodym derivative

$$\log \rho_\mu(x, x') = \sum_{\mathbf{m} \in \mathbb{Z}^d} \phi(\sigma_{\mathbf{m}}(x)) - \phi(\sigma_{\mathbf{m}}(x')) \tag{3.5}$$

for μ-a.e. $(x, x') \in \Delta_X$. The assumption that ϕ has $(d-1)$-summable variation is sufficient—but not necessary—to guarantee that the sum on the right hand side of (3.5) converges for every $(x, x') \in \Delta_X$; in many interesting cases this sum converges under much weaker assumptions on ϕ, in which case one can still speak of Gibbs measures in the sense of (3.5) for such functions. We write $M_1^\phi(X)$ for the set of Gibbs measures of ϕ, observe exactly as in the case where $d = 1$ that $M_1^\phi(X)$ is nonempty (cf. e.g. [9]), and conclude that the set $M_1^\phi(X)^\sigma$ of shift-invariant Gibbs measures of ϕ is nonempty. We are interested in the case where $M_1^\phi(X)^\sigma$ contains nonatomic measures which are ergodic under Δ_X. For such measures to exist, Δ_X obviously has to be reasonably large; however, even if Δ_X is topologically transitive or minimal (i.e. if $\Delta_X(x)$ is dense in X for some or every $x \in X$), the existence and possible uniqueness of such measures is a difficult and intriguing problem (cf. e.g. [1])

The following theorem is an immediate consequence of Definition 2.2 and Theorem 2.3.

Theorem 3.1. *Let $d \geq 1$, A a finite set, $X \subset A^{\mathbb{Z}^d}$ a* SFT, *and let μ be a shift-invariant probability measure on $\mathfrak{B}_X$ which is nonsingular and ergodic under Δ_X. Then the following is true for every nonzero $\mathbf{m} \in \mathbb{Z}^d$.*

(1) *$\sigma_{\mathbf{m}}$ is an asymptotically central automorphism of (Δ_X, μ);*
(2) *$\sigma_{\mathbf{m}}$ is a K-automorphism of (X, μ).*

Proof. The only statement going beyond Theorem 2.3 is that $\sigma_{\mathbf{m}}$ is a K-automorphism (and not just mixing). However, if $\mathcal{P}_{\mathbf{0}} = \{\pi_{\{\mathbf{0}\}}^{-1}(\{a\}) : a \in A\}$ is the state partition of X, then the ergodicity of μ under Δ_X guarantees that, for every $k \geq 0$, every set in the two-sided tail-sigma-algebra $\bigcap_{n \geq 0} \bigvee_{|l| \geq n} \bigvee_{\mathbf{m} \in B(k)} \sigma_{-\mathbf{m}-l\mathbf{n}}(\mathcal{P}_{\mathbf{0}})$ is saturated under Δ_X and hence of measure zero or one. □

Corollary 3.2. *Suppose that the assumptions of* Theorem 3.1 *are satisfied, and that G is a locally compact, second countable, abelian group and $c\colon \Delta_X \longmapsto G$ a cocycle of Δ_X which is shift-invariant, i.e. invariant under every $\sigma_{\mathbf{m}}$, $\mathbf{m} \in \mathbb{Z}^d$. Then the skew product relation $\Delta_X^{(c)}$ defined in* (2.2) *is ergodic. In particular, if G is discrete, then μ is ergodic under the subrelation $\Delta_X^{(c,0)}$ defined in* (2.3).

Proof. Apply Theorem 2.3 and Corollary 2.5. □

As a special case of Corollary 3.2 we obtain the following statement.

Corollary 3.3. *Suppose that the assumptions of* Theorem 3.1 *are satisfied, and that G is a discrete, abelian group, $\psi\colon X \longmapsto G$ a continuous map (which therefore has finite range), and $c_\psi\colon \Delta_X \longmapsto G$ the cocycle defined by*

$$c_\psi(x, x') = \sum_{\mathbf{m} \in \mathbb{Z}^d} \psi(\sigma_{\mathbf{m}}(x)) - \psi(\sigma_{\mathbf{m}}(x')) \tag{3.6}$$

for every $(x, x') \in \Delta_X$. Then μ is ergodic under the subrelation $\Delta_X^{(c_\psi,0)} \subset \Delta_X$.

In [9], Corollary 3.3 was proved for $d = 1$. The distinguishing feature of the subrelations covered by Corollary 3.3 is that they are defined by means of continuous functions ψ taking values in discrete, abelian groups. The next example uses a similar construction of subrelations, but with the help of a discontinuous function ψ.

Example 3.4. *The relation associated with finite permutations of monochromatic sets.* Let $d \geq 1$, A a finite set, $X \subset A^{\mathbb{Z}^d}$ a *SFT*, $\Delta_X \subset X \times X$ the Gibbs equivalence relation of X, and let μ be a shift-invariant probability measure on $\mathfrak{B}_X$ which is quasi-invariant and ergodic under Δ_X.

For every $\mathbf{m} = (m_1, \dots m_d) \in \mathbf{Z}^d$ we set $|\mathbf{m}| = \sum_{i=1}^d |m_i|$, and we call two points $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$ *adjacent* if $|\mathbf{m} - \mathbf{n}| = 1$. A subset $F \subset \mathbb{Z}^d$ is *connected* if there exists, for all $\mathbf{m}, \mathbf{n} \in F$, a finite sequence $\mathbf{m}_0 = \mathbf{m}, \mathbf{m}_1, \dots, \mathbf{m}_k = \mathbf{n}$ in F such that $\mathbf{m}_j$ and $\mathbf{m}_{j+1}$ are adjacent for $j = 0, \dots, k-1$. The *connected component* $C_F(\mathbf{m})$ of a point $\mathbf{m} \in F$ is the largest connected subset of F containing $\mathbf{m}$. If $d = 1$, then $C_F(\mathbf{m})$ is the longest interval in F containing $\mathbf{m}$.

If we interpret the elements of A as colours, then every point $x \in X$ represents a colouring of $\mathbb{Z}^d$, and we write

$$F(x) = \{\mathbf{m} \in \mathbb{Z}^d : x_{\mathbf{m}} = x_{\mathbf{0}}\}$$

for the set of coordinates with the same colour as the origin $\mathbf{0} = (0, \dots, 0)$ and denote by $C(x)$ the connected component of $\mathbf{0}$ in $F(x)$. Let $\mathcal{F}$ be the collection of all finite, connected subsets of $\mathbb{Z}^d$ containing $\mathbf{0}$, denote by $G \cong \mathbb{Z}^\infty$ the free abelian group whose set of generators is $\{\mathbf{e}(a, F) : a \in A, F \in \mathcal{F}\}$, and define a Borel map $f \colon X \longmapsto G$ by setting

$$f(x) = \begin{cases} \mathbf{e}(x_0, C(x)) & \text{if } C(x) \text{ is finite} \\ 0 & \text{otherwise.} \end{cases}$$

We define a shift-invariant cocycle $c \colon \Delta_X \longmapsto G$ by setting

$$c(x, x') = \sum_{\mathbf{m} \in \mathbb{Z}^d} f(\sigma_{\mathbf{m}}(x)) - f(\sigma_{\mathbf{m}}(x')) \tag{3.7}$$

for every $(x, x') \in \Delta_X$ (note that the sum on the right hand side of (3.7) contains only finitely many nonzero differences). According to Corollary 3.2, μ is ergodic under the equivalence relation $\Delta_X^{(c,0)}$ defined as in (2.3).

For an intuitive interpretation of this ergodicity property we again consider a typical point $x \in X$ as a colouring and assume that $x' \in X$ is a second colouring which differs from x only in a finite region of space (i.e. $x' \in \Delta_X(x)$). Suppose that we know that x' has (up to permutation) exactly the same finite, connected, monochromatic sets as x. Can one draw any conclusions as to what colour any particular coordinate (or set of coordinates) of x' is? (We know by assumption the colour of each coordinate of x.) According to Corollary 3.2, our knowledge of x reveals no information whatsoever about x'. If $d = 1$ this may not be too surprising, since one can fit together the alternating monochromatic intervals of a point $x \in X$ in many different ways (note that under our assumptions each monochromatic interval has finite length for μ-a.e. $x \in X$). If $d > 1$, the sets $C(x)$ will—in general—no longer be finite. However, in some interesting examples, like the full two-dimensional two-shift $X = \{0,1\}^{\mathbb{Z}^2}$ with uniform Bernoulli measure, the monochromatic component $C(x)$ is finite for μ-a.e. $x \in X$ (cf. [5]), but generally has an irregular shape. Nevertheless these monochromatic 'tiles' of x can be fitted together in many different ways; in particular one can rearrange, for typical points $x, x' \in X$, the tiles defined by x in a finite region of space so that the tiles in this rearrangement still alternate (i.e. no tile of a given colour 'touches' any other tile of the same colour), and that this new arrangement looks locally like the tiling arising from x'. Figure 1 shows a typical partial configuration in $X = \{0,1\}^{\mathbb{Z}^2}$, in which the zeros and ones are represented by white and black squares, and gives an impression of the random tiling arising from the connected, monochromatic subsets of $\mathbb{Z}^2$ defined by a point $x \in \{0,1\}^{\mathbb{Z}^2}$.

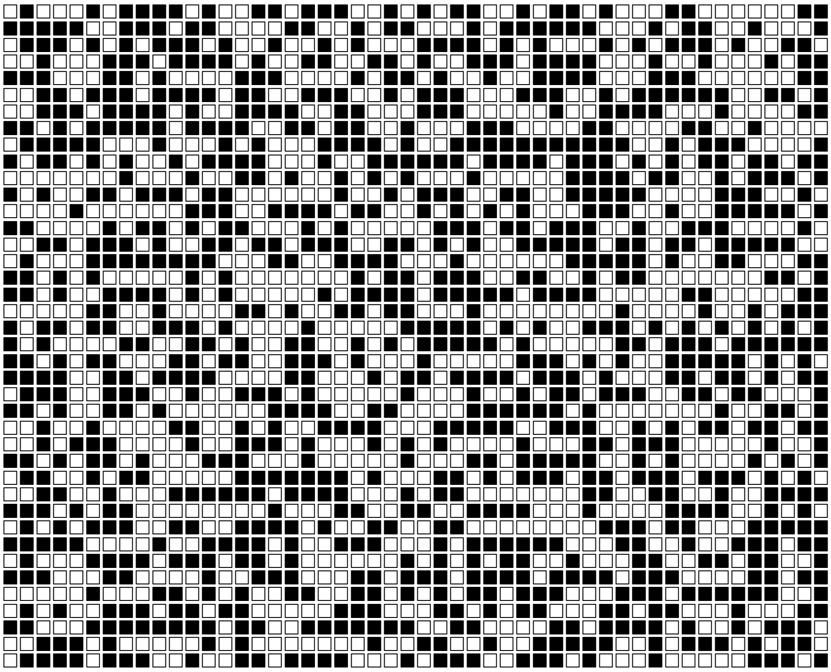

FIGURE 1

Any pair of points $(x, x') \in \Delta_X^{(c,0)}$ also has the property that, for every $a \in A$ and every sufficiently large $k \geq 0$,

$$\begin{aligned} N(x,a,k) &= |\{\mathbf{m} \in \mathbb{Z}^d : |\mathbf{m}| \leq k \ \text{ and } \ x_{\mathbf{m}} = a\}| \\ &= |\{\mathbf{m} \in \mathbb{Z}^d : |\mathbf{m}| \leq k \ \text{ and } \ x'_{\mathbf{m}} = a\}| = N(x',a,k). \end{aligned}$$

In particular, the coordinates of x and x' differ by a finite permutation. The ergodicity of the subrelation of Δ_X consisting of all pairs $(x, x') \in X \times X$ which differ by such finite permutations (which was called the *super-K property* in [9]) is a consequence of the ergodicity of the relation $\Delta_X^{(c,0)}$ defined above. For further discussion of the super-K property and related topics we refer to [9].

Example 3.5. *Marker relations and the strong Markov property.* Let A be a finite set, $X \subset A^{\mathbb{Z}}$ a *SFT*, and let μ be the unique Gibbs measure of a function $\phi\colon X \longmapsto \mathbb{R}$ with summable variation (cf. [3]). We fix a Borel set $B \subset X$ with $\mu(B) > 0$ and set, for every $x \in X$,

$$m^-(x) = \begin{cases} \min\{n \geq 0 : \sigma^{-n}(x) \in B\} & \text{if } x \in \cup_{n\geq 0}\sigma^n(B), \\ \infty & \text{otherwise,} \end{cases},$$

$$m^+(x) = \begin{cases} \min\{n \geq 1 : \sigma^{n}(x) \in B\} & \text{if } x \in \cup_{n\geq 0}\sigma^{-n}(B), \\ \infty & \text{otherwise.} \end{cases}$$

Denote by $\Omega(A) = \bigcup_{k\geq 1} A^k$ the disjoint union of the sets A^k, $k \geq 1$, write G for the free abelian group with generators $\{\mathbf{e}(\omega) : \omega \in \Omega(A)\}$, and define a map

$\psi\colon X \longmapsto G$ by setting

$$\psi(x) = \begin{cases} \mathbf{e}(x_{-m^-(x)}, \dots, x_{m^+(x)-1}) \in A^{m^-(x)+m^+(x)} \subset \Omega(A) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } m^-(x)+m^+(x) < \infty, \\ 0 \qquad\qquad\qquad\qquad\qquad\quad\; \text{otherwise.} \end{cases}$$

Again we would like to define a cocycle $c_B\colon \Delta_X \longmapsto G$ by setting

$$c_B(x,x') = \sum_{k\in\mathbb{Z}} \psi(\sigma^k(x)) - \psi(\sigma^k(x')) \tag{3.8}$$

for every $(x,x') \in \Delta_X$. However, if B is an arbitrary Borel set, there is no guarantee that the right hand side of (3.8) is well defined μ-*a.e.* on Δ_X. In order to ensure that (3.8) makes sense μ-*a.e.* we write $\mathcal{D}(m) \subset \mathfrak{B}_X$, $m \ge 0$, for the finite algebra of sets generated by all cylinder sets of the form $[i_{-m}, \dots, i_m]_{-m} = \{x = (x_n) \in X : x_j = i_j \text{ for } j = -m, \dots, m\}$ with $(i_{-m}, \dots, i_m) \in A^{2m+1}$, and call B *well approximable* if

$$\sum_{m\ge 0} \min_{E\in\mathcal{D}(m)} \mu(B\triangle E) < \infty. \tag{3.9}$$

In particular, if $(E_n,\, n \ge 0)$ is a sequence of sets such that $E_m \in \mathcal{D}(m)$ for every $m \ge 0$ and $\sum_{m\ge0} \mu(E_m) < \infty$, then the sets $\bigcup_{m\ge0} E_m$ or $\lim_{n\ge0} E_0\triangle\dots\triangle E_n$ are well approximable. Such sets occur, for example, in the study of isomorphisms of *SFT's* with finite expected code lengths (cf. [7]).

Suppose that $B \in \mathfrak{B}_X$ is well approximable. Let Γ be the countable group of homeomorphisms γ of X for which there exists an $N(\gamma) \ge 0$ with $x_n = (\gamma x)_n$ for every $x = (x_n) \in X$ and every $n \ge N(\gamma)$, and observe that $\Gamma x = \Delta_X(x)$ for every $x \in X$ (the elements of Γ are sometimes called *uniformly finite dimensional homeomorphisms* of X). For every $\gamma \in \Gamma$ and every $m \in \mathbb{Z}$ with $|m| \ge N(\gamma)$ we obtain that

$$\mu(\sigma^m(B)\triangle\gamma\sigma^m(B)) \le (1 + e^{(2N(\gamma)+1)\max|\phi|}) \min_{E\in\mathcal{D}(|m|-N(\gamma))} \mu(B\triangle E)$$

(cf. (3.5)), so that

$$\sum_{m\in\mathbb{Z}} \mu(\sigma^m(B)\triangle\gamma\sigma^m(B)) < \infty.$$

A straightforward calculation shows that the map $x \mapsto c_B(\gamma x, x)$ in (3.8) is well defined μ-*a.e.* for every $\gamma \in \Gamma$, since the right hand side of (3.8) contains only finitely many nonzero differences. By setting $c_B = 0$ on $\Delta_X \cap ((X \smallsetminus N) \times (X \smallsetminus N))$ for a suitable null set $N \in \mathfrak{B}_X$ we obtain from (3.8) a well defined, shift-invariant cocycle $c_B\colon \Delta_X \longmapsto G$. By Corollary 2.5, μ is ergodic under the equivalence relation $\Delta_X^{(c,0)}$ appearing in (2.3). If $B = [i_0, \dots, i_{l-1}] = \{x = (x_m) \in X : x_j = i_j \text{ for } j = 0, \dots, l-1\}$ is a cylinder set, and if μ is an l-step Markov measure, then the ergodicity of $\Delta_X^{(c,0)}$ is an immediate consequence of the strong Markov property, i.e. of the fact that the strings between successive visits to B form an infinite state Bernoulli process. The ergodicity of the relation $\Delta_X^{(c,0)}$ implies a weaker form of the strong Markov property for every Gibbs measure and every well approximable 'marker' set $B \in \mathfrak{B}_X$: the equivalence relation defined by all possible (allowed) rearrangements of the times between successive visits to B is ergodic (but in general not measure preserving).

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