ALGEBRAIC $\mathbb{Z}^d$-ACTIONS ON ZERO-DIMENSIONAL COMPACT ABELIAN GROUPS

KLAUS SCHMIDT

Dedicated to Anatole Katok on the occasion of his 60th birthday

1. INTRODUCTION

In 1967 Furstenberg proved that every infinite closed subset of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ simultaneously invariant under multiplication by 2 and by 3 is equal to $\mathbb{T}$ (cf. [8]), which motivated the still unresolved question whether this scarcity of invariant sets is paralleled by a corresponding scarcity of invariant probability measures: *is Lebesgue measure the only nonatomic probability measure on* $\mathbb{T}$ *which is invariant under multiplication by* 2 *and by* 3*?* Furstenberg's question remained dormant until 1988, when Lyons [18] proved that any probability measure on $\mathbb{T}$ which has completely positive entropy under either of these maps is equal to Lebesgue measure. In 1990 Rudolph weakened Lyons' hypotheses and proved the same result for any probability measure which is invariant and ergodic under the $\mathbb{N}^2$-action generated by multiplication by 2 and by 3 and has positive entropy under either of these maps.

In 1996 Katok and Spatzier [10] introduced a remarkable extension of the scope of Furstenberg's question to certain $\mathbb{Z}^d$-actions by automorphisms of compact abelian groups with $d > 1$.[1] They proved that any probability measure μ on a finite-dimensional torus $X = \mathbb{T}^n$ which is invariant and mixing under a topologically mixing, irreducible (Definition 4.2) and expansive[2] algebraic $\mathbb{Z}^d$-action α, and which has positive entropy under some element of the action, is a translate of Lebesgue measure on an α-invariant subtorus of X (the hypotheses in [10] are actually much weaker, but somewhat technical). The definitive version of this result is due to Einsiedler and Lindenstrauss [6] and implies that, for any probability measure μ on a finite-dimensional torus or solenoid X which is invariant and weakly mixing under a topologically mixing algebraic $\mathbb{Z}^d$-action α, there exists a closed α-invariant subgroup $Y \subset X$ such that $\mu = \mu * \lambda_Y$ and the action induced by each $\alpha^{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^d$, on X/Y has zero entropy with respect to the measure $\overline{\mu} = \mu\pi^{-1}$ (cf. Theorem 5.3). Here $\pi\colon X \longrightarrow X/Y$ is the quotient map.

Instead of pursuing further the many fascinating extensions of these *measure rigidity* results due to Katok and others let me turn to *isomorphism rigidity* of algebraic $\mathbb{Z}^d$-actions. Suppose that α and β are topologically mixing algebraic $\mathbb{Z}^d$-actions on finite-dimensional tori or solenoids X and Y, respectively. By following a suggestion of Thouvenot and applying the results in [10] or [6] to

---

I would like to thank the Mathematics Department, University of Washington, Seattle, for hospitality while writing up these notes, and to the FWF Project P16004-MAT and the NSF grant DMS 0222452 for partial support.

Much of this article forms part of the electronically published lecture notes of a series of Pacific Institute for the Mathematical Sciences Distinguished Chair Lectures given at the Mathematics Department, University of Victoria, BC, in November 2002 (`http://www.pims.math.ca/publications/distchair/schmidtnotes.pdf`).

[1] $\mathbb{Z}^d$-actions by continuous automorphisms of compact abelian groups will be referred to as *algebraic* $\mathbb{Z}^d$*-actions* throughout this article, and we shall always assume that $d > 1$.

[2] An algebraic $\mathbb{Z}^d$-action α on a compact abelian group X is *expansive* if there exists a neighbourhood $\mathcal{U}$ of the identity element $0 \in X$ with $\bigcap_{\mathbf{n}\in\mathbb{Z}^d} \alpha^{\mathbf{n}}(\mathcal{U}) = \{0\}$.

appropriate $\alpha \times \beta$-invariant joinings of the Haar measures λ_X and λ_Y one obtains that any measurable conjugacy of α and β is affine, i.e. that *isomorphism rigidity* holds for topologically mixing algebraic $\mathbb{Z}^d$-actions on finite-dimensional tori or solenoids for $d > 1$ (cf. [9]). Such actions obviously have zero entropy, but not all zero entropy mixing algebraic $\mathbb{Z}^d$-action on a compact connected abelian group live on *finite-dimensional* tori or solenoids. This raises the natural question whether all zero entropy mixing algebraic $\mathbb{Z}^d$-action on compact connected abelian groups exhibit isomorphism rigidity — a question which remains open at this stage (cf. Conjecture 5.1 and Problem 5.2).[3]

For mixing algebraic $\mathbb{Z}^d$-actions on zero-dimensional compact abelian groups the picture has recently become much clearer, and the results leading to this clarification are the subject of this article.

In 1978 Ledrappier [15] gave a simple example of a mixing algebraic $\mathbb{Z}^2$-action on a compact zero-dimensional abelian group which is not three-mixing. In 1993 Kitchens and the author investigated further the class of algebraic $\mathbb{Z}^d$-actions on zero-dimensional compact abelian groups and exhibited a number of invariants of measurable conjugacy of such actions related to the higher order mixing behaviour (in the sense of [15]) and to certain partially invariant sigma-algebras of such actions (cf. [12]). These invariants again suggested a close link between the measurable and the algebraic structure of such actions.

The results in [12] imply that an algebraic $\mathbb{Z}^d$-action α on a zero-dimensional compact abelian group X is mixing of every order if and only if it has completely positive entropy (cf. Theorem 3.3), and that every such action which is not mixing of every order has *nonmixing sets* which describe a very regular breakdown of mixing of a particular order (cf. (3.4)). In spite of Theorem 3.4 by Masser, which ties nonmixing sets to the algebraic structure of the action α, the explicit determination of the nonmixing sets of an algebraic $\mathbb{Z}^d$-action is generally a nontrivial task.

From the definition of nonmixing sets it is clear that an algebraic $\mathbb{Z}^d$-action α is not r-mixing if it has a nonmixing set of size r. The converse had been an open problem for some time and was only proved recently by Masser (Theorem 3.7).

The connection between the apparently unrelated notions of the order of mixing and isomorphism rigidity for irreducible and mixing algebraic $\mathbb{Z}^d$-actions on zero-dimensional compact abelian groups was established in 2000 by Kitchens and the author: if α and β are measurably conjugate irreducible and mixing algebraic $\mathbb{Z}^d$-actions on zero-dimensional compact abelian groups X and Y, then their nonmixing sets coincide, and every $\alpha \times \beta$-invariant joining μ of the Haar measures λ_X and λ_Y on $X \times Y$ which has the same nonmixing sets as λ_X and λ_Y is a translate of the Haar measure on some $\alpha \times \beta$-invariant closed subgroup $Z \subset X \times Y$. As in the connected case one can now use a joinings argument to prove isomorphism rigidity for such actions.

Up to this stage of the story isomorphism rigidity of mixing algebraic $\mathbb{Z}^d$-actions with zero entropy had only been established under the additional hypothesis of irreducibility or, somewhat more generally, of *entropy rank* 1.[4] The first step beyond this hypothesis in the zero-dimensional case is due to Bhattacharya in [2], where he uses the bounded order of mixing for arbitrary (i.e. not necessarily of entropy rank 1) zero entropy algebraic $\mathbb{Z}^d$-actions on zero-dimensional compact abelian groups to prove that measurable conjugacies between such actions are automatically continuous. With this information one can bring a variety of further tools into play, and by considering homoclinic points of certain sub-actions one can prove isomorphism rigidity of mixing zero entropy algebraic $\mathbb{Z}^d$-actions

[3] Positive entropy algebraic $\mathbb{Z}^d$-actions have Bernoulli factors and can therefore not be expected to exhibit isomorphism rigidity — cf. [17] and [25].

[4] A mixing algebraic $\mathbb{Z}^d$-action α on a compact abelian group X has *entropy rank* 1 if $0 < h_{\lambda_X}(\alpha^{\mathbf{n}}) < \infty$ for every nonzero $\mathbf{n} \in \mathbb{Z}^d$ (cf. [5]).

α on zero-dimensional compact abelian groups for which there exists a d' with $1 \leq d' < d$ such that the restriction of α to a subgroup $\Gamma \subset \mathbb{Z}^d$ of rank d' is expansive and has completely positive entropy (Theorem 6.11). The necessity of this condition is illustrated with a series of examples (Examples 6.15) which show that *isomorphism rigidity does not hold in general for zero entropy algebraic $\mathbb{Z}^d$-actions on zero-dimensional compact abelian groups*. However, the phenomenon underlying this possible breakdown of isomorphism rigidity in the zero-dimensional case (namely the existence of polynomial maps of degree > 1) is absent in the connected case (cf. Proposition 6.4 (3)), so that one currently cannot draw any further conclusions or conjectures from it.

2. Algebraic $\mathbb{Z}^d$-actions

An *algebraic $\mathbb{Z}^d$-action* is an action $\alpha\colon \mathbf{n} \mapsto \alpha^{\mathbf{n}}$ of $\mathbb{Z}^d$, $d \geq 1$, by continuous automorphisms of a compact abelian group X with Borel field $\mathcal{B}_X$ and normalized Haar measure λ_X. Let α and β be algebraic $\mathbb{Z}^d$-actions on compact abelian groups X and Y, respectively. The action β is a *measurable factor* of α if there exists a surjective Borel map $\phi\colon X \longrightarrow Y$ with $\lambda_X \phi^{-1} = \lambda_Y$ such that

$$\phi \cdot \alpha^{\mathbf{n}} = \beta^{\mathbf{n}} \cdot \phi \tag{2.1}$$

λ_X-*a.e.* for every $\mathbf{n} \in \mathbb{Z}^d$. If the map ϕ in (2.1) is continuous, then β is a *topological factor* of α, and if ϕ is a group homomorphism, β is an *algebraic factor* of α. If the factor map ϕ in (2.1) is invertible it is a (measurable, topological or algebraic) *conjugacy* and the actions α and β are (measurably, topologically or algebraically) *conjugate*.

In [11] and [24], Pontryagin duality was shown to imply a one-to-one correspondence between algebraic $\mathbb{Z}^d$-actions (up to algebraic conjugacy) and modules over the ring of Laurent polynomials $R_d = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ with integral coefficients in the commuting variables $u_1, \dots, u_d$ (up to module isomorphism). In order to describe this correspondence we write a typical element $f \in R_d$ as

$$f = \sum_{\mathbf{m} \in \mathbb{Z}^d} f_{\mathbf{m}} u^{\mathbf{m}} \tag{2.2}$$

with $u^{\mathbf{m}} = u_1^{m_1} \cdots u_d^{m_d}$ and $f_{\mathbf{m}} \in \mathbb{Z}$ for every $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{Z}^d$, where $f_{\mathbf{m}} = 0$ for all but finitely many $\mathbf{m}$. If α is an algebraic $\mathbb{Z}^d$-action on a compact abelian group X, then the additively-written dual group $M = \widehat{X}$ is a module over the ring R_d with operation

$$f \cdot a = \sum_{\mathbf{m} \in \mathbb{Z}^d} f_{\mathbf{m}} \widehat{\alpha^{\mathbf{m}}} a \tag{2.3}$$

for $f \in R_d$ and $a \in M$, where $\widehat{\alpha^{\mathbf{m}}}$ is the automorphism of $M = \widehat{X}$ dual to $\alpha^{\mathbf{m}}$. In particular,

$$u^{\mathbf{m}} \cdot a = \widehat{\alpha^{\mathbf{m}}} a \tag{2.4}$$

for $\mathbf{m} \in \mathbb{Z}^d$ and $a \in M$. This module $M = \widehat{X}$ is called the *dual module* of α. For every $f \in R_d$, the group homomorphism

$$f(\alpha) = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} \alpha^{\mathbf{n}}\colon X \longrightarrow X \tag{2.5}$$

is dual to multiplication by f on $M = \widehat{X}$ (or, equivalently, $\widehat{f(\alpha)} a = f \cdot a$ in (2.3)). In particular, $f(\alpha)$ is surjective if and only if f does not lie in any prime ideal associated[5] with M.

[5] A prime ideal $\mathfrak{p} \subset R_d$ is *associated with* an R_d-module M if

$$\mathfrak{p} = \operatorname{ann}(a) = \{f \in R_d : f \cdot a = 0_M\}$$

Conversely, any R_d-module M determines an algebraic $\mathbb{Z}^d$-action α_M on the compact abelian group $X_M = \widehat{M}$ with $\alpha_M^{\mathbf{m}}$ dual to multiplication by $u^{\mathbf{m}}$ on M for every $\mathbf{m} \in \mathbb{Z}^d$ (cf. (2.4)). Note that X_M is metrizable if and only if the dual module M of α_M is countable.

Examples 2.1. (1) Let $M = R_d$. Since R_d is isomorphic to the direct sum $\sum_{\mathbb{Z}^d} \mathbb{Z}$ of copies of $\mathbb{Z}$, indexed by $\mathbb{Z}^d$, the dual group $X = \widehat{R_d}$ is isomorphic to the Cartesian product $\mathbb{T}^{\mathbb{Z}^d}$ of copies of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We write a typical element $x \in \mathbb{T}^{\mathbb{Z}^d}$ as $x = (x_{\mathbf{n}})$ with $x_{\mathbf{n}} \in \mathbb{T}$ for every $\mathbf{n} \in \mathbb{Z}^d$ and choose the following identification of $X_{R_d} = \widehat{R_d}$ with $\mathbb{T}^{\mathbb{Z}^d}$: for every $x \in \mathbb{T}^{\mathbb{Z}^d}$ and $f \in R_d$,

$$\langle x, f\rangle = e^{2\pi i \sum_{\mathbf{n}\in\mathbb{Z}^d} f_{\mathbf{n}} x_{\mathbf{n}}},$$

where f is given by (2.2). Under this identification the $\mathbb{Z}^d$-action α_{R_d} on $X_{R_d} = \mathbb{T}^{\mathbb{Z}^d}$ becomes the shift-action

$$(\sigma^{\mathbf{m}} x)_{\mathbf{n}} = x_{\mathbf{m}+\mathbf{n}}. \tag{2.6}$$

(2) Let $I \subset R_d$ be an ideal and $M = R_d/I$. Since M is a quotient of the additive group R_d by an $\widehat{\alpha_{R_d}}$-invariant subgroup (i.e. by a submodule), the dual group $X_M = \widehat{M}$ is the closed α_{R_d}-invariant subgroup

$$\begin{aligned} X_{R_d/I} &= \{x \in X_{R_d} = \mathbb{T}^{\mathbb{Z}^d} : \langle x, f\rangle = 1 \ \text{ for every } \ f \in I\} \\ &= \Bigl\{x \in \mathbb{T}^{\mathbb{Z}^d} : \sum_{\mathbf{n}\in\mathbb{Z}^d} f_{\mathbf{n}} x_{\mathbf{m}+\mathbf{n}} = 0 \pmod 1 \\ &\qquad\qquad \text{for every } \ f \in I \ \text{ and } \ \mathbf{m} \in \mathbb{Z}^d \Bigr\} \\ &= \bigcap_{f\in I} \ker f(\alpha_{R_d}) = \bigcap_{i=1}^{m} \ker f_i(\alpha_{R_d}), \end{aligned} \tag{2.7}$$

where $f_1, \dots, f_m$ is a set of generators of I and $f(\alpha_{R_d})$ is defined by (2.5) for every $f \in I$. The $\mathbb{Z}^d$-action $\alpha_{R_d/I}$ is the restriction of the shift-action $\sigma = \alpha_{R_d}$ in (2.6) to the shift-invariant subgroup $X_{R_d/I} \subset \mathbb{T}^{\mathbb{Z}^d}$.

Conversely, let $X \subset \mathbb{T}^{\mathbb{Z}^d} = \widehat{R_d}$ be a closed subgroup, and let

$$X^{\perp} = \{f \in R_d : \langle x, f\rangle = 1 \ \text{ for every } \ x \in X\}$$

be the annihilator of X in $\widehat{R_d}$. Then X is shift-invariant if and only if $X^{\perp}$ is an ideal in R_d.

(3) Let $p > 1$ be a rational prime, denote by $R_d^{(p)} = F_p[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ the ring of Laurent polynomials in $u_1, \dots, u_d$ with coefficients in the prime field $F_p = \mathbb{Z}/p\mathbb{Z}$, and write every $f \in R_d^{(p)}$ as $f = \sum_{\mathbf{n}\in\mathbb{Z}^d} f_{\mathbf{n}} u^{\mathbf{n}}$ with $f_{\mathbf{n}} \in F_p$ for every $\mathbf{n} \in \mathbb{Z}^d$. For every $f = \sum_{\mathbf{n}\in\mathbb{Z}^d} f_{\mathbf{n}} u^{\mathbf{n}} \in R_d$ we denote by

$$f_{/p} = \sum_{\mathbf{n}\in\mathbb{Z}^d} (f_{\mathbf{n}} \ (\mathrm{mod}\, p)) u^{\mathbf{n}} \in R_d^{(p)} \tag{2.8}$$

the Laurent polynomial obtained by reducing each coefficient of f modulo p. For every ideal $I \subset R_d^{(p)}$, $\overline{I} = \{f \in R_d : f_{/p} \in I\}$ is an ideal in R_d, and $R_d^{(p)}/I \cong R_d/\overline{I}$. Furthermore, $\overline{I} \subset R_d$ is a prime ideal if and only if $I \subset R_d^{(p)}$ is a prime ideal.

---

for some $a \in M$. The set of all prime ideals associated with M is denoted by $\mathrm{asc}(M)$ and satisfies that

$$\bigcup_{\mathfrak{p}\in\mathrm{asc}(M)} \mathfrak{p} = \bigcup_{0\neq a\in M} \mathrm{ann}(a).$$

If M is Noetherian, then $\mathrm{asc}(M)$ is finite. For details see [14].

The additive group $R_d^{(p)}$ can be identified with the dual group of $(\mathbb{Z}/p\mathbb{Z})^{\mathbb{Z}^d}$ by setting

$$\langle h, \omega\rangle = e^{2\pi i \left(\sum_{\mathbf{n}\in\mathbb{Z}^d} h_{\mathbf{n}}\omega_{\mathbf{n}}\right)/p}$$

for every $h \in R_d^{(p)}$ and $\omega \in (\mathbb{Z}/p\mathbb{Z})^{\mathbb{Z}^d}$. With this identification the shift $\sigma^{\mathbf{m}}\colon (\mathbb{Z}/p\mathbb{Z})^{\mathbb{Z}^d} \longrightarrow (\mathbb{Z}/p\mathbb{Z})^{\mathbb{Z}^d}$ defined as in (2.6) is dual to multiplication by $u^{\mathbf{m}}$ on $R_d^{(p)}$, and $h(\sigma)$ is dual to multiplication by h on $R_d^{(p)}$ for every $h \in R_d^{(p)}$ (cf. (2.5)).

If $\mathfrak{q} \subset R_d^{(p)}$ is an ideal with generators $\{h^{(1)}, \dots, h^{(k)}\}$ we can rewrite (2.7) as

$$\begin{aligned}\widehat{R_d^{(p)}/\mathfrak{q}} = X_{R_d^{(p)}/\mathfrak{q}} &= \{\omega \in (\mathbb{Z}/p\mathbb{Z})^{\mathbb{Z}^d} : \langle h, \omega\rangle = 1 \text{ for every } h \in \mathfrak{q}\} \\ &= \bigcap_{h\in\mathfrak{q}} \ker(h(\sigma)) = \bigcap_{i=1}^{k} \ker(h^{(i)}(\sigma)),\end{aligned} \tag{2.9}$$

and

$$\alpha_{R_d^{(p)}/\mathfrak{q}} = \sigma_{X_{R_d^{(p)}/\mathfrak{q}}} \tag{2.10}$$

is the restriction of the shift-action σ to $X_{R_d^{(p)}/\mathfrak{q}} \subset (\mathbb{Z}/p\mathbb{Z})^{\mathbb{Z}^d}$.

The correspondence between algebraic $\mathbb{Z}^d$-actions $\alpha = \alpha_M$ and R_d-modules M yields a correspondence (or 'dictionary') between dynamical properties of α_M and algebraic properties of the module M (cf. [25]). It turns out that many of the principal dynamical properties of α_M can be expressed entirely in terms of the prime ideals associated with the module M (cf. Footnote 5 on the facing page). Here we need only a few entries from this dictionary.

Theorem 2.2. *Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X with dual module $M = \widehat{X}$.*

1. *The group X is connected if and only if* no *prime ideal $\mathfrak{p} \in \operatorname{asc}(M)$ contains a nonzero constant, and X is zero-dimensional if and only if* every *$\mathfrak{p} \in \operatorname{asc}(M)$ contains a nonzero constant;*
2. *The action α is mixing if and only no prime ideal $\mathfrak{p} \in \operatorname{asc}(M)$ contains a polynomial of the form $u^{\mathbf{m}} - 1$ with $\mathbf{m} \in \mathbb{Z}^d \smallsetminus \{\mathbf{0}\}$;*
3. *If X is zero-dimensional, then the action α has completely positive entropy if and only if every prime ideal $\mathfrak{p} \in \operatorname{asc}(M)$ is principal (and hence equal to $p(\mathfrak{p})R_d$ for some nonzero prime constant $p(\mathfrak{p}) > 1$).*
4. *If X is zero-dimensional, then the action α is expansive if and only if the dual module $M = \widehat{X}$ is Noetherian.*

Proof. If M contains a nonzero element a of finite order $n \geq 2$, say, then $\langle a, x\rangle$ is an n-th root of unity for every $x \in X$, and the continuous map map $x \mapsto \langle a, x\rangle$ sends X onto a finite set containing more than one element. Hence X is not connected.

Conversely, suppose that every nonzero element of M has infinite order, and that X is not connected. We fix a metric δ on X and choose two complementary open sets $\mathcal{O}_1, \mathcal{O}_2$ in X. By compactness there exists an $\varepsilon > 0$ such that $x + B_\delta(\varepsilon) \subset \mathcal{O}_i$ for every $x \in \mathcal{O}_i$, $i = 1, 2$, where $B_\delta(\varepsilon) = \{y \in X : \delta(y, 0) < \varepsilon\}$.

Choose an increasing sequence of finitely generated subgroups (A_n) in M with $\bigcup_{n\geq 1} A_n = M$. The annihilators $Y_n = A_n^{\perp}$ form a decreasing sequence of closed subgroups of X with $\bigcap_{n\geq 1} Y_n = \{0\}$, and hence with $Y_n \subset B_\delta(\varepsilon)$ for all $n \geq n_0$, say. Our choice of ε implies that $x + Y_{n_0} \subset \mathcal{O}_i$ for every

$x \in \mathcal{O}_i$, $i = 1,2$, and hence that the quotient group X/Y_{n_0} is not connected. As $\widehat{X/Y_{n_0}} = A_{n_0}$ is finitely generated and has no nonzero elements of finite order, $A_{n_0} \cong \mathbb{Z}^m$ and $\widehat{A_{n_0}} = X/Y_{n_0} \cong \mathbb{T}^m$ for some $m \geq 1$, which contradicts the diconnectedness of X/Y_{n_0}.

We have established the well known fact that X is disconnected if and only if $\widehat{X} = M$ contains an element $a \neq 0$ of finite order. If the latter condition holds we set $N = R_d \cdot a$ and choose a nonzero $b \in N$ whose annihilator $\operatorname{ann}(b) = \{f \in R_d : f \cdot b = 0\}$ is maximal (this is possible since the ring R_d is Noetherian). Then $\mathfrak{p} = \operatorname{ann}(b)$ is a prime ideal which is obviously associated with M and contains a nonzero constant by assumption.

Conversely, if some $\mathfrak{p} \in \operatorname{asc}(M)$ contains a nonzero constant, then M obviously contains elements of finite order.

Essentially the same argument as above shows that the following conditions are equivalent:

(i) X is zero-dimensional,
(ii) X contains no nontrivial connected subgroups,
(iii) Every element $a \in M$ has finite order,
(iv) Every prime ideal $\mathfrak{p} \in \operatorname{asc}(M)$ contains a nonzero constant.

This completes the proof of (1).

The second assertion is [25, Theorem 6.5 (2)] and (3) follows from [25, Theorem 20.8].

In order to prove (4) we note that $\alpha_{R_d/\mathfrak{p}}$ is obviously expansive for every every prime ideal $\mathfrak{p} \subset R_d$ containing a rational prime constant $p > 1$, since it is the shift-action on some closed, shift-invariant subgroup of $F_p^{\mathbb{Z}^d}$ (cf. Example 2.1 (3)). If X is zero-dimensional, then (1) implies that every $\mathfrak{p} \in \operatorname{asc}(M)$ contains a prime constant, and our assertion is a consequence of [25, Corollary 4.7, Proposition 5.4 and Theorem 6.5 (4)]. □

3. Multiple mixing of algebraic $\mathbb{Z}^d$-actions on zero-dimensional groups

In this section we describe the connection between higher order mixing properties of algebraic $\mathbb{Z}^d$-actions and certain diophantine results on additive relations in fields due to David Masser ([12], [19]).

Recall that an algebraic $\mathbb{Z}^d$-action α on a compact abelian group X is *mixing of order $r \geq 2$* if

$$\lim_{\substack{\mathbf{n}_1,\dots,\mathbf{n}_r \in \mathbb{Z}^d \\ \|\mathbf{n}_i - \mathbf{n}_j\| \to \infty \text{ for } 1 \leq i < j \leq d}} \lambda_X\left(\bigcap_{i=1}^r \alpha^{-\mathbf{n}_i} B_i\right) = \prod_{i=1}^r \lambda_X(B_i) \tag{3.1}$$

for all Borel sets $B_i \subset X$, $i = 1, \dots, r$.

Let $\mathfrak{p} \subset R_d$ be a prime ideal, and let $\alpha = \alpha_{R_d/\mathfrak{p}}$ be the algebraic $\mathbb{Z}^d$-action with dual module $M = R_d/\mathfrak{p} = \widehat{X}$. If α is not mixing (i.e. not mixing of order 2 in the sense of (3.1)), then there exist Borel sets $B_1, B_2 \subset X$ and a sequence $(\mathbf{n}_k, k \geq 1)$ in $\mathbb{Z}^d$ with $\lim_{k\to\infty} \mathbf{n}_k = \infty$ and

$$\lim_{k\to\infty} \lambda_X(B_1 \cap \alpha^{-\mathbf{n}_k} B_2) = c$$

for some $c \neq \lambda_X(B_1)\lambda_X(B_2)$. Fourier expansion implies that the latter condition is equivalent to the existence of nonzero elements $a_1, a_2 \in M$ such that

$$a_1 + u^{\mathbf{n}_k} \cdot a_2 = 0$$

for infinitely many $k \geq 1$. In particular,

$$(u^{\mathbf{m}} - 1) \cdot a_2 = 0 \tag{3.2}$$

for some nonzero $\mathbf{m} \in \mathbb{Z}^d$. A very similar (but a little more careful) argument shows that α is not mixing of order $r \geq 2$ if and only if there exist elements $a_1, \dots, a_r$ in M, not all equal to zero, and a sequence $((\mathbf{n}_k^{(1)}, \dots, \mathbf{n}_k^{(r)}), k \geq 1)$ in $(\mathbb{Z}^d)^r$ with $\lim_{k\to\infty} \|\mathbf{n}_k^{(i)} - \mathbf{n}_k^{(j)}\| = \infty$ for all i, j with $1 \leq i < j \leq r$, such that

$$u^{\mathbf{n}_k^{(1)}} \cdot a_1 + \cdots + u^{\mathbf{n}_k^{(r)}} \cdot a_r = 0 \tag{3.3}$$

for every $k \geq 1$.

Below we shall see that higher order mixing of an algebraic $\mathbb{Z}^d$-action α on a compact abelian group X can break down in a particularly regular way (cf. Examples 3.2). We call a nonempty finite subset $S \subset \mathbb{Z}^d$ *mixing* for α if

$$\lim_{k\to\infty} \lambda_X \left(\bigcap_{\mathbf{n}\in S} \alpha^{-k\mathbf{n}} B_{\mathbf{n}} \right) = \prod_{\mathbf{n}\in S} \lambda_X(B_{\mathbf{n}}) \tag{3.4}$$

for all Borel sets $B_{\mathbf{n}} \subset X$, $\mathbf{n} \in S$, and *nonmixing* otherwise. A set $S \subset \mathbb{Z}^d$ is *minimal nonmixing* if it is nonmixing, but every nonempty subset $S' \subsetneq S$ is mixing.

As in (3.3) one sees that a nonempty finite set $S \subset \mathbb{Z}^d$ is nonmixing if and only if there exist elements $a_{\mathbf{n}} \in M$, $\mathbf{n} \in S$, not all equal to zero, such that

$$\sum_{\mathbf{n}\in S} u^{k\mathbf{n}} \cdot a_{\mathbf{n}} = 0 \quad \text{for infinitely many } k \geq 1. \tag{3.5}$$

Our next result shows that the higher order mixing behaviour of an algebraic $\mathbb{Z}^d$-action α with dual module M is again completely determined by that of the actions $\alpha_{R_d/\mathfrak{p}}$ with $\mathfrak{p} \in \operatorname{asc}(M)$ ([12] and [27]).

Theorem 3.1. *Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X with dual module $M = \widehat{X}$.*

(1) *For every $r \geq 2$, the following conditions are equivalent:*
   (a) *α is r-mixing (i.e. mixing of order r),*
   (b) *$\alpha_{R_d/\mathfrak{p}}$ is r-mixing for every $\mathfrak{p} \in \operatorname{asc}(M)$.*

(2) *For every nonempty finite set $S \subset \mathbb{Z}^d$, the following conditions are equivalent:*
   (a) *S is α-mixing,*
   (b) *S is $\alpha_{R_d/\mathfrak{p}}$-mixing for every $\mathfrak{p} \in \operatorname{asc}(M)$.*

The following examples show some of the mechanisms which can lead to nonmixing sets for mixing algebraic $\mathbb{Z}^d$-actions.

Examples 3.2. (1) (Ledrappier's Example [15]) Let $\mathfrak{p} = (2, 1+u_1+u_2) = 2R_2 + (1+u_1+u_2)R_2$, $M = R_2/\mathfrak{p}$, and let $\alpha = \alpha_M$ be the algebraic $\mathbb{Z}^2$-action on $X = X_M = \widehat{M}$ defined in Example 2.1 (2). Then α is mixing by Theorem 2.2 (2), but the set $S = \{(0,0),(1,0),(0,1)\} \subset \mathbb{Z}^2$ is nonmixing.

Indeed, $(1+u_1+u_2)^{2^n} \cdot a = 0$ for every $n \geq 0$ and $a \in M$. For $a = 1 + (2, 1+u_1+u_2) \in M$ our identification of M with $\widehat{X}$ in Example 2.1 (2) implies that $x_{(0,0)} + x_{(2^n,0)} + x_{(0,2^n)} = 0 \pmod 1$ for every $x \in X$ and $n \geq 0$. For $B = \{x \in X : x_{(0,0)} = 0\}$ it follows that

$$B \cap \alpha^{-(2^n,0)}(B) \cap \alpha^{-(0,2^n)}(B) = B \cap \alpha^{-(2^n,0)}(B),$$

and hence that

$$\lambda_X(B \cap \alpha^{-(2^n,0)}(B) \cap \alpha^{-(0,2^n)}(B)) = \lambda_X(B \cap \alpha^{-(2^n,0)}(B)) = 1/4$$

for every $n \geq 0$. If the set $S = \{(0,0),(1,0),(0,1)\} \subset \mathbb{Z}^2$ were α-mixing, we would have that

$$\lim_{n\to\infty} \lambda_X(B \cap \alpha^{-(2^n,0)}(B) \cap \alpha^{-(0,2^n)}(B)) = \lambda_X(B)^3 = 1/8.$$

By comparing this with (3.4) we see that S is indeed nonmixing (cf. [15]).

(2) In order to generalize Example (1) we fix a rational prime $p > 1$ and an ideal $I \subset R_d^{(p)}$, and observe as in Example (1) that the *support*

$$\mathcal{S}(h) = \{\mathbf{n} \in \mathbb{Z}^d : h_{\mathbf{n}} \neq 0\} \tag{3.6}$$

of every nonzero $h \in I$ is a nonmixing set for $\alpha_{R_d^{(p)}/I}$.

The two following examples show that nonmixing sets can also arise in a much less obvious manner.

(3) ([12]) Let $f = 1 + u_1 + u_2 + u_1^2 + u_1u_2 + u_2^2 \in R_2^{(2)}$, and let $\mathfrak{p} = (f) = fR_2^{(2)} \subset R_2^{(2)}$. Since f is irreducible, $\mathfrak{p}$ is a prime ideal. We set $\alpha = \alpha_{R_2^{(2)}/\mathfrak{p}}$ and $X = X_{R_2^{(2)}/\mathfrak{p}}$ (cf. (2.9)–(2.10)).

A direct calculation shows that

$$\begin{aligned} (u_1+u_2) + (1+u_2)u_1 + (1+u_1)u_2 = 0,\\ (1+u_1)^3 = (1+u_2)^3 = (u_1+u_2)^3 \pmod{\mathfrak{p}}. \end{aligned} \tag{3.7}$$

By raising the first of these equations to the fourth power and substituting terms according to the second equation we obtain that

$$\begin{aligned} 0 &= (u_1+u_2)^4 + (1+u_2)^4u_1^4 + (1+u_1)^4u_2^4 \\ &= (u_1+u_2)^4 + (1+u_2)(u_1+u_2)^3u_1^4 + (1+u_1)(u_1+u_2)^3u_2^4 \pmod{\mathfrak{p}}. \end{aligned}$$

It follows that

$$(u_1+u_2) + (1+u_2)u_1^4 + (1+u_1)u_2^4 \in \mathfrak{p},$$

and by repeating this argument we see that

$$(u_1+u_2) + (1+u_2)u_1^{4^k} + (1+u_1)u_2^{4^k} \in \mathfrak{p} \tag{3.8}$$

for every $k \geq 0$. A glance at (3.5) reveals that we have proved that the set $S = \{(0,0),(1,0),(0,1)\}$ is α-nonmixing, although it is not the support of any element of $\mathfrak{p}$.

Theorem 3.4 below will explain what is going on here: if we choose a primitive third root of unity in $\bar{F}_2$, the algebraic closure of the prime field F_2, and set $F_4 = F_2[\omega]$, then the polynomial $f \in F_2[u_1^{\pm1}, u_2^{\pm1}]$ is no longer irreducible in the ring $F_4[u_1^{\pm1}, u_2^{\pm1}]$:

$$1 + u_1 + u_2 + u_1^2 + u_1u_2 + u_2^2 = (1 + \omega u_1 + \omega^2 u_2)(1 + \omega^2 u_1 + \omega u_2).$$

For every $h \in R_2^{(2)}$ we set $[h] = h + \mathfrak{p} \in R_2^{(2)}/\mathfrak{p}$. If $K = Q(R_2^{(2)}/\mathfrak{p})$ is the field of fractions of the integral domain $R_2^{(2)}/\mathfrak{p}$, then the second equation in (3.7) is equivalent to saying that $\omega = \frac{[1+u_2]}{[u_1+u_2]}$ is a primitive third root of unity in K and hence that $K \supset F_4$. Equation (3.8) translates as

$$1 + \omega^{4^k}[u_1]^{4^k} + (\omega^2)^{4^k}[u_2]^{4^k} = 1 + \omega[u_1]^{4^k} + \omega^2[u_2]^{4^k} = 0$$

for every $k \geq 0$.

(4) ([12]) Let $f = 1 + u_1 + u_2 + u_1^2 + u_1u_2 + u_2^2 + u_1^3 + u_1^2u_2 + u_1u_2^2 + u_2^3 \in R_2^{(2)}$, $g = 1 + u_1 + u_2 \in R_2^{(2)}$, $\mathfrak{p} = (f) \subset R_2^{(2)}$, $\mathfrak{q} = (g) \subset R_2^{(2)}$, and let $\alpha = \alpha_{R_2^{(2)}/\mathfrak{p}}$ and $X = X_{R_2^{(2)}/(f)}$ as in Example 2.1 (3). We claim that the set $S = \{(0,0),(1,0),(0,1)\}$ is nonmixing for α.

In contrast to Example (3), the polynomial f is irreducible not only in $R_2^{(2)}$, but also in $\bar{F}_2[u_1^{\pm1}, u_2^{\pm1}]$, i.e. f is *absolutely irreducible*. However,

$$f(u_1^3,u_2^3) = 1+u_1^3+u_2^3+u_1^6+u_1^3u_2^3+u_2^6+u_1^9+u_1^6u_2^3+u_1^3u_2^6+u_2^9 = gh$$

for some $h \in R_2^{(2)}$.

We denote by $K = Q(R_2^{(2)}/\mathfrak{p})$ and $L = Q(R_2^{(2)}/\mathfrak{q})$ the fields of fractions of the integral domains $R_2^{(2)}/\mathfrak{p}$ and $R_2^{(2)}/\mathfrak{q}$, respectively, and set $[h] = h+\mathfrak{q} \in R_2^{(2)}/\mathfrak{q} \subset L$ for every $h \in R_2^{(2)}$. The ring homomorphism $\eta \colon R_2^{(2)} \longrightarrow L$, defined by setting $\eta(u_i) = [u_i^3] = [u_i]^3 \in R_2^{(2)}/\mathfrak{q} \subset L$ for $i = 1,2$, satisfies that $\ker\eta = \mathfrak{p} = (f)$. Hence η induces an embedding $\eta' \colon K \longrightarrow L$ of K as a subfield $K' = \eta'(K) \subset L$.

By assumption, $1+[u_1]^{2^k}+[u_2]^{2^k} = 0$ in L for every $k \geq 0$. As $2^{2k} = 1 \pmod 3$ for every $k \geq 0$, the sequence of integers $l_k = \frac{2^{2k}-1}{3}$, $k \geq 0$, satisfies that

$$1+[u_1]^{2^{2k}}+[u_2]^{2^{2k}} = 1+[u_1^3]^{l_k}[u_1]+[u_2^3]^{l_k}[u_2] = 0$$

for every $k \geq 0$. This shows that the nonzero vector $\mathbf{v} = (1,[u_1],[u_2])$ is orthogonal to all the vectors $\mathbf{w}_k = (1,[u_1^3]^{l_k},[u_2^3]^{l_k})$, $k \geq 0$, in L^3. As $\mathbf{w}_k \in K'^3$ for every $k \geq 0$, there also exists a nonzero vector $\mathbf{v}' = (a,b,c) \in K'^3$ which is orthogonal to every $\mathbf{w}_k$. After identifying K' with K and multiplying out denominators we obtain a nonzero vector $(a',b',c') \in (R_2^{(2)}/\mathfrak{p})^3$ such that

$$a'+u_1^{l_k}\cdot b'+u_2^{l_k}\cdot c' = 0$$

in $R_2^{(2)}/\mathfrak{p}$ for every $k \geq 0$. According to (3.5) this shows that the set $S = \{(0,0),(1,0),(0,1)\}$ is indeed nonmixing for α.

In contrast to the connected case, where every mixing algebraic $\mathbb{Z}^d$-action is mixing of every order by [27], all zero entropy algebraic $\mathbb{Z}^d$-actions on zero-dimensional compact abelian groups have nonmixing sets.

Theorem 3.3. *A mixing algebraic $\mathbb{Z}^d$-action α on a totally disconnected compact abelian group X has nonmixing sets* (*and is thus not mixing of every order*) *if and only if it does not have completely positive entropy.*

Proof. Theorem 3.1 shows that α has no nonmixing sets if and only if the same is true for each $\alpha_{R_d/\mathfrak{p}}$, $\mathfrak{p} \in \operatorname{asc}(M)$, where $M = \widehat{X}$ is the dual module of α.

As X is zero-dimensional, every $\mathfrak{p} \in \operatorname{asc}(M)$ contains a rational prime $p = p(\mathfrak{p}) > 0$ by Theorem 2.2 (1). If some $\mathfrak{p} \in \operatorname{asc}(M)$ is principal, then it is of the form $\mathfrak{p} = p(\mathfrak{p})R_d$, $\alpha_{R_d/\mathfrak{p}}$ is the shift action of $\mathbb{Z}^d$ on the full shift space $X_{R_d/\mathfrak{p}} = (\mathbb{Z}/p(\mathfrak{p})\mathbb{Z})^{\mathbb{Z}}$, $h(\alpha_{R_d/\mathfrak{p}}) = \log p(\mathfrak{p}) > 0$, and $\alpha_{R_d/\mathfrak{p}}$ is mixing of every order.

If the ideal $\mathfrak{p} \in \operatorname{asc}(M)$ is nonprincipal, we set $\mathfrak{q} = \{f_{/p(\mathfrak{p})} : f \in \mathfrak{p}\} \subset R_d^{(p(\mathfrak{p}))}$ and observe that $\mathfrak{q} \neq \{0\}$ and $\alpha_{R_d/\mathfrak{p}} = \alpha_{R_d^{(p(\mathfrak{p}))}/\mathfrak{q}}$. Example 3.2 (2) shows that the support $\mathcal{S}(h)$ of every nonzero Laurent polynomial $h \in \mathfrak{q}$ is a nonmixing set for $\alpha_{R_d/\mathfrak{p}} = \alpha_{R_d^{(p(\mathfrak{p}))}/\mathfrak{q}}$ and hence, by Theorem 3.1, for α.

If α has completely positive entropy, then Theorem 2.2 (3) implies that *every* $\mathfrak{p} \in \operatorname{asc}(M)$ is principal, and Theorem 3.1 and the discussion above show that α is mixing of every order. If α does not have completely positive entropy, at least one $\mathfrak{p} \in \operatorname{asc}(M)$ is nonprincipal, and α therefore has nonmixing sets. $\square$

The description of the nonmixing sets of an algebraic $\mathbb{Z}^d$-action α is facilitated by the following theorem of David Masser ([12], [19]).

Theorem 3.4. *Let K be an algebraically closed field of characteristic $p>0$, $r\ge 2$, and let $(x_1,\dots,x_r) \in (K^\times)^r$. The following conditions are equivalent:*

(1) *There exists a nonzero element $(c_1,\dots,c_r)\in K^r$ such that*

$$\sum_{i=1}^{r} c_i x_i^k = 0$$

for infinitely many $k\ge 0$;

(2) *There exists a rational number $s>0$ such that the set $\{x_1^s,\dots,x_r^s\}$ is linearly dependent over the algebraic closure $\overline{F_p}\subset K$ of the prime field $F_p=\mathbb{Z}/p\mathbb{Z}$.*

Corollary 3.5. *Let $\mathfrak{p}\subset R_d$ be a prime ideal containing a rational prime $p>1$, and let $\alpha=\alpha_{R_d/\mathfrak{p}}$ be the algebraic $\mathbb{Z}^d$-action on $X=X_{R_d/\mathfrak{p}}$ defined in Example 2.1 (2). We denote by $K=Q(R_2/\mathfrak{p})\supset R_d/\mathfrak{p}$ the field of fractions of the integral domain $R_d/\mathfrak{p}$, write $\overline{K}$ for its algebraic closure, and set $x_{\mathbf{n}}=u^{\mathbf{n}}+\mathfrak{p}\in R_d/\mathfrak{p}\subset K\subset\overline{K}$ for every $\mathbf{n}\in\mathbb{Z}^d$. If $S\subset\mathbb{Z}^d$ is a nonempty finite set, then the following conditions are equivalent:*

(1) *S is not α-mixing;*

(2) *There exists a rational number $s>0$ such that the set $\{x_{\mathbf{n}}^s:\mathbf{n}\in S\}\subset\overline{K}$ is linearly dependent over $\overline{F_p}\subset\overline{K}$.*

Proof of Corollary 3.5*, given Theorem* 3.4*.* If a nonempty finite subset $S\subset\mathbb{Z}^d$ is not mixing for α, then (3.5) implies that there exist elements $\{a_{\mathbf{n}}:\mathbf{n}\in S\}$ in $R_d/\mathfrak{p}$, not all equal to zero, and infinitely many $k\ge 1$ such that

$$\sum_{\mathbf{n}\in S} u^{k\mathbf{n}}\cdot a_{\mathbf{n}}=0.$$

If we set $x_{\mathbf{n}}=u^{\mathbf{n}}+\mathfrak{p}\in R_d/\mathfrak{p}\subset K$ for every $\mathbf{n}\in S$, we obtain Condition (1) in Theorem 3.4 and hence Condition (2) in our corollary.

Conversely, if $\{x_{\mathbf{n}}^s:\mathbf{n}\in S\}$ is linearly dependent over $\overline{F_p}$ for some rational number $s>0$, then we obtain a nontrivial equation of the form

$$\sum_{\mathbf{n}\in S}\omega_{\mathbf{n}}x_{\mathbf{n}}^s=0$$

with $\omega_{\mathbf{n}}\in\overline{F_p}$ for every $\mathbf{n}\in S$. By Theorem 3.4 there exists a nonzero element $(c_{\mathbf{n}},\mathbf{n}\in S)\in\overline{K}^S$ with

$$\sum_{\mathbf{n}\in S}c_{\mathbf{n}}x_{\mathbf{n}}^k=0$$

for infinitely many $k\ge 0$. Hence there exists a nonzero element $(c_{\mathbf{n}}',\mathbf{n}\in S)\in K^S$ with

$$\sum_{\mathbf{n}\in S}c_{\mathbf{n}}'x_{\mathbf{n}}^k=0$$

for infinitely many $k\ge 0$, and after clearing denominators we obtain a nonzero element $(a_{\mathbf{n}},\mathbf{n}\in S)\in(R_d/\mathfrak{p})^S$ with

$$\sum_{\mathbf{n}\in S}u^{k\mathbf{n}}\cdot a_{\mathbf{n}}=0$$

for infinitely many $k\ge 0$. This shows that the set S is α-nonmixing. □

In order to illustrate the dynamical implications of Corollary 3.5 we return to the Examples 3.2 on page 7.

Examples 3.6. (1) In Example 3.2 (2) we used the fact that $f = 1+u_1+u_2+u_1^2+u_1u_2+u_2^2 \in R_2^{(2)}$ is irreducible over F_2, but not over $\bar{F_2}$. We define $\mathfrak{p} = (f) \subset R_2^{(2)}$ as in that example, set $R_2^{(4)} = F_4[u_1^{\pm 1}, u_2^{\pm 1}]$ and put $\mathfrak{q} = (1+\omega u_1+\omega^2 u_2) \subset R_2^{(4)}$. If $\iota\colon R_2^{(2)} \longrightarrow R_2^{(4)}$ is the inclusion map and $\pi\colon R_2^{(4)} \mapsto R_2^{(4)}/\mathfrak{q}$ the quotient map, then $\ker(\pi\circ\iota) = \mathfrak{p}$, and the map $\pi\circ\iota$ induces an embedding of the field of fractions $K = Q(R_2^{(2)}/\mathfrak{p})$ in the field of fractions $L = Q(R_2^{(4)}/\mathfrak{p})$. As we saw in Example 3.2 (2),

$$1+\omega u_1^{2^{2k}} + \omega^2 u_2^{2^{2k}} = 0$$

in L for every $k \geq 0$, i.e the vector $(1,\omega,\omega^2) \in L^3$ is orthogonal to $(1, u_1^{2^{2k}}, u_2^{2^{2k}}) \in K \subset L$ for every $k \geq 0$. Hence there exists an nonzero $\mathbf{v} \in K^3$ which is orthogonal to every $(1, u_1^{2^{2k}}, u_2^{2^{2k}})$, and $\mathbf{v} = (u_1+u_2, 1+u_2, 1+u_1)$ corresponds to an explicit choice of such a vector.

The injection $\hat{\eta}\colon R_2^{(2)}/\mathfrak{p} \longrightarrow R_2^{(4)}/\mathfrak{p}$ induced by the map $\pi\circ\iota\colon R_2^{(2)} \longrightarrow R_2^{(4)}/\mathfrak{q}$ above embeds the R_2-module $M = R_2^{(2)}/\mathfrak{p}$ as a submodule of index 2 in the R_2-module $N = R_2^{(4)}/\mathfrak{q}$. The corresponding dual factor map $\eta\colon X_N \longrightarrow X_M$ sends $\alpha = \alpha_M$ to $\beta = \alpha_N$ and is two-to-one. We shall return to these two algebraic $\mathbb{Z}^2$-actions in Example 4.9 on page 16.

(2) In the notation of Example 3.2 (4) we set $\mathfrak{p} = (f) \subset R_2^{(2)}$, $\mathfrak{q} = (g) \subset R_2^{(2)}$, $\alpha = \alpha_{R_2^{(2)}/\mathfrak{p}}$, $X = X_{R_2^{(2)}/\mathfrak{p}} = \widehat{R_2^{(2)}/\mathfrak{p}}$, $\beta = \alpha_{R_2^{(2)}/\mathfrak{q}}$ and $Y = X_{R_2^{(2)}/\mathfrak{q}} = \widehat{R_2^{(2)}/\mathfrak{q}}$. We put $\Gamma = 3\mathbb{Z}^3$ and write $\pi_\Gamma\colon Y \longrightarrow (\mathbb{Z}/2\mathbb{Z})^\Gamma$ for the projection onto the coordinates in Γ. By identifying Γ with $\mathbb{Z}^2$ we view $\pi_\Gamma(Y)$ as a closed shift-invariant subgroup of $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$, and a little calculation shows that $\pi_\Gamma(Y) = X$ and that $\pi_\Gamma\colon Y \longrightarrow X$ is two-to-one.

The set $S = \{(0,0),(1,0),(0,1)\} \subset \mathbb{Z}^2$ is obviously β-nonmixing. We write $\beta_\Gamma\colon \mathbf{n} \mapsto \beta^{3\mathbf{n}}$ for the Γ-action obtained from β by restriction and observe that the two-to-one factor map $\pi_\Gamma\colon Y \longrightarrow X$ sends β_Γ to α. Furthermore, the set S is also β_Γ-nonmixing, and this property of S survives under the factor map $\pi_\Gamma\colon Y \longrightarrow X$ (this is the essence of the calculation in Example 3.2 (4)).

If an algebraic $\mathbb{Z}^d$-action α is r-mixing, then every set $S \subset \mathbb{Z}^d$ with cardinality $|S| \leq r$ is obviously mixing. The converse is far from obvious: if α is not mixing of order $r \geq 2$, and if r is the smallest integer with this property, does there exist a nonmixing set $S \subset \mathbb{Z}^d$ of size r? Remarkably, this turns out to be the case, as a consequence of a second theorem by David Masser.

Theorem 3.7 ([20]). *Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X, and let $r \geq 2$. If every subset $S \subset \mathbb{Z}^d$ of cardinality r is mixing, then α is r-mixing.*

In order to explain the connection between Theorem 3.7 and an appropriate statement about additive relations in fields in the spirit of Theorem 3.4 we need a definition.

Definition 3.8. Let G be a multiplicative abelian group and n a positive integer. An infinite subset $\Xi \subset G^n$ is *broad* if it satisfies the following conditions.

(1) If $g \in G$ and $1 \leq i \leq n$, then there are at most finitely many $(\xi_1,\dots,\xi_n) \in \Xi$ with $\xi_i = g$;
(2) If $n \geq 2$, $g \in G$ and $1 \leq i < j \leq n$, then there are at most finitely many $(\xi_1,\dots,\xi_n) \in \Xi$ with $\xi_i/\xi_j = g$.

Theorem 3.9 ([20]). *Let K be a field of characteristic $p > 1$ and $G \subset K^\times$ a finitely generated subgroup. Suppose that $n \geq 1$, and that the equation*

$$a_1x_1 + \cdots + a_nx_n = 1 \tag{3.9}$$

has a broad set of solutions $(x_1,\dots,x_n)\in G^n$ *for some* $(a_1,\dots,a_n)\in (K^\times)^n$*. Then there exist a positive integer* $m\le n$ *and elements* $(b_1,\dots,b_m)\in (K^\times)^m$*,* $(g_1,\dots,g_m)\in G^m$*, with the following properties.*

(1) $g_i\ne 1$ *for* $i=1,\dots,m$*;*
(2) $g_i/g_j\ne 1$ *for* $1\le i<j\le m$*;*
(3) *There exist infinitely many* $k\ge 1$ *with*

$$b_1g_1^k+\dots+b_mg_m^k=1. \tag{3.10}$$

Proof of Theorem 3.7*, given Theorem* 3.9*.* The translation of Theorem 3.7 into Theorem 3.9 works exactly as in Corollary 3.5. If α is an algebraic $\mathbb{Z}^d$-action on a compact abelian group X which is not mixing of order $r\ge 2$, and if r is the smallest integer with this property, then Theorem 3.1 guarantees the existence of a prime ideal $\mathfrak{p}$ associated with the dual module $M=\widehat{X}$ of α such that $\alpha_{R_d/\mathfrak{p}}$ is not r-mixing.

If $r=2$, Theorem 2.2 (2) implies that $u^{\mathbf{n}}-1\in\mathfrak{p}$ for some nonzero $\mathbf{n}\in\mathbb{Z}^d$. Hence $u^{k\mathbf{n}}-1\in\mathfrak{p}$ and $a-u^{k\mathbf{n}}\cdot a=0$ for every $k\ge 0$ and $a\in R_d/\mathfrak{p}$, and (3.5) shows that the set $S=\{\mathbf{0},\mathbf{n}\}\subset\mathbb{Z}^d$ is nonmixing for $\alpha_{R_d/\mathfrak{p}}$ and hence, by Theorem 3.1, for α.

If $r>2$ we denote by K the field of fractions of the integral domain $R_d/\mathfrak{p}$, embed $R_d/\mathfrak{p}$ in K in the obvious manner, and write $G\subset K^\times$ for the multiplicative group generated by $\{x_{\mathbf{n}}=u^{\mathbf{n}}+\mathfrak{p}:\mathbf{n}\in\mathbb{Z}^d\}$. Since $\alpha_{R_d/\mathfrak{p}}$ is mixing, $G\cong\mathbb{Z}^d$ by Theorem 2.2 (2). Equation (3.3) shows that there exist elements $a_1,\dots,a_r\in R_d/\mathfrak{p}$, not all equal to zero, and a sequence $((\mathbf{n}_k^{(1)},\dots,\mathbf{n}_k^{(r)}),k\ge 1)$ in $(\mathbb{Z}^d)^r$ such that $\lim_{k\to\infty}\|\mathbf{n}_k^{(i)}-\mathbf{n}_k^{(j)}\|=\infty$ for all i,j with $1\le i<j\le r$, and

$$u^{\mathbf{n}_k^{(1)}}\cdot a_1+\dots+u^{\mathbf{n}_k^{(r)}}\cdot a_r=0$$

for every $k\ge 1$. The minimality of r implies that the a_i are all nonzero, and we may obviously assume in addition that $\mathbf{n}_k^{(r)}=\mathbf{0}$ for every $k\ge 1$.

We set $\xi_k=(\xi_k^{(1)},\dots,\xi_k^{(r-1)})=(u^{\mathbf{n}_k^{(1)}}+\mathfrak{p},\dots,u^{\mathbf{n}_k^{(r-1)}}+\mathfrak{p})\in G^{r-1}$ for every $k\ge 1$. Then $\Xi=\{\xi_k: k\ge 1\}$ is a broad set of solutions of the equation

$$\frac{a_1}{a_r}x_1+\dots+\frac{a_{r-1}}{a_r}x_{r-1}=1.$$

Theorem 3.9 yields a positive integer $m\le r-1$ and elements $(b_1,\dots,b_m)\in(K^\times)^m$, $(g_1,\dots,g_m)\in G^m$ with the properties listed there, such that

$$b_1g_1^k+\dots+b_mg_m^k=1$$

for infinitely many $k\ge 1$. Since each $g_i=u^{\mathbf{t}_i}+\mathfrak{p}$ for some unique nonzero $\mathbf{t}_i\in\mathbb{Z}^d$ we obtain after clearing denominators that

$$u^{k\mathbf{t}_1}\cdot b_1'+\dots+u^{k\mathbf{t}_m}\cdot b_m'=b_{m+1}'$$

for some nonzero elements $b_i'\in R_d/\mathfrak{p}$ and infinitely many $k\ge 1$. An application of (3.5) shows that the set $S=\{\mathbf{0},\mathbf{t}_1,\dots,\mathbf{t}_m\}$ is nonmixing for $\alpha_{R_d/\mathfrak{p}}$ and hence, by Theorem 3.1, for α. The minimality of r implies that $|S|=m+1=r$. This completes the proof of the theorem. □

In order to appreciate the difficulty in proving Theorem 3.7 one should once again consider Ledrappier's Example 3.2 (1). As we saw there, the set $S=\{(0,0),(1,0),(0,1)\}$ is nonmixing (and obviously minimal) for the $\mathbb{Z}^2$-action $\alpha=\alpha_{R_2^{(2)}/(f)}$ defined in that example. However, for every

$k_0,k_1,k_2,k_3 \geq 0$ with $2^{k_0} > 2^{k_1}+2^{k_2}+2^{k_3}$, say, the set

$$\begin{aligned} S_{k_0,k_1,k_2,k_3} = \{&(2^{k_1},0),(0,2^{k_1}),(2^{k_0}-2^{k_2},0),(2^{k_0}-2^{k_2},2^{k_2}),\\ &(0,2^{k_0}-2^{k_3}),(2^{k_3},2^{k_0}-2^{k_3})\} \end{aligned}$$

is also minimal nonmixing: it is the support of the polynomial

$$\begin{aligned} g_{k_0,k_1,k_2,k_3} = (1+u_1+u_2)^{2^{k_0}} + (1+u_1+u_2)^{2^{k_1}} + u_1^{2^{k_0}-2^{k_2}}(1+u_1+u_2)^{2^{k_2}}\\ + u_2^{2^{k_0}-2^{k_3}}(1+u_1+u_2)^{2^{k_3}} \in \mathfrak{p}. \end{aligned}$$

By choosing appropriate increasing sequences $k_i^{(n)}$, $n \geq 1$, $i=0,\dots,3$, we obtain minimal nonmixing sets $S_n = S_{k_0^{(n)},k_1^{(n)},k_2^{(n)},k_3^{(n)}}$, $n \geq 1$, of varying shapes without any resemblance to linear multiples of a single nonmixing set $S' \subset \mathbb{Z}^2$. Nevertheless one can extract sufficient information from any such sequence to obtain a nonmixing set for α; for details we refer to [20].

Theorem 3.7 reduces the problem of determining the order of mixing to finding nonmixing sets of smallest cardinality. However, even with Corollary 3.5 at hand, the latter problem remains nontrivial: I am not aware of any good general algorithm for determining polynomials with minimal support in a given ideal.

4. Isomorphism rigidity of algebraic $\mathbb{Z}^d$-actions: the irreducible case

In this section we turn to a problem of an apparently quite unrelated nature from that of the last section. Every algebraic $\mathbb{Z}^d$-action α with completely positive entropy is measurably conjugate to a Bernoulli shift (cf. [23]). Since entropy is a complete invariant for measurable conjugacy of Bernoulli shifts by [21], α is measurably conjugate to the $\mathbb{Z}^d$-action

$$\alpha^A \colon \mathbf{n} \mapsto \alpha^{A\mathbf{n}}$$

for every $A \in \mathrm{GL}(d,\mathbb{Z})$, since the entropies of all these actions coincide. In general, however, α and α^A are not topologically conjugate.

Every algebraic $\mathbb{Z}^d$-action α with positive entropy has Bernoulli factors by [17] and [23], and two such actions may again be measurably conjugate without being topologically conjugate. For zero entropy actions, however, there is some evidence for a very strong form of isomorphism rigidity. In order to formulate this property we introduce a definition.

Definition 4.1. Let α and β be algebraic $\mathbb{Z}^d$-actions on compact abelian groups X and Y, respectively. The actions α and β (or (X,α) and (Y,β)) are *finitely* (*algebraically*) *equivalent* if each of them is an algebraic factor of the other one with a finite-to-one factor map.

A map $\phi \colon X \longrightarrow Y$ is *affine* if it is of the form $\phi(x) = \psi(x) + y$ for every $x \in X$, where $\psi \colon X \longrightarrow Y$ is a continuous surjective group homomorphism and $y \in Y$. If there exists an affine map $\phi \colon X \longrightarrow Y$ satisfying (2.1), then β is obviously an algebraic factor of α.

We say that *isomorphism rigidity* holds for a class of algebraic $\mathbb{Z}^d$-actions if any measurable conjugacy between two actions in this class coincides *a.e.* with an affine map. Let us begin with the class of irreducible $\mathbb{Z}^d$-actions to illustrate a much more general phenomenon.

Definition 4.2. An algebraic $\mathbb{Z}^d$-action α on a compact abelian group X is *irreducible* if every closed α-invariant subgroup $Y \subsetneq X$ is finite.

Irreducible $\mathbb{Z}^d$-actions were called *almost minimal* in [25].

Proposition 4.3. *Let α be an irreducible and ergodic algebraic $\mathbb{Z}^d$-action on a compact abelian group X, and let β be an algebraic $\mathbb{Z}^d$-action on a compact abelian group $Y \neq \{0\}$ such that (Y,β) is an algebraic factor of (X,α). Then the factor map is finite-to-one, and β is irreducible, ergodic and finitely equivalent to α. Furthermore there exists a unique prime ideal $\mathfrak{p} \subset R_d$ with the following properties.*

1. *$\alpha_{R_d/\mathfrak{p}}$ is ergodic (and hence $R_d/\mathfrak{p}$ is infinite);*
2. *For every ideal $I \supsetneq \mathfrak{p}$ in R_d, R_d/I is finite;*
3. *α is finitely equivalent to $\alpha_{R_d/\mathfrak{p}}$.*

Conversely, if $\mathfrak{p} \subset R_d$ is a prime ideal satisfying Condition (2) above, then the $\mathbb{Z}^d$-action $\alpha = \alpha_{R_d/\mathfrak{p}}$ on the group $X_{R_d/\mathfrak{p}}$ is irreducible.

Proof. Let $\phi\colon X \longrightarrow Y$ be an algebraic factor map from (X,α) to (Y,β). The kernel $K = \ker\phi$ is an α-invariant closed subgroup of X. As $Y \neq \{0\}$ by assumption, K is a proper α-invariant subgroup and thus finite by irreducibility.

Let Z be a proper closed β-invariant subgroup of Y. The subgroup $\phi^{-1}(Z) \subset X$ is finite by irreducibility. This shows that $Z = \phi(\phi^{-1}(Z))$ is finite. The (obviously ergodic) action β is therefore irreducible.

The ergodicity of α also implies that every nonzero submodule $N \subset M$ of the dual module $M = \widehat{X}$ of α is infinite: otherwise $Z = \widehat{N} = X/N^{\perp}$ would be a finite quotient of X by an α-invariant subgroup, contrary to ergodicity. As the inclusion $N \subset M$ is dual to a factor map ψ from (X,α) to (X_N,α_N), the beginning of this proof shows that α_N is irreducible and $|M/N| = |\ker\psi|$ is finite. In particular, if $\mathfrak{p}$ is a prime ideal associated with M, and if $a \in M$ satisfies that $\mathrm{ann}(a) = \mathfrak{p}$ and hence $N = R_d \cdot a \cong R_d/\mathfrak{p}$, then N is infinite, M/N is finite and $\alpha_N = \alpha_{R_d/\mathfrak{p}}$ is ergodic and irreducible.

If $I \supsetneq \mathfrak{p}$ is an ideal, then $N' = I \cdot a \cong I/\mathfrak{p}$ is a submodule of N and hence — again by irreducibility — of finite index in N. It follows that R_d/I is finite, as claimed in (2).

If $\mathfrak{q} \neq \mathfrak{p}$ is a second prime ideal associated with M then $\mathfrak{q} = \mathrm{ann}(b)$ for some $b \in M \smallsetminus N$. Every nonzero $b' \in N' = R_d \cdot b$ has $\mathfrak{q}$ as its annihilator. However, since $R_d/\mathfrak{q} \cong N'$ is infinite by ergodicity and $N'/N = N'/(N \cap N')$ is finite, there exists an $h \in R_d \smallsetminus \mathfrak{q}$ with $h \cdot b \in N$ and hence $\mathrm{ann}(h \cdot b) = \mathfrak{p}$. This contradiction implies that $\mathfrak{p}$ is the only prime ideal associated with M.

In order to complete the proof that α and $\alpha_N = \alpha_{R_d/\mathfrak{p}}$ are finitely equivalent we have to find a (necessarily finite-to-one) algebraic factor map $\phi'\colon (X_N,\alpha_N) \longrightarrow (X,\alpha)$. As in the preceding paragraph we note that there exists, for every $b \in M \smallsetminus N$, an element $h_b \in R_d \smallsetminus \mathfrak{p}$ with $h_b \cdot b \in N$. The polynomial

$$h = \prod_{b \in M \smallsetminus N} h_b \in R_d \smallsetminus \mathfrak{p}$$

satisfies that $h \cdot M \subset N$. The map $m_h\colon M \longrightarrow N$ consisting of multiplication by h is injective by Footnote 5 on page 4, and the surjective homomorphism $\phi'\colon X_N \longrightarrow X$ dual to m_h is an algebraic factor map from (X_N,α_N) to (X,α). This proves (3).

We return to the first assertion of this proposition. We have proved that β is irreducible and hence finitely equivalent to $\alpha_{R_d/\mathfrak{p}}$ for some prime ideal $\mathfrak{p} \subset R_d$ satisfying the conditions (1) and (2). The factor map $\phi\colon (X,\alpha) \longrightarrow (Y,\beta)$ is dual to an embedding $\hat{\phi}\colon \widehat{Y} \longrightarrow M$. Since $\mathfrak{p}$ is the only prime ideal associated with M, $\mathfrak{p}$ is also associated with $\widehat{Y}$, and β is finitely equivalent to $\alpha_{R_d/\mathfrak{p}}$ and hence to α.

The final assertion has already been verified in the course of this proof. $\square$

Irreducibility is an extremely strong hypothesis: if α is mixing it implies that $\alpha^{\mathbf{n}}$ is Bernoulli with finite entropy for every nonzero $\mathbf{n} \in \mathbb{Z}^d$, and hence, if $d > 1$, that α has zero entropy. If β is a second irreducible and mixing algebraic $\mathbb{Z}^d$-action on a compact abelian group Y such that $h(\alpha^{\mathbf{n}}) = h(\beta^{\mathbf{n}})$

for every $\mathbf{n} \in \mathbb{Z}^d$, then $\alpha^{\mathbf{n}}$ is measurably conjugate to $\beta^{\mathbf{n}}$ for every $\mathbf{n} \in \mathbb{Z}^d$. However, if $d > 1$, then the actions α and β are generally not measurably conjugate, as the following theorem and the examples below show.

Theorem 4.4 (Isomorphism rigidity for irreducible $\mathbb{Z}^d$-actions)**.** *Let $d > 1$, and let α_1 and α_2 be irreducible and mixing algebraic $\mathbb{Z}^d$-actions on compact abelian groups X_1 and X_2, respectively. If $\phi\colon X_1 \longrightarrow X_2$ is a measurable conjugacy of α_1 and α_2, then ϕ is λ_{X_1}-a.e. equal to an affine map. In particular, measurable conjugacy implies algebraic conjugacy.*

Theorem 4.4 is a combination of two theorems in [9] and [13], respectively, and follows from a result on invariant measures of algebraic $\mathbb{Z}^d$-actions with $d \geq 2$ whose scope is still something of a mystery. We state a very special case which will be sufficient for proving Theorem 4.4; possible ramifications of Theorem 4.5 will be discussed in Section 5.

Theorem 4.5. *Let $d \geq 2$, and let α_1 and α_2 be irreducible and mixing algebraic $\mathbb{Z}^d$-actions on compact abelian groups X_1 and X_2 with normalized Haar measures λ_{X_1} and λ_{X_2}, respectively. We write $\alpha = \alpha_1 \times \alpha_2$ for the product-$\mathbb{Z}^d$-action on $X = X_1 \times X_2$ and assume that μ is an α-invariant probability measure on X with the following property: if $\pi_i\colon X \longrightarrow X_i$ denotes the i-th coordinate projection, then $\mu\pi_i^{-1} = \lambda_{X_i}$, and π_i is a measurable conjugacy of the $\mathbb{Z}^d$-actions (X,μ,α) and $(X_i,\lambda_{X_i},\alpha_i)$.*

Then there exists a closed α-invariant subgroup $Y \subset X$ such that μ is a translate of the Haar measure λ_Y.

Proof. Since the $\mathbb{Z}^d$-actions α_i are irreducible, Proposition 4.3 shows that the groups X_i have to be either zero-dimensional or connected (depending on whether or not the prime ideal $\mathfrak{p} \subset R_d$ appearing there contains a nonzero constant). If X_1 and X_2 are finite-dimensional tori, Theorem 4.5 follows from Corollary 5.2′ in [10, Corrections] (cf. [9, Theorem 5.1]), and this result can be extended to irreducible $\mathbb{Z}^d$-actions on compact connected abelian groups without much difficulty, using the structure theorems about irreducible $\mathbb{Z}^d$-actions in [24] and [7]. If X_1 and X_2 are zero-dimensional, Theorem 4.5 follows from the main result in [13].

The case where one of the groups is connected and the other is zero-dimensional is impossible: if X_1 is connected and X_2 zero-dimensional, then the main result in [27] implies that α_1 has no nonmixing sets, whereas α_2 has nonmixing sets by Theorem 3.3, since it has entropy zero. Since the hypotheses of Theorem 4.5 imply that α_1 and α_2 are measurably conjugate we obtain a contradiction. □

Proof of Theorem 4.4, *given Theorem* 4.5. Suppose that $\phi\colon X_1 \longrightarrow X_2$ is a measurable conjugacy of α_1 and α_2. We set $X = X_1 \times X_2$, consider the product $\mathbb{Z}^d$-action $\alpha = \alpha_1 \times \alpha_2$ on $X = X_1 \times X_2$, and denote by μ the unique α-invariant probability measure on the graph $\Gamma(\phi) = \{(x,\phi(x)) : x \in X_1\} \subset X$ which satisfies that $\mu\pi_i^{-1} = \lambda_{X_i}$ for $i = 1,2$, where $\pi_i\colon X \longrightarrow X_i$ are the coordinate projections. Since all the hypotheses of Theorem 4.4 are satisfied we conclude that μ is a translate of the Haar measure of a closed subgroup of X and hence that ϕ is *a.e.* equal to an affine map. □

The papers [9] and [13] contain many examples of pairs of irreducible algebraic $\mathbb{Z}^d$ which look very similar, but which can can be shown to be measurably nonconjugate by Theorem 4.4. Here we restrict ourselves to some zero-dimensional examples. We start with two definitions.

Definition 4.6. Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X. The *algebraic centralizer* $\mathcal{C}_0(\alpha)$ is the group of all continuous group automorphisms of X which commute with α.

The *affine centralizer* $\mathcal{C}_{\mathrm{aff}}(\alpha)$ is the group of all affine bijections of X which commute with α, and is of the form $\mathcal{C}_{\mathrm{aff}}(\alpha) = \mathcal{C}_0(\alpha) \times \mathrm{Fix}(\alpha)$, where $\mathrm{Fix}(\alpha)$ is the group of fixed points of α.

The *measurable centralizer* $\mathcal{C}_{\lambda_X}(\alpha)$ is the group of all Haar measure preserving bijective Borel maps $\phi\colon X \longrightarrow X$ which commute with α λ_X*-a.e.*

Definition 4.7. Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X. The dual group $\widehat{X}$ of X is *cyclic* under the dual action $\hat{\alpha}$ of α (or α has *cyclic dual*) if there exists a character $a \in \widehat{X}$ such that $\widehat{X}$ — as a group — is generated by the set $\{\hat{\alpha}^{\mathbf{n}} a : \mathbf{n} \in \mathbb{Z}\}$.

Example 4.8 (The trivial centralizer of Ledrappier's example)**.** In Example 3.2 (1) we considered the $\mathbb{Z}^2$-action $\alpha = \alpha_{R_2^{(2)}/(f)}$ with $f = 1 + u_1 + u_2 \in R_2^{(2)}$. We claim that

$$\mathcal{C}_0(\alpha) = \mathcal{C}_{\text{aff}}(\alpha) = \mathcal{C}_{\lambda_X}(\alpha) = \{\alpha^{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^2\}.$$

Since 0 is the only fixed point of α, $\mathcal{C}_0(\alpha) = \mathcal{C}_{\text{aff}}(\alpha) = \mathcal{C}_{\lambda_X}(\alpha)$ by Theorem 4.4. As α has cyclic dual, every automorphism $\beta \in \mathcal{C}_0(\alpha)$ is completely determined by the element $g + (f) = \hat{\beta}(1 + (f)) \in \widehat{X} = R_2^{(2)}/(f)$, where β is the automorphism of $\widehat{X}$ dual to β. As β is a group automorphism, its kernel is trivial, which translates into the statement that the varieties

$$\begin{aligned} V(f) &= \{(c_1, c_2) \in (\bar{F}_2)^\times \times (\bar{F}_2)^\times : f(c_1, c_2) = 0\} \\ &= \{(c_1, 1 + c_1) : c_1 \in (\bar{F}_2)^\times, 1 + c_1 \in (\bar{F}_2)^\times\}, \\ V(g) &= \{(c_1, c_2) \in (\bar{F}_2)^\times \times (\bar{F}_2)^\times : g(c_1, c_2) = 0\} \end{aligned}$$

of f and g do not intersect (this statement is meaningful in spite of the fact that g is determined only up to addition of an element in (f)). After modifying g by an element of (f) we may assume that

$$g(u_1, u_2) = \sum_{\mathbf{m} = (m_1, m_2) \in F} u_1^{m_1} (1 + u_1)^{m_2} = h(u_1),$$

say, for some finite subset $F \subset \mathbb{Z}^2$. Our hypothesis on the intersection of varieties guarantees that $h(u_1) \neq 0$ for every $u_1 \in (\bar{F}_2)^\times$, and hence that $h(u_1) = u_1^{k_1}(1 + u_1)^{k_2}$ and $g = u^{\mathbf{k}} \pmod{(f)}$ for some $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2$. This proves that $\beta = \alpha^{\mathbf{k}}$ for some $\mathbf{k} \in \mathbb{Z}^2$.

Example 4.9. Consider the $\mathbb{Z}^2$-action $\alpha = \alpha_M$ and α_N with $M = R_2^{(2)}/\mathfrak{p}$ and $N = R_2^{(4)}/\mathfrak{q}$ in Example 3.6 (1), where $\mathfrak{p} = (1 + u_1 + u_2 + u_1^2 + u_1 u_2 + u_2^2) \subset R_2^{(2)}$ and $\mathfrak{q} = (1 + \omega u_1 + \omega^2 u_2) \subset R_2^{(4)}$. There we found a two-to-one algebraic factor map from $(X', \alpha') = (X_N, \alpha_N)$ to $(X, \alpha) = (X_M, \alpha_M)$. However, the dual module $\widehat{X} = M$ is obviously cyclic in the sense of Definition 4.7, whereas the module $\widehat{X'} = N$ is not. Theorem 4.4 shows that the finitely equivalent actions α and α' are not measurably conjugate.

By exploiting the fact that the polynomials $f' = 1 + u_1^2 + u_2 + u_1 u_2 + u_2^2$ and $f'' = 1 + u_1 + u_1^2 + u_2 + u_2^2$ are irreducible in $R_2^{(2)}$, but not in $R_2^{(4)}$, one can construct further examples of this kind.

Example 4.10 (Nonconjugacy of $\mathbb{Z}^2$-actions with positive entropy)**.** Let

$$\begin{aligned} f_1 &= 1 + u_1 + u_1^2 + u_1 u_2 + u_2^2, \\ f_2 &= 1 + u_1^2 + u_2 + u_1 u_2 + u_2^2, \\ f_3 &= 1 + u_1 + u_1^2 + u_2 + u_2^2, \\ f_4 &= 1 + u_1 + u_1^2 + u_2 + u_1 u_2 + u_2^2, \end{aligned}$$

in R_2, put $\mathfrak{p}_i = (2, f_i) \subset R_2$, $J_i = (4, 2f_i) \subset R_2$, $M_i = R_2/J_i$, $N_i = R_2/\mathfrak{p}_i$, and define the algebraic $\mathbb{Z}^2$-actions $\alpha_i = \alpha_{M_i}$ on $X_i = X_{M_i}$ and $\beta_i = \alpha_{N_i}$ on $Y_i = X_{N_i}$ as in Example 2.1 (2). For every $i = 1, \dots, 4$, the prime ideals associated with the module M_i are $(2) = 2R_2$ and $\mathfrak{p}_i$, and the inclusion of $2M_i \cong N_i$ in M_i is dual to an algebraic factor map $\phi_i\colon X_i \longrightarrow Y_i$ from (X_i, α_i) to (Y_i, β_i). Since

$\ker \phi_i \cong \widehat{R_2/2R_2} = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$ and the actions β_i have zero entropy, the Pinsker algebra $\pi(\alpha_i)$ of α_i is the sigma-algebra $\mathcal{B}_{X_i/\ker\phi_i}$ of $\ker\phi_i$-invariant Borel sets in X_i. In other words, the $\mathbb{Z}^2$-action induced by α_i on the Pinsker algebra $\pi(\alpha_i)$ is measurably conjugate to β_i.

Since any measurable conjugacy of α_i and α_j would map $\pi(\alpha_i)$ to $\pi(\alpha_j)$ and induce a conjugacy of β_i and β_j, Theorem 4.4 implies that α_i and α_j are measurably nonconjugate for $1 \leq i < j \leq 4$.

5. ISOMORPHISM RIGIDITY: THE GENERAL CASE

In Section 4 we investigated the isomorphism problem for irreducible algebraic $\mathbb{Z}^d$-actions. Although the discussion below shows that one can relax the hypothesis of irreducibility in Theorem 4.4 to some extent, the methods currently do not extend significantly beyond the class of expansive and mixing algebraic $\mathbb{Z}^d$-actions α on compact abelian groups X with the property that $h(\alpha^{\mathbf{n}}) < \infty$ for every $\mathbf{n} \in \mathbb{Z}^d$ (i.e. the *rank one case* in the terminology of [5]). For example, if $\mathfrak{p}, \mathfrak{q} \subset R_3$ are nonprincipal prime ideals with 2 generators such that the zero-entropy $\mathbb{Z}^3$-actions $\alpha = \alpha_{R_3/\mathfrak{p}}$ and $\beta = \alpha_{R_3/\mathfrak{q}}$ are measurably conjugate (cf. Theorem 2.2 (3)), and if the groups $X = X_{R_d/\mathfrak{p}}$ and $Y = X_{R_d/\mathfrak{q}}$ are connected, there are at present no general results about isomorphism rigidity of such actions. As far as I know, the following 'cautious conjecture' from [26] may have a positive answer under the hypothesis that the groups X and Y are connected (it is now known to be wrong without this hypothesis by [1] and [2]).

Conjecture 5.1. *Let $d > 1$, and let α and β be expansive and mixing algebraic $\mathbb{Z}^d$-actions on compact connected abelian groups X and Y, respectively. If α and β have zero entropy, then any measurable conjugacy between them is a.e. equal to an affine map.*

Conjecture 5.1 would be implied by a positive answer to the following problem.

Problem 5.2. Let $d \geq 2$, and let α_1 and α_2 be expansive and mixing algebraic $\mathbb{Z}^d$-actions on compact abelian groups X_1 and X_2 with normalized Haar measures λ_{X_1} and λ_{X_2}, respectively. We write $\alpha = \alpha_1 \times \alpha_2$ for the product-$\mathbb{Z}^d$-action on $X = X_1 \times X_2$ and assume that μ is an α-invariant probability measure on X with the following property: if $\pi_i \colon X \longrightarrow X_i$ denotes the i-th coordinate projection, then $\mu\pi_i^{-1} = \lambda_{X_i}$, and π_i is a measurable conjugacy of the $\mathbb{Z}^d$-actions α on (X, μ) and α_i on (X_i, λ_{X_i})?

Does there exists a closed α-invariant subgroup $Y \subset X$ such that μ is a translate of the Haar measure λ_Y.

Theorem 4.5 is, of course, a special case of this problem, which is in turn part of a much more general quest to determine all invariant and ergodic probability measures of a zero entropy mixing algebraic $\mathbb{Z}^d$-action α with $d \geq 2$ (where the mixing hypothesis is imposed only to ensure that there is no single group automorphism β such that $\alpha^{\mathbf{n}}$ is a power of β for all $\mathbf{n}$ in some subgroup of finite index in $\mathbb{Z}^d$). The first instance of this problem is due to Furstenberg (cf. [8]): *Is every nonatomic probability measure μ on $\mathbb{T}$ which is simultaneously invariant under multiplication by 2 and by 3 equal to Lebesgue measure?* In spite of some remarkable progress due to Rudolph in [22], who proved that any such measure with positive entropy under either of these multiplications has to be equal to $\lambda_{\mathbb{T}}$, Furstenberg's original question is still open, and several ingenious proofs by Host and others depend in a very crucial way on positive entropy. For extensions of Rudolph's results to commuting automorphisms of finite-dimensional tori or solenoids we refer to the paper by Katok and Spatzier [10] and to recent work in progress by Einsiedler and Lindenstrauss [6], which contains the currently most general statement about invariant probability measures for irreducible and mixing algebraic $\mathbb{Z}^d$-actions on compact connected abelian groups.

Theorem 5.3. *Let $d \geq 2$, and let α be an irreducible and mixing algebraic $\mathbb{Z}^d$-action on a finite-dimensional torus or solenoid X. If μ is an α-invariant and ergodic probability measure on X which has positive entropy under some $\alpha^{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^d$, then there exists a finite index subgroup $\Lambda \subset \mathbb{Z}^d$ with the following properties.*

(1) *Let $\mathbf{n}_1, \dots, \mathbf{n}_k \in \mathbb{Z}^d$ be a complete set of representatives of $\mathbb{Z}^d/\Lambda$, let α^Λ be the restriction of α to Λ, and let $\mu = \frac{1}{k}\sum_{i=1}^k \mu_i$ be the α^Λ-ergodic decomposition of μ. There exists an infinite closed α^Λ-invariant subgroup $Y \subset X$ such that each μ_i is invariant under translation by the subgroup $Y_i = \alpha^{\mathbf{n}_i}(Y)$.*

(2) *For every $i = 1, \dots, k$, the measure μ_i and the Λ-action α^Λ descend naturally to the factor X/Y_i, and every $\alpha^{\mathbf{n}}$, $\mathbf{n} \in \Lambda$, has zero entropy on X/Y_i with respect to μ_i.*

Although much more is known about isomorphism rigidity of algebraic $\mathbb{Z}^d$-actions on zero-dimensional compact abelian groups than in the connected case (cf. Section 6), the problem of describing the invariant probability measures of even the simplest examples is in no better state than in the connected case. Here are two unresolved questions about Ledrappier's Example 3.2 (1).

Problem 5.4. Let $\alpha = \alpha_{R_2^{(2)}/(1+u_1+u_2)}$ be the shift-action on the group $X = X_{R_2^{(2)}/(1+u_1+u_2)}$ in Example 3.2 (1).

(1) If μ is an α-invariant probability measure on X with full support (i.e. with $\mu(\mathcal{O}) > 0$ for every nonempty open subset $\mathcal{O} \subset X$), is $\mu = \lambda_X$?

(2) If μ is a nonatomic α-invariant probability measure on X which is ergodic under some $\alpha^{\mathbf{n}}$, is $\mu = \lambda_X$?

6. Isomorphism rigidity: the disconnected case

This chapter is devoted to isomorphism rigidity results (and counterexamples) for expansive and mixing algebraic $\mathbb{Z}^d$-actions on zero-dimensional compact abelian groups. The exposition follows [2] and [3].

6.1. **Measurable polynomials.**

Definition 6.1. Let X, Y be compact abelian groups, and let $U(X,Y)$ be the group of all λ_X-equivalence classes of Borel maps $f\colon X \longrightarrow Y$, furnished with pointwise addition as composition and the topology of convergence in Haar measure. For every $x \in X$ we denote by $\partial_x\colon U(X,Y) \longrightarrow U(X,Y)$ the continuous map defined by

$$\partial_x(f)(x') = f(x+x') - f(x')$$

for every $x' \in X$ and $f \in U(X,Y)$, and we set

$$\partial_{\mathbf{x}} = \partial_{x_1} \circ \partial_{x_2} \circ \cdots \circ \partial_{x_k}\colon U(X,Y) \longrightarrow U(X,Y)$$

for every $k \geq 1$ and $\mathbf{x} = (x_1, \dots, x_k) \in X^k$.

An element $f \in U(X,Y)$ is a *measurable polynomial* if there exists an integer $k \geq 1$ with $\partial_{\mathbf{x}}(f) = 0 \pmod{\lambda_X}$ for every $\mathbf{x} \in X^k$. If k is the smallest such integer, then the *degree* $\deg(f)$ of the measurable polynomial f is equal to $k-1$.

For every $a \in \widehat{Y}$ and $f \in U(X,Y)$ we denote by $\langle a, f\rangle \in U(X,\mathbb{S})$ the map $x \mapsto \langle a, f(x)\rangle$, where $\langle a, x\rangle$ is the value of the character $a \in \widehat{Y}$ at the point $x \in X$.

Lemma 6.2. *An element $f \in U(X,Y)$ is a measurable polynomial if and only if $\langle a, f\rangle \in U(X,\mathbb{S})$ is a measurable polynomial for every $a \in \widehat{Y}$, and f has degree $\le k$ if and only if $\deg(\langle a, f\rangle) \le k$ for every $a \in \widehat{Y}$. Finally, f is continuous if and only if $\langle a, f\rangle$ is continuous for every $a \in \widehat{Y}$.*

Proof. We set $\Omega = \mathbb{S}^{\widehat{Y}}$ and write every $\omega \in \Omega$ as $\omega = (\omega_a,\, a \in \widehat{Y})$ with $\omega_a \in \mathbb{S}$ for every $a \in \widehat{Y}$. Define a continuous injective group homomorphism $\Phi\colon Y \longrightarrow \mathbb{S}^{\widehat{Y}}$ by setting

$$\Phi(y)_a = \langle a,y\rangle$$

for every $a \in \widehat{Y}$ and $y \in Y$. Then $Z = \Phi(Y)$ is a closed subgroup of Ω, and the map $f' = \Phi \circ f\colon X \longrightarrow Z$ is a measurable polynomial (of degree $\le k$) if and only if each coordinate $x \mapsto f'(x)_a = \langle a, f(x)\rangle$ of the map f' is an $\mathbb{S}$-valued measurable polynomial (of degree $\le k$) for every $a \in \widehat{Y}$. Since $\Phi\colon X \longrightarrow Z$ is a topological group isomorphism, the last statement is obvious. □

Lemma 6.3. *Let $f \in U(X,Y)$ and $k \ge 1$. Then the map $\mathbf{x} \mapsto \partial_{\mathbf{x}}(f)$ from X^k to $U(X,Y)$ is continuous.*

Proof. The same argument as in Lemma 6.2 allows us to assume without loss in generality that $Y = \mathbb{S}$.

Consider the special case where $k = 2$. For any $f \in U(X,\mathbb{S})$ and $x \in X$ we denote by $\overline{f}$ the complex conjugate of f and write $f_x \in U(X,\mathbb{S})$ for the map given by $f_x(x') = f(x+x')$. Define maps $S_1,\dots,S_4\colon X^2 \longrightarrow U(X,\mathbb{S})$ by

$$S_1(x_1,x_2) = f_{x_1+x_2},\;\; S_2(x_1,x_2) = \overline{f_{x_1}},\;\; S_3(x_1,x_2) = \overline{f_{x_2}},\;\; S_4(x_1,x_2) = f,$$

where the bar denotes complex conjugation. For every $\mathbf{x} \in X^2$, $\partial_{\mathbf{x}}(f) = S_1(\mathbf{x}) \cdot S_2(\mathbf{x}) \cdot S_3(\mathbf{x}) \cdot S_4(\mathbf{x})$. Since the right regular representation of X on $L^2(X,\lambda_X)$ is continuous, each S_i is a continuous map from X^2 into $L^2(X,\lambda_X)$ and hence also a continuous map from X^2 into $U(X,\mathbb{S})$. As multiplication is continuous in $U(X,\mathbb{S})$, this proves our assertion for $k = 2$. In the general case we define $S_1,\dots,S_{2^k}$ in an analogous way and apply the same argument as above. □

Proposition 6.4 ([2]). *Let X,Y be compact abelian groups, and let $f \in U(X,Y)$ be a measurable polynomial.*

1. *There exists a unique continuous map $f'\colon X \longrightarrow Y$ such that $f = f' \pmod{\lambda_X}$.*
2. *The map f' is constant if and only if $\deg(f) = 0$, and affine if and only if $\deg(f) \le 1$.*
3. *If X is connected, then f has degree ≤ 1.*

Proof. For $k \ge 0$ we denote by $P_k \subset U(X,Y)$ the topological space consisting of all measurable polynomials $p\colon X \longrightarrow Y$ of degree at most k, furnished with the subspace topology. If f is a measurable polynomial of degree 0, then f is λ_X-*a.e.* equal to a constant $y \in Y$. If $\deg(f) = 1$, then there exists, for every $x \in X$, a unique constant $c(x) \in Y$ with $\partial_x(f) = c(x) \pmod{\lambda_X}$, and the map $x \mapsto c(x)$ is a Borel measurable — and thus continuous — group homomorphism. Hence there exists, for every $x \in X$, a Borel set $B_x \subset X$ with $\lambda_X(B_x) = 1$ such that

$$f(x+x') = c(x) + f(x') \tag{6.1}$$

for every $x \in X$ and $x' \in B_x$. Fubini's Theorem implies that there exists a Borel set $B \subset X$ with $\lambda_X(B) = 1$ such that (6.1) holds for every $x' \in B$ and λ_X-*a.e.* $x \in X$, which shows that f is *a.e.* equal to an affine map.

We have proved that every map in P_1 is *a.e.* equal to a continuous map. Continuing by induction, we assume that k is a positive integer such that every measurable polynomial of degree $\le k$ is *a.e.* equal to a continuous map and consider a polynomial $f \in P_{k+1} \subset U(X,Y)$. According to Lemma

6.3 it suffices to prove the continuity of f in the special case where $Y = \mathbb{S}$, and we assume therefore without loss in generality that $f \in U(X, \mathbb{S})$.

Since the characters form an orthonormal basis of $L^2(X, \lambda_X)$ we deduce that P_1 is homeomorphic to $P_0 \times \widehat{X}$, where $\widehat{X}$ is equipped with the discrete topology, and we write $\theta\colon P_1 \longrightarrow \widehat{X}$ for the projection map. The map

$$\mathbf{x} \mapsto q(\mathbf{x}) = \theta \circ \partial_{\mathbf{x}}(f)$$

from X^k to $\widehat{X}$ is continuous by Lemma 6.3. Since $\widehat{X}$ is discrete, $q(X^k)$ is finite, and there exists an open subgroup $K_1 \subset X^k$ such that q is constant on each coset of K_1 in X^k. We choose an open subgroup $K \subset X$ with $K^k \subset K_1$. Then $\partial_{\mathbf{x}}(f)$ lies in P_0 for all $\mathbf{x} \in K^k$, so that the restriction of f to K is a measurable polynomial of degree at most k. Let $K + z_1, \dots, K + z_l$ be the distinct cosets of K in X, and let, for $i = 1, \dots, l$, $f_i\colon X \longrightarrow \mathbb{S}$ be the map defined by $f_i(x) = f(z_i + x)$. Since $\partial_{\mathbf{x}}(f_i)(x) = \partial_{\mathbf{x}}(f)(z_i + x)$ for each i, we conclude that restriction of each f_i to K is a measurable polynomial of degree at most k. By the induction hypothesis, the restriction of each f_i to K agrees λ_K-a.e. with a continuous map, i.e. f agrees λ_X-a.e. with a continuous map.

If X is connected then q is trivial, i.e. the degree of f is $\leq k$. By a slight modification of the above induction argument, f agrees λ_X-a.e. with an affine map. □

6.2. **Topological rigidity.**

Theorem 6.5 ([2])**.** *Let α and β be mixing algebraic $\mathbb{Z}^d$-actions on compact abelian groups X and Y, respectively. Suppose furthermore that there exists an integer $k \geq 2$ with the following property: for every closed β-invariant subgroup $Z \subset Y$, the restriction β_Z of β to Z is not $(k+1)$-mixing. Then every equivariant Borel map $\phi\colon (X, \alpha) \longrightarrow (Y, \beta)$ is a measurable polynomial of degree $\leq k-1$ and hence* a.e. *equal to a continuous map.*

We begin the proof of Theorem 6.5 with a lemma.

Lemma 6.6. *Let α be a mixing algebraic $\mathbb{Z}^d$-action on a compact abelian group X, $k \geq 1$, and let $f_i\colon X^k \longrightarrow \mathbb{R}_+$, $i = 0, \dots, k$, be continuous maps with the following properties.*

(1) *For every $i = 1, \dots, k$ and $(x_1, \dots, x_k) \in X^k$, $f_i(x_1, \dots, x_k) = 0$ whenever $x_j = 0$ for some $j \in \{1, \dots, k\}$;*

(2) *There exist sequences $(\mathbf{n}_m^{(i)}, m \geq 1)$, $i = 1, \dots, k$, in $\mathbb{Z}^d$ with*

$$\lim_{m \to \infty} \mathbf{n}_m^{(i)} = \infty$$

for $i = 1, \dots, k$, and

$$f_0 \leq \sum_{i=1}^{k} f_i \circ \bar{\alpha}^{\mathbf{n}_m^{(i)}} \tag{6.2}$$

for every $m \geq 1$, where $\bar{\alpha}\colon \mathbf{n} \longrightarrow \alpha^{\mathbf{n}} \times \dots \times \alpha^{\mathbf{n}}$ is the diagonal $\mathbb{Z}^d$-action on X^k induced by α.

Then $f_0 \equiv 0$.

Proof. If $f_0 \not\equiv 0$, then there exist nonempty open subsets $\mathcal{U}_1, \dots, \mathcal{U}_k$ in X and an $\varepsilon > 0$ such that

$$f_0(x_1, \dots, x_k) > \varepsilon \ \text{ for every } \ (x_1, \dots, x_k) \in \mathcal{U}_1 \times \dots \times \mathcal{U}_k. \tag{6.3}$$

Since each f_i is continuous, it is uniformly continuous on X^k, and there exists an open neighbourhood $\mathcal{U}$ of 0 in X such that

$$f_i(x_1, \dots, x_k) < \varepsilon / k \tag{6.4}$$

whenever $i \in \{1, \dots, k\}$ and $x_j \in \mathcal{U}$ for some $j \in \{1, \dots, k\}$.

As α is mixing, there exists an integer $M \geq 1$ with $\alpha^{-\mathbf{n}_m^{(i)}}(\mathcal{U}) \cap \mathcal{U}_i \neq \varnothing$ for every $i = 1, \dots, k$ and $m \geq M$. Fix $x_i \in \alpha^{-\mathbf{n}_M^{(i)}}(\mathcal{U}) \cap \mathcal{U}_i$ for $i = 1, \dots, k$. Then $\alpha^{\mathbf{n}_M^{(i)}} x_i \in \mathcal{U}$ and hence, by (6.3),

$$f_i \circ \overline{\alpha}^{\mathbf{n}_M^{(i)}}(x_1, \dots, x_k) < \varepsilon/k$$

for $i = 1, \dots, k$, which violates (6.2)–(6.3). □

Proof of Theorem 6.5. It suffices to show that $\langle a, \phi \rangle \colon X \longrightarrow Y$ is a measurable polynomial of degree $\leq k-1$ for every character $a \in \widehat{Y}$. We set

$$A = \{a \in \widehat{Y} : \langle a, \phi \rangle \text{ is a measurable polynomial of degree } \leq k-1\}$$

and assume that $A \subsetneq \widehat{Y}$.

The group A is obviously invariant under $\widehat{\beta}$, and its annihilator

$$Z = A^{\perp} = \{y \in Y : \langle a, y \rangle = 1 \text{ for every } a \in A\}.$$

is a closed β-invariant subgroup of Y.

By assumption, β_Z is not $(k+1)$-mixing. Hence there exist characters $b_0, \dots, b_k \in \widehat{Z}$ with $b_0 \neq 0$, and sequences $(\mathbf{n}_m^{(i)}, m \geq 1)$, $i = 1, \dots, k$, in $\mathbb{Z}^d$ with

$$\lim_{m \to \infty} \mathbf{n}_m^{(i)} = \infty$$

for $i = 1, \dots, k$, such that

$$b_0 = \sum_{i=1}^{k} \widehat{\beta}_Z^{\mathbf{n}_m^{(i)}} b_i$$

for every $m \geq 1$. We extend each $b_i \in \widehat{Z}$ to an element $b_i' \in \widehat{Y}$ and obtain elements $a_m \in A$, $m \geq 1$, with

$$b_0' = \sum_{i=1}^{k} \widehat{\beta}^{\mathbf{n}_m^{(i)}} b_i' + a_m$$

for every $m \geq 1$. By composing this equation with ϕ we obtain that

$$\langle b_0', \phi \rangle = \langle a_m, \phi \rangle \cdot \prod_{i=1}^{k} \langle \widehat{\beta}^{\mathbf{n}_m^{(i)}} b_i', \phi \rangle = \langle a_m, \phi \rangle \cdot \prod_{i=1}^{k} \langle b_i', \phi \circ \alpha^{\mathbf{n}_m^{(i)}} \rangle$$

for every $m \geq 1$. Put

$$f_i(x_1, \dots, x_k) = \|\partial_k(x_1, \dots, x_k)(\langle b_i', \phi \rangle) - 1\|_2$$

for every $(x_1, \dots, x_k) \in X^k$ and $i = 0, \dots, k$, and note that

$$f_0 \leq \sum_{i=1}^{k} f_i \circ \overline{\alpha}^{\mathbf{n}_m^{(i)}} + \|\partial_k(x_1, \dots, x_k)(\langle a_m, \phi \rangle) - 1\|_2 \tag{6.5}$$

for every $m \geq 1$, where we are using the same notation as in Lemma 6.6. As $a_m \in A$, $\langle a_m, \phi \rangle$ is a measurable polynomial of degree $\leq k$, and hence $\partial_k(x_1, \dots, x_k)(\langle a_m, \phi \rangle) = 1$ λ_Y*-a.e.* The inequality (6.5) thus reduces to

$$f_0 \leq \sum_{i=1}^{k} f_i \circ \overline{\alpha}^{\mathbf{n}_m^{(i)}}$$

for every $m \geq 1$, and Lemma 6.6 guarantees that $f_0 \equiv 0$. This shows that $b_0' \in A$ and hence $b_0 = 0$, and the resulting contradiction to our choice of b_0 implies that $A = \widehat{Y}$ and that ϕ is a measurable polynomial of degree $\leq k-1$, as claimed. □

Corollary 6.7. *Let $d > 1$, and let α and β be algebraic $\mathbb{Z}^d$-actions on compact abelian groups X and Y, respectively. Suppose that Y is zero-dimensional and that β has zero entropy. Then there exists a continuous factor map $\phi' \colon (X,\alpha) \longrightarrow (Y,\beta)$ such that $\phi = \phi'$ λ_X-a.e.*

Proof. Let $N = \widehat{Y}$ be the dual module of β. Then there exists an increasing sequence $(N_k, k \geq 1)$ of submodules of N such that $N = \bigcup_{k\geq 1} N_k$ and each N_k is Noetherian. For every $k \geq 1$, the annihilator $Y_k = N_k^{\perp} \subset Y$ is a closed β-invariant subgroup, and we denote by $\pi_k \colon Y \longrightarrow Y/Y_k$ the quotient map.

Let $\phi \colon (X,\alpha) \longrightarrow (Y,\beta)$ be a measurable factor map such that $\phi_k = \pi_k \circ \phi$ is a measurable polynomial for every $k \geq 1$. Then $\pi_k \circ \phi$ is λ_X-*a.e.* equal to a continuous factor map $\phi_k \colon (X,\alpha) \longrightarrow (Y/Y_k, \beta_{Y/Y_k})$ for every $k \geq 1$, where β_{Y/Y_k} is the $\mathbb{Z}^d$-action on Y/Y_k induced by β. As $\bigcap_{k\geq 1} Y_k = \{0_Y\}$, compactness implies that there exists, for every neighbourhood $\mathcal{U}$ of the identity in Y, an integer $K \geq 1$ with $Y_k \subset \mathcal{U}$ for every $k \geq K$. If ϕ is not equal to a continuous map λ_X-*a.e.*, then the same is true for some ϕ_k, which leads to a contradiction. This observation allows us to assume without loss in generality that $N = \widehat{Y}$ is Noetherian.

As $\bigcup_{k\geq 1} N_k = N$ we know that $\bigcap_{k\geq 1} Y_k = \{0_Y\}$. By compactness there exists, for every neighbourhood $\mathcal{U}$ of the identity in Y, an integer $K \geq 1$ with $Y_k \subset \mathcal{U}$ for every $k \geq K$. If ϕ is not equal to a continuous map λ_X-*a.e.*, then the same is true for some $\phi_k = \pi_k \circ \phi$, which contradicts the hypothesis in preceding paragraph. This allows us to assume without loss in generality that $N = \widehat{Y}$ is Noetherian.

Let therefore N be Noetherian, and let $\mathrm{Asc}(N)$ be the set of associated prime ideals of N. Since Y is zero-dimensional, every $\mathfrak{p} \in \mathrm{Asc}(N)$ contains a rational prime constant $p(\mathfrak{p}) > 1$ by Theorem 2.2 (1), and Theorem 2.2 (3) implies that $\mathfrak{p} \supsetneq (p(\mathfrak{p})) = p(\mathfrak{p})R_d$, since β has zero entropy. We choose and fix, for every $\mathfrak{p} \in \mathrm{Asc}(N)$, a Laurent polynomial $f(\mathfrak{p}) \in \mathfrak{p} \smallsetminus (p(\mathfrak{p}))$, observe that the polynomial $f(\mathfrak{p})_{/p(\mathfrak{p})} \in R_d^{(p(\mathfrak{p}))}$ in (2.8) is nonzero, and denote by $K = \max_{\mathfrak{p}\in\mathrm{Asc}(N)} |\mathcal{S}(f(\mathfrak{p})_{/p(\mathfrak{p})})|$ the maximal cardinality of the supports of these polynomials.

Suppose that $Z \subset Y$ is a closed β-invariant subgroup. We write $L = \widehat{Z}$ for the dual module of Z, choose a prime ideal $\mathfrak{q} \in \mathrm{Asc}(L)$ and an element $a \in L$ with $\mathfrak{q} = \mathrm{ann}(a)$, and set $L' = R_d \cdot a \cong R_d/\mathfrak{q}$. Since L is a quotient of N, $\mathfrak{q}$ contains some $\mathfrak{p} \in \mathrm{Asc}(N)$, and Example 3.2 (2) shows that $\alpha_{L'} \cong \alpha_{R_d/\mathfrak{q}}$ — and hence $\beta_Z = \alpha_L$ — is not mixing of order $|\mathcal{S}(f(\mathfrak{q}))| \leq K$. By Theorem 6.5, ϕ is a measurable polynomial and thus coincides λ-*a.e.* with a continuous factor map. □

6.3. **Homoclinic points and isomorphism rigidity.** Once we know that measurable conjugacies and factor maps between two algebraic $\mathbb{Z}^d$-actions (X,α) and (Y,β) are automatically continuous it is not too difficult to verify that they have to be polynomials (the approach using homoclinic points described below is one such method). If the groups X and Y are connected, these polynomials are affine by Proposition 6.4, which proves isomorphism rigidity. However, if the groups X and Y are zero-dimensional, polynomials may have degrees > 1, and one needs additional hypotheses (whose necessity will be illustrated below in Example 6.15) to ensure that the measurable conjugacies and factor maps are affine.

Definition 6.8. Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X, and let $\Gamma \subset \mathbb{Z}^d$ be a subgroup. An element $x \in X$ is (α,Γ)-*homoclinic* (to the identity element 0_X of X), if

$$\lim_{\substack{\mathbf{n}\to\infty\\ \mathbf{n}\in\Gamma}} \alpha^{\mathbf{n}} x = 0_X.$$

The α-invariant subgroup $\Delta_{(\alpha,\Gamma)}(X) \subset X$ of all (α,Γ)-homoclinic points is an R_d-module under the operation

$$f \cdot x = f(\alpha)(x)$$

for every $f \in R_d$ and $x \in \Delta_{(\alpha,\Gamma)}(X)$ (cf. (2.5)), and is called the Γ-*homoclinic module of* α (cf. [16]).

Proposition 6.9. *Let α be an expansive algebraic $\mathbb{Z}^d$-action on a compact abelian group X, and let $\Gamma \subset \mathbb{Z}^d$ be a subgroup. Then $\Delta_{(\alpha,\Gamma)} \neq \{0_X\}$ if and only if the entropy $h(\alpha^\Gamma)$ of the algebraic Γ-action α^Γ on X is positive, and $\Delta_{(\alpha,\Gamma)}$ is dense in X if and only if α^Γ has completely positive entropy* (*where entropy is always taken with respect to Haar measure*).

Proof. This is [16, Theorems 4.1 and 4.2]. □

If an expansive and mixing algebraic $\mathbb{Z}^d$-action α on a compact abelian group X has zero entropy, then the homoclinic group $\Delta_\alpha(X)$ of this $\mathbb{Z}^d$-action is trivial by Proposition 6.9, but $\Delta_{(\alpha,\Gamma)}$ will be dense in X for appropriate subgroups $\Gamma \subset \mathbb{Z}^d$. We investigate this phenomenon in the special case where $p > 1$ is a rational prime, $f \in R_d^{(p)}$ an irreducible Laurent polynomial such that the convex hull $\mathcal{C}(f) \subset \mathbb{R}^d$ of the support $\mathcal{S}(f) \subset \mathbb{Z}^d$ of f contains an interior point (cf. (3.6)), and where $\alpha = \alpha_{R_d^{(p)}/(f)}$ is the shift-action of $\mathbb{Z}^d$ on the compact abelian group $X = X_{R_d^{(p)}/(f)} \subset F_p^{\mathbb{Z}^d}$ defined in (2.9)–(2.10).

We write $[\,.\,,.\,]$ and $\|\cdot\|$ for the Euclidean inner product and norm on $\mathbb{R}^d$ and

$$\mathsf{S}_{d-1} = \{\mathbf{v} \in \mathbb{R}^d : \|\mathbf{v}\| = 1\}$$

for the unit sphere in $\mathbb{R}^d$ and set, for every nonzero element $\mathbf{m} \in \mathbb{Z}^d$,

$$\begin{aligned} \mathbf{m}^* &= \tfrac{\mathbf{m}}{\|m\|}, \\ \Gamma_{\mathbf{m}} &= \{\mathbf{n} \in \mathbb{Z}^d : [\mathbf{m},\mathbf{n}] = 0\}. \end{aligned} \tag{6.6}$$

Proposition 6.10. [3] *Let $d > 1$, $p > 1$ a rational prime, $f \in R_d^{(p)}$ an irreducible Laurent polynomial such that the shift-action $\alpha = \alpha_{R_d^{(p)}/(f)}$ of $\mathbb{Z}^d$ on the compact abelian group $X = X_{R_d^{(p)}/(f)} \subset F_p^{\mathbb{Z}^d}$ in* (2.9)–(2.10) *is mixing, and let $\mathbf{m} \subset \mathbb{Z}^d$ be a nonzero element such that the restriction $\alpha^{\Gamma_{\mathbf{m}}}$ of α to the subgroup $\Gamma_{\mathbf{m}}$ in* (6.6) *is expansive. Then the homoclinic group $\Delta_{(\alpha,\Gamma_{\mathbf{m}})}(X)$ is dense in X. Furthermore there exists an open subset $W \subset \mathsf{S}_{d-1}$ such that every nonzero element $\mathbf{n} \in \mathbb{Z}^d$ with $\mathbf{n}^* \in \mathsf{S}_{d-1}$ has the following properties.*

(1) $\Delta_{(\alpha,\Gamma_{\mathbf{n}})}(X)$ *is dense in X;*
(2) $\Delta_{(\alpha,\Gamma_{\mathbf{m}})}(X) \cap \Delta_{(\alpha,\Gamma_{\mathbf{n}})}(X) = \{0_X\}$.

The proof of Proposition 6.10 is given in [3]. By using this proposition and some algebraic structure theory one obtains the following rigidity result for measurable factor maps between algebraic $\mathbb{Z}^d$-actions on zero-dimensional compact abelian groups.

Theorem 6.11. *Let $d > 1$, and let α and β be mixing algebraic $\mathbb{Z}^d$-actions on zero-dimensional compact abelian groups X and Y, respectively. Suppose that there exists a subgroup $\Gamma \subset \mathbb{Z}^d$ of of infinite index such that the restriction α^Γ of α to Γ is expansive and has completely positive entropy. Then every measurable factor map $\phi\colon (X,\alpha) \longrightarrow (Y,\beta)$ is λ_X-a.e. equal to an affine map.*

Theorem 6.11 was proved independently in [3] and [4]; the latter proof depends on a characterization of invariant measures analogous to the connection between the Theorems 4.5 and 4.4. Here we follow the 'homoclinic' route in [3]; however, before turning to the proof of this result, we mention a couple of corollaries which generalize the main result in [12] in different directions.

Corollary 6.12. *Let $d > 1$, and let α and β be mixing algebraic $\mathbb{Z}^d$-actions on zero-dimensional compact abelian groups X and Y, respectively. Suppose that there exists a nonzero element $\mathbf{n} \in \mathbb{Z}^d$ such that the automorphism $\alpha^{\mathbf{n}}$ is expansive. Then every measurable factor map $\phi\colon (X,\alpha) \longrightarrow (Y,\beta)$ is λ_X-a.e. equal to an affine map.*

Proof. Since every mixing (= ergodic) group automorphism has completely positive entropy, this is Theorem 6.11 with Γ of rank one. $\square$

Corollary 6.13. *Let $d>1$, p a rational prime, and $\mathfrak{p},\mathfrak{q}\subset R_d^{(p)}$ nonzero prime ideals such that the $\mathbb{Z}^d$-actions $\alpha=\alpha_{R_d^{(p)}/\mathfrak{p}}$ and $\beta=\alpha_{R_d^{(p)}/\mathfrak{q}}$ on the compact zero-dimensional groups $X=X_{R_d^{(p)}/\mathfrak{p}}$ and $Y=X_{R_d^{(p)}/\mathfrak{q}}$ in* (2.9)–(2.10) *are mixing. Then α and β are measurably conjugate if and only if they are algebraically conjugate, and hence if and only if $\mathfrak{p}=\mathfrak{q}$. Furthermore, every measurable conjugacy $\phi\colon(X,\alpha)\longrightarrow(Y,\beta)$ is λ_X-a.e. equal to an affine map.*

Proof. The existence of a subgroup $\Gamma\subset\mathbb{Z}^d$ of infinite index with the properties required by Theorem 6.11 is proved in [5] (the rank of Γ is the maximal number of algebraically independent elements in the set $\{u^{\mathbf{n}}+\mathfrak{p}\colon\mathbf{n}\in\mathbb{Z}^d\}\subset R_d^{(p)}/\mathfrak{p}$). Let $\phi\colon(X,\alpha)\longrightarrow(Y,\beta)$ be a measurable conjugacy. By Theorem 6.11, there exist $y\in Y$ and a continuous homomorphism $\theta\colon X\longrightarrow Y$ such that $\phi(x)=y+\theta(x)$ for λ_X-*a.e.* $x\in X$. It is easy to verify that θ is an algebraic conjugacy of (X,α) and (Y,β).

In order to see that algebraic conjugacy implies that $\mathfrak{p}=\mathfrak{q}$ we note that, for every $f\in R_d^{(p)}$, the maps $f(\alpha)$ and $f(\beta)$ in (2.5) are surjective if and only if $f\notin\mathfrak{p}$ (resp. $f\notin\mathfrak{q}$). $\square$

We begin our sketch of the proof of Theorem 6.11 with a lemma.

Lemma 6.14. *For $i=1,2,3$, let α_i be a mixing algebraic $\mathbb{Z}^d$-action on a compact abelian group X_i, and let $\phi\colon(X_1\times X_2,\alpha_1\times\alpha_2)\longrightarrow(X_3,\alpha_3)$ be a continuous factor map such that $\phi(x_1,x_2)=0_{X_3}$ whenever $x_1=0_{X_1}$ or $x_2=0_{X_2}$. Suppose furthermore that there exist subgroups Γ_1,Γ_2 in $\mathbb{Z}^d$ such that the homoclinic groups $\Delta_{(\alpha_i,\Gamma_i)}(X_i)$ are dense in X_i for $i=1,2$, and that $\Delta_{(\alpha_3,\Gamma_1)}(X_3)\cap\Delta_{(\alpha_3,\Gamma_2)}(X_3)=\{0_{X_3}\}$. Then $\phi(X_1\times X_2)=\{0_{X_3}\}$.*

Proof. Since ϕ is a continuous factor map,

$$\begin{aligned}\lim_{\substack{\mathbf{m}\to\infty\\ \mathbf{m}\in\Gamma_1}}\alpha_3^{\mathbf{m}}\phi(x_1,x_2)&=\lim_{\substack{\mathbf{m}\to\infty\\ \mathbf{m}\in\Gamma_1}}\phi(\alpha_1^{\mathbf{m}}x_1,\alpha_2^{\mathbf{m}}x_2)=0_{X_3}\\&=\lim_{\substack{\mathbf{n}\to\infty\\ \mathbf{n}\in\Gamma_2}}\alpha_3^{\mathbf{n}}\phi(x_1,x_2)=\lim_{\substack{\mathbf{n}\to\infty\\ \mathbf{n}\in\Gamma_2}}\phi(\alpha_1^{\mathbf{n}}x_1,\alpha_2^{\mathbf{n}}x_2)\end{aligned}$$

for every $x_i\in\Delta_{(\alpha_i,\Gamma_i)}(X_i)$, $i=1,2$. Hence

$$\phi(x_1,x_2)\in\Delta_{(\alpha_3,\Gamma_1)}(X_3)\cap\Delta_{(\alpha_3,\Gamma_2)}(X_3)=\{0_{X_3}\}.$$

As $\Delta_{(\alpha_i,\Gamma_i)}(X_i)\subset X_i$ is dense for $i=1,2$ and ϕ is continuous this implies our assertion. $\square$

Leaving technicalities and a bit of algebra aside, the basic idea of the proof of Theorem 6.11 is the fact that there exist two subgroups $\Gamma_1,\Gamma_2\subset\mathbb{Z}^d$ such that each action α_{Γ_i} has a dense group of homoclinic points and there are no nonzero common homoclinic points for the actions β_{Γ_i}. Since we know already that the factor map $\phi\colon X\longrightarrow Y$ is continuous, we can form a new map $\psi\colon X\times X\longrightarrow Y$ by setting

$$\psi(x_1,x_2)=\psi(x_1+x_2)-\psi(x_1)-\psi(x_1)+\psi(0).$$

Since $\psi\circ(\alpha^{\mathbf{n}}\times\alpha^{\mathbf{n}})=\beta^{\mathbf{n}}\circ\psi$ for all $\mathbf{n}\in\mathbb{Z}^d$, and since ψ is continuous and hence uniformly continuous, $\psi(x_1,x_2)\in\Delta_{(\beta,\Gamma_1)}\cap\Delta_{(\beta,\Gamma_2)}=\{0\}$ whenever $x_i\in\Delta_{(\alpha,\Gamma_i)}$, $i=1,2$. Hence ψ vanishes on the dense set $\Delta_{(\alpha,\Gamma_1)}\times\Delta_{(\alpha,\Gamma_1)}\subset X\times X$ and is thus equal to zero by continuity. This shows that ϕ is affine.

The crucial point in this argument is that *two* such subgroups Γ_1,Γ_2 suffice under the hypotheses of Theorem 6.11. In general one can find finitely many such subgroups $\Gamma_1,\dots,\Gamma_n\subset\mathbb{Z}^d$ such that each action α_{Γ_i} has a dense group of homoclinic points and there are no nonzero common homoclinic points for the actions β_{Γ_i}, $i=1,\dots,n$, and obtains that the map $\psi\colon X^n \longrightarrow Y$ with

$$\psi(x_1,\dots,x_n)=\sum_{F\subset\{1,\dots,n\}}(-1)^{|F|}\,\phi\Biggl(\sum_{i\in F}x_i\Biggr)$$

vanishes on X^n. This implies that ϕ is a polynomial of degree $n-1$, but not necessarily of degree 1.

The following examples from [3] show that Theorem 6.11 and Corollary 6.13 need not hold if any of the assumptions are dropped.

Examples 6.15. (1) *A non-surjective and non-affine equivariant map.* Let $d=3$, $p=2$, and consider the polynomials $f_1,f_2\in R_3^{(2)}$ defined by $f_1=1+u_1+u_2$, $f_2=1+u_1+u_2+u_1^2+u_1u_2+u_2^2+u_3$. Let $\mathfrak{p}=(f_1,f_2)\subset R_3^{(2)}$ denote the ideal generated by f_1 and f_2, and let $\mathfrak{q}=(f_2)\subset R_3^{(2)}$ be the principal ideal generated by f_2. It is easy to see that $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals. We define the shift-actions $\alpha_1=\alpha_{R_3^{(2)}/\mathfrak{p}}$ and $\alpha_2=\alpha_{R_3^{(2)}/\mathfrak{q}}$ on $X_1=X_{R_3^{(2)}/\mathfrak{p}}\subset F_2^{\mathbb{Z}^3}$ and $X_2=X_{R_3^{(2)}/\mathfrak{q}}\subset F_2^{\mathbb{Z}^3}$, respectively, by (2.9)–(2.10). From Theorem 2.2 it is clear that α_1 and α_2 are mixing and have zero entropy.

We write $\star$ for the component-wise multiplication $(z\star z')_{\mathbf{n}}=z_{\mathbf{n}}z'_{\mathbf{n}}$ in $F_2^{\mathbb{Z}^3}$ and observe that

$$\sigma^{\mathbf{n}}(z\star z')=(\sigma^{\mathbf{n}}z)\star(\sigma^{\mathbf{n}}z')$$

for every $z,z'\in F_2^{\mathbb{Z}^3}$ and $\mathbf{n}\in\mathbb{Z}^3$ (cf. (2.6)). We claim that

$$x\star x'\in X_2 \text{ for every } x,x'\in X_1. \tag{6.7}$$

In order to verify this we define subsets $S_i\subset\mathbb{Z}^3$, $i=0,\dots,3$, by

$$\begin{gathered}S_0=\mathcal{S}(f_2),\ \ S_1=\mathcal{S}(f_1),\\ S_2=\{(1,0,0),(1,1,0),(2,1,0)\}=\mathcal{S}(u_1f_1),\\ S_3=\{(0,1,0),(0,2,0),(1,1,0)\}=\mathcal{S}(u_2f_1),\end{gathered}$$

and consider the set Z of all $z\in F_2^{S_0}$ with $\sum_{\mathbf{n}\in S_i}z_{\mathbf{n}}=0$ for $i=0,\dots,3$. A calculation shows that, for every $z,z'\in Z$, the component-wise product $w=z\star z'\in F_2^{S_0}$ satisfies that $\sum_{\mathbf{n}\in S_0}w_{\mathbf{n}}=0$. This implies (6.7).

Take a non-zero $\mathbf{m}\in\mathbb{Z}^3$ such that $\alpha_1^{\mathbf{m}}z=z$ for some non-zero $z\in X_1$ and define $\phi\colon X_1\longrightarrow X_2$ by $\phi(x)=x\star\alpha_1^{\mathbf{m}}x$. Clearly ϕ is a $\mathbb{Z}^3$-equivariant map from (X_1,α_1) to (X_2,α_2). We choose $y\in X_1$ such that $z\star(\alpha_1^{\mathbf{m}}y-y)\neq 0_{X_2}$. Since $\phi(0_{X_1})=0_{X_2}$ and $\phi(z+y)-\phi(z)-\phi(y)=z\star(\alpha_1^{\mathbf{m}}y-y)\neq 0_{X_2}$, the map ϕ is not affine.

(2) *A non-affine factor map $\psi\colon(X,\alpha)\longrightarrow(X',\alpha')$ between expansive and mixing zero-entropy algebraic $\mathbb{Z}^3$-actions, where α' has an expansive $\mathbb{Z}^2$-sub-action with completely positive entropy.* We use the same notation as in Example (1). Let $\mathfrak{r}=\mathfrak{p}\mathfrak{q}=(f_1f_2,f_2^2)\subset R_3^{(2)}$ be the ideal generated by f_1f_2 and f_2^2 and let β denote the algebraic $\mathbb{Z}^3$-action $\alpha_{R_3^{(2)}/\mathfrak{r}}$ on $Y=X_{R_3^{(2)}/\mathfrak{r}}\subset F_2^{\mathbb{Z}^3}$. From Theorem 2.2 it follows that the action (Y,β) is mixing and has zero entropy. We define continuous group homomorphisms $\theta_1\colon Y\longrightarrow X_1$ and $\theta_2\colon Y\longrightarrow X_2$ by

$$\theta_1(y)=f_2(\sigma)(y),\qquad \theta_2(y)=f_1(\sigma)(y).$$

It is easy to verify that for $i=1,2$, $\theta_i\colon(Y,\beta)\longrightarrow(X_i,\alpha_i)$ is an algebraic factor map. Let $\psi\colon(Y,\beta)\longrightarrow(X_2,\alpha_2)$ be the $\mathbb{Z}^3$-equivariant continuous map defined by

$$\psi(x)=\theta_2(x)+\phi\circ\theta_1(x),$$

where $\phi\colon X_1 \longrightarrow X_2$ is as in the previous example. Since θ_1 is a surjective homomorphism and ϕ is non-affine, it follows that $\phi\circ\theta_1$ is non-affine, i.e. that ψ is a non-affine map. It is easy to see that the restriction of θ_2 to X_2 is a surjective map from X_2 to itself. Since $\theta_1(x)=0$ for all $x\in X_2\subset Y$, this shows that ψ is a non-affine factor map from (Y,β) to (X_2,α_2).

(3) *Two measurably conjugate expansive and mixing zero-entropy algebraic $\mathbb{Z}^3$-actions on non-isomorphic compact zero-dimensional abelian groups.* Let (X_1,α_1) and (X_2,α_2) be as in Example (1), and let (X,α) denote the product action $(X_1,\alpha_1)\times(X_2,\alpha_2)$. Following [1] we define a zero-dimensional compact abelian group Y and an algebraic $\mathbb{Z}^3$-action β on Y by setting $Y=X_1\times X_2$ with composition

$$(x,y)\odot(x',y')=(x+x',x\star x'+y+y')$$

for every $(x,x'),(y,y')\in Y$, and by letting

$$\beta^{\mathbf{n}}(x,y)=(\alpha_1^{\mathbf{n}}x,\alpha_2^{\mathbf{n}}y)$$

for every $(x,y)\in Y$ and $\mathbf{n}\in\mathbb{Z}^3$. The 'identity' map $\phi\colon X\longrightarrow Y$, defined by

$$\phi(x,y)=(x,y)$$

for every $(x,y)\in X$, is obviously a topological conjugacy of (X,α) and (Y,β) with $\lambda_X\phi^{-1}=\lambda_Y$ (by Fubini's theorem). However, ϕ is not a group isomorphism. In fact, the groups X and Y are not isomorphic: since X is a subgroup $(F_2\oplus F_2)^{\mathbb{Z}^3}$, every element in X has order 2, whereas $(x,0_{X_2})\in Y$ and $(x,0_{X_2})\odot(x,0_{X_2})=(0_{X_2},x)\neq 0_Y$ for every nonzero $x\in X_1$.

References

[1] S. Bhattacharya, *Zero entropy* $\mathbf{Z}^d$*-actions which do not exhibit rigidity*, Duke Math. J. **116** (2003), 471–476.

[2] S. Bhattacharya, *Higher Order Mixing and Rigidity of Algebraic Actions on Compact Abelian Groups*, Israel J. Math. (to appear),
ftp://ftp.esi.ac.at/pub/Preprints/esi1128.ps.

[3] S. Bhattacharya and K. Schmidt, *Homoclinic points and isomorphism rigidity of algebraic* $\mathbf{Z}^d$*-actions on zero-dimensional compact abelian groups*, Israel J. Math. (to appear),
ftp://ftp.esi.ac.at/pub/Preprints/esi1127.ps.

[4] M. Einsiedler, *Isomorphism and measure rigidity for algebraic actions on zero-dimensional groups*, Preprint (2002).

[5] M. Einsiedler, D. Lind, R. Miles and T.B. Ward, *Expansive subdynamics for algebraic* $\mathbf{Z}^d$*-actions*, Ergod. Th. & Dynam. Sys. **21** (2001), 1695–1729.

[6] M. Einsiedler and E. Lindenstrauss, *Measurable rigidity properties of* $\mathbf{Z}^d$*-actions on tori and solenoids*, in preparation.

[7] M. Einsiedler and K. Schmidt, *Irreducibility, homoclinic points and adjoint actions of algebraic* $\mathbf{Z}^d$*-actions of rank one*, in: Nonlinear Phenomena and Complex Systems, ed. A. Maass, S. Martinez and J. San Martin, Kluwer Academic Publishers, Dordrecht, 2002, 95–124.

[8] H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in diophantine approximation*, Math. Systems Theory **1** (1967), 1–49.

[9] A. Katok, S. Katok and K. Schmidt, *Rigidity of measurable structure for algebraic actions of higher-rank abelian groups*, Comment. Math. Helv. **77** (2002), 718–745.

[10] A. Katok and R.J. Spatzier, *Invariant measures for higher-rank hyperbolic abelian actions*, Ergod. Th. & Dynam. Sys. **16** (1996), 751–778; *Corrections*, **18** (1998), 507–507.

[11] B. Kitchens and K. Schmidt, *Automorphisms of compact groups*, Ergod. Th. & Dynam. Sys. **9** (1989), 691–735.

[12] B. Kitchens and K. Schmidt, *Mixing sets and relative entropies for higher dimensional Markov shifts*, Ergod. Th. & Dynam. Sys. **13** (1993), 705–735.

[13] B. Kitchens and K. Schmidt, *Isomorphism rigidity of irreducible algebraic* $\mathbf{Z}^d$*-actions*, Invent. Math. **142** (2000), 559–577.

[14] S. Lang, *Algebra*, 2nd edn., Addison-Wesley, Reading, Massachusetts, 1984.

[15] F. Ledrappier, *Un champ markovien peut être d'entropie nulle et mélangeant*, C. R. Acad. Sci. Paris Sér. I Math. **287** (1978), 561–562.

[16] D. Lind and K. Schmidt, *Homoclinic points of algebraic* $\mathbb{Z}^d$*-actions*, J. Amer. Math. Soc. **12** (1999), 953–980.

[17] D. Lind, K. Schmidt and T. Ward, *Mahler measure and entropy for commuting automorphisms of compact groups*, Invent. Math. **101** (1990), 593–629.
[18] R. Lyons, *On measures simultaneously 2- and 3-invariant*, Israel J. Math. **61** (1988), 219–224.
[19] D. Masser, *Two letters to D. Berend*, dated 12th and 19th September, 1985.
[20] D. Masser, *Mixing and equations over groups in positive characteristic*, ftp://ftp.esi.ac.at/pub/Preprints/esi1216.ps, 2002.
[21] D.S. Ornstein and B. Weiss, *Entropy and isomorphism theorems for actions of amenable groups*, J. Analyse Math. **48** (1987), 1–141.
[22] D.J. Rudolph, $\times 2$ *and* $\times 3$ *invariant measures and entropy*, Ergod. Th. & Dynam. Sys. **10** (1990), 395–406.
[23] D.J. Rudolph and K. Schmidt, *Almost block independence and Bernoullicity of* $\mathbb{Z}^d$*-actions by automorphisms of compact groups*, Invent. Math. **120** (1995), 455–488.
[24] K. Schmidt, *Automorphisms of compact abelian groups and affine varieties*, Proc. London Math. Soc. **61** (1990), 480–496.
[25] K. Schmidt, *Dynamical Systems of Algebraic Origin*, Birkhäuser Verlag, Basel-Berlin-Boston, 1995.
[26] K. Schmidt, *The dynamics of algebraic* $\mathbf{Z}^d$*-actions*, in: European Congress of Mathematics (Barcelona 2000), Vol. I, ed. C. Casacuberta, R.M. Miró-Roig, J. Verdera and S. Xambó-Descamps, Progress in Mathematics, vol. 201, Birkhäuser Verlag, Basel-Berlin-Boston, 2001, 543–553.
[27] K. Schmidt and T. Ward, *Mixing automorphisms of compact groups and a theorem of Schlickewei*, Invent. Math. **111** (1993), 69–76.

ERWIN SCHRÖDINGER INSTITUTE FOR MATHEMATICAL PHYSICS, BOLTZMANNGASSE 9, A-1090 VIENNA, AUSTRIA