FROM INFINITELY DIVISIBLE REPRESENTATIONS TO COHOMOLOGICAL RIGIDITY

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1. Introduction

After completing my Ph.D. on some problems in the theory of uniform distribution in Vienna in early 1968 I felt in need of a change of mathematical direction and started reading K.R. Parthasarathy's book *Probability measures on metric spaces* ([20]), which impressed me so much that I decided to go to Manchester and work with Parthasarathy. When I eventually arrived there in 1969 I discovered that Partha—as I learned to call him—had moved on to the mathematical foundations of quantum mechanics and quantum field theory, and was working on certain connections between continuous tensor products and classical central limit theorems. The fact that he was starting on a new mathematical venture, combined with his extraordinary willingness to share problems and ideas, allowed me to begin working with him on these problems immediately after my arrival. It was a happy time for me: his and Shyama's kindness, warmth and hospitality (as well as Shyama's fiery Indian cooking) all contributed to the fond memories I still have of my stay in Manchester. Much to my regret Partha and his family went back to India in 1970, and I went on to Bedford College in London. In 1972 Partha came back to England to spend some time at Warwick, and I joined him there for a while, continuing our earlier collaboration on topics related to continuous tensor products and noncommutative versions of the central limit theorem (cf. [26]). In 1974 I moved to Warwick, and in 1975/76 Partha invited me to spend 7 months at the Indian Statistical Institute in Delhi (he was then working at the Indian Institute of Technology in Delhi, but had close links with the ISI and was therefore able to arrange an invitation for me). Although our mathematical interests had begun to diverge by then, this stay in India was a memorable time both for my wife and me, not only because of the wonderful sights, sounds and smells of India, but also because of the company of Partha and Shyama, and the kindness and help of Partha's students Bhatia, Rana and Subramaniam.

Since then Partha and I have met only infrequently, but his intellectual and spiritual generosity certainly has left its mark on my way—and, indeed, my enjoyment—of doing mathematics, and I will always remain grateful to him for that. Although I am not an expert on Partha's more recent work on noncommutative stochastic processes and quantum differential equations I have great respect for his achievements, and the international recognition he has received during the past years proves that they are of the very highest quality. Other mathematicians more knowledgeable in these matters will probably discuss Partha's more recent work in noncommutative probability theory in detail in this volume; all I can do is to try and trace the influence of our joint work on some of my subsequent mathematical interests. Let me begin with the problem Partha greeted me with when I arrived in Manchester.

Dedicated to K.R. Parthasarathy on the occasion of his 60th birthday.

2. Infinitely divisible representations

Let G be a Polish (i.e. complete, separable metric) group, $\mathcal{H}$ a complex, separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let $V: g \mapsto V_g$ be a continuous, unitary, cyclic representation of G on $\mathcal{H}$ with a cyclic unit vector v. If we set

$$\phi(g) = \langle V_g v, v \rangle \tag{2.1}$$

for every $g \in G$, then the function $\phi\colon G \longmapsto \mathbb{C}$ is continuous, $\phi(1_G) = 1$, where 1_G is the identity element of G, and ϕ is *positive definite*, i.e.

$$\sum_{i=1}^{m} c_i \bar{c}_j \phi(g_j^{-1} g_i) = \left\| \sum_{i=1}^{m} c_i V_g v \right\|^2 \geq 0$$

for every $m \geq 1$ and every choice of $g_1, \dots, g_m$ in G and $c_1, \dots, c_m$ in $\mathbb{C}$. Conversely, if $\phi\colon G \longmapsto \mathbb{C}$ is a continuous positive definite function with $\phi(1_G) = 1$, then there exists a continuous, unitary, cyclic representation V of G on a Hilbert space $\mathcal{H}$ with a cyclic unit vector v satisfying (2.1) for every $g \in G$; furthermore, if V' is a second cyclic representation of G on a Hilbert space $\mathcal{H}'$ with a cyclic unit vector v' such that $\phi(g) = \langle V'_g v', v' \rangle$ for every $g \in G$, then there exists a unitary operator $W\colon \mathcal{H} \longmapsto \mathcal{H}'$ with $Wv = v'$ and $WV_g = V'_g W$ for every $g \in G$. In other words, ϕ determines the cyclic representation V and the cyclic vector v up to unitary equivalence. If the group G is abelian and locally compact, then *Bochner's Theorem* allows us to find a probability measure μ_ϕ on the dual group $\hat{G}$ of G such that

$$\phi(g) = \int_{\hat{G}} \chi(g) \, d\mu_\phi(\chi) = \hat{\mu}_\phi(g) \tag{2.2}$$

for every $g \in G$. Conversely, if μ is a probability measure on $\hat{G}$, then the function $\hat{\mu}\colon G \longmapsto \mathbb{C}$ in (2.2) is continuous and positive definite on G.

One can reconcile the equations (2.1)–(2.2) by setting $\mathcal{H} = L^2(\hat{G}, \mu_\phi)$, and by considering the unitary representation V of G on $\mathcal{H}$ given by

$$(V_g f)(\chi) = \chi(g) f(\chi)$$

for every $f \in \mathcal{H}$, $g \in G$ and $\chi \in \hat{G}$. The constant function $v = 1 \in \mathcal{H}$ is cyclic for V, and

$$\langle V_g v, v \rangle = \int_{\hat{G}} \chi(g) \, d\mu_\phi(\chi) = \phi(g)$$

for every $g \in G$.

Now suppose that we have two continuous, unitary, cyclic representation V, V' of G on Hilbert spaces $\mathcal{H}, \mathcal{H}'$ with cyclic vectors v, v', and put

$$\phi(g) = \langle V_g v, v \rangle, \qquad \phi'(g) = \langle V'_g v', v' \rangle$$

for every $f \in G$. The tensor product representation $V \otimes V'$ on $\mathcal{H} \otimes \mathcal{H}'$, restricted to the cyclic subspace of $\mathcal{H} \otimes \mathcal{H}'$ generated by $v \otimes v'$, satisfies that

$$\langle (V_g \otimes V'_g)\,(v \otimes v'), (v \otimes v') \rangle = \phi(g) \phi'(g) \tag{2.3}$$

for every $g \in G$. This shows that the pointwise product $\phi\phi'$ of two continuous, positive definite functions is again positive definite and determines the tensor product of the representations arising from ϕ and ϕ'.

If G is abelian and μ, μ' are probability measures on $\hat{G}$, then product of the positive definite functions $\hat{\mu}, \hat{\mu}'$ in (2.2) is again positive definite, and

$$\hat{\mu}\hat{\mu}' = \widehat{\mu * \mu'}, \tag{2.4}$$

where $\mu * \mu'$ is the *convolution* of μ and μ', i.e. the probability measure on $\hat{G}$ with

$$\int_{\hat{G}} f d(\mu * \mu') = \int_{\hat{G}} \int_{\hat{G}} f(\chi\chi')\, d\mu(\chi)\, d\mu'(\chi')$$

for every bounded Borel function $f\colon \hat{G} \longmapsto \mathbb{C}$.

A probability measure μ on $\hat{G}$ is *infinitely divisible* if there exists, for every $n \geq 1$, a probability measure ν_n on $\hat{G}$ with $\mu = \nu_n^{*n} = \nu_n * \cdots * \nu_n$. If μ on $\hat{G}$ is infinitely divisible, then (2.4) shows that $\hat{\mu} = (\hat{\nu}_n)^n$ for every $n \geq 1$, i.e. that $\hat{\mu}$ is, for every $n \geq 1$, the n-th power of a positive definite function, and is again *infinitely divisible*. Since the description of all infinitely divisible probability measures on $\mathbb{R}$ (or, more generally, on Polish groups) is one of the central questions of classical probability theory, the infinitely divisible positive definite functions on a locally compact, abelian group G are of great importance: they are described by the *Lévy-Khinchine Formula* (cf. [9] for $G = \mathbb{R}$ and [20] for arbitrary locally compact, second countable groups G).

If G is an arbitrary Polish group, then (2.3) implies that every infinitely divisible positive definite function ϕ on G defines a continuous, unitary, cyclic representation V of G on a complex, separable Hilbert space $\mathcal{H}$ which can, for every $n \geq 1$, be written as the tensor product of n copies of some other cyclic representation V_n of G on some Hilbert space $\mathcal{H}_n$ (cf. (2.3)). R. Streater ([36]), motivated by papers by Araki and Woods ([1], [2]), had shown that such infinitely divisible positive definite functions and the corresponding 'infinitely divisible representations' allow the construction of factorisable representations of current groups and lead to interesting mathematical models of quantum fields. Partha's earlier derivation of the Lévy-Khinchine formula for positive definite functions on locally compact, second countable, abelian groups led him to be immediately captivated by the problem of finding an analogous formula for arbitrary locally compact groups. When I arrived in Manchester Partha had already achieved some partial answers: a Lévy-Khinchine formula for infinitely divisible positive definite functions on compact groups ([22]), and a first glimpse of the connection between infinitely divisible positive definite functions on G and 1-cocycles for unitary representations of G which was soon to provide the desired general Lévy-Khinchine formula.

Let G be a Polish group and $\phi\colon G \longmapsto \mathbb{C}$ an infinitely divisible positive definite function. Then there exists a continuous, unitary representation U of G on a complex, separable Hilbert space $\mathcal{H}$ and a continuous map $a\colon G \longmapsto \mathcal{H}$ with the following properties for every $g_1, g_2 \in G$:

(i) $a(g_1 g_2) = U_{g_1} a(g_2) + a(g_1)$,
(ii) $\phi(g_1 g_2)\phi(g_1^{-1})\phi(g_2^{-1}) = \exp\big(\langle a(g_2), a(g_1^{-1})\rangle\big)$.

Any reader familiar with elementary cohomology will recognise (i) as a *cocycle equation*: a is a 1-cocycle of G with values in the G-module $\mathcal{H}$, where G acts on $\mathcal{H}$ by U. A 1-cocycle $a\colon G \longmapsto \mathcal{H}$ is a *coboundary* and is regarded as boring if there exists a $b \in \mathcal{H}$ with

$$a(g) = U_g b - b \tag{2.5}$$

for every $g \in G$. From (i) it is clear that ϕ determines the representation W and the cocycle a essentially uniquely. Unfortunately, not every cocycle $a\colon G \longmapsto \mathcal{H}$ admits a continuous map $\phi\colon G \longmapsto \mathbb{C}$ satisfying (ii); however, if such a map exists, then it is positive definite and infinitely divisible, and it is determined uniquely up to multiplication by a continuous homomorphism of G into $\mathbf{S}$. When Partha told me about this problem I had just read his notes [21] on projective representations, and we realised that the obstruction to solving (ii) was a 2-cocycle $s\colon G \times G \longmapsto \mathbb{R}$: indeed, if

$$L(g_1, g_2) = \langle a(g_2), a(g_1^{-1})\rangle,$$

then

$$\begin{aligned}\Re L(g_1,g_2) &= -\frac{1}{2}(\|a(g_1g_1)\| - \|a(g_1)\| - \|a(g_2)\|),\\ L(g_1,g_2)+L(g_1g_2,g_3) &= L(g_1,g_2g_3)+L(g_2,g_3)\end{aligned} \tag{2.6}$$

for every $g_1, g_2, g_3 \in G$, where $\Re$ stands for the real part. The second equation in (2.6) is the familiar equation for a 2-*cocycle* of G with values in the trivial G-module $\mathbb{C}$. In particular, (ii) has a continuous solution ϕ if and only if the map $L\colon G\times G \longmapsto \mathbb{R}$ is of the form

$$L(g_1,g_2) = \psi(g_1g_2) - \psi(g_1) - \psi(g_2) \tag{2.7}$$

for some continuous (or, indeed, Borel) map $\psi\colon G \longmapsto \mathbb{C}$, i.e. if and only if L is a continuous 2-*coboundary*. Although I don't wish to go into details I should just remark that in the absence of such a map ψ the cocycle a determines a *factorisable projective representation* of G on a Hilbert space $\mathcal{H}$. For background and details see [23]–[25].

For the purpose of this discussion I shall ignore the problem of the existence of the map ψ in (2.7). Then there is essentially a one-to-one correspondence between 1-cocycles of continuous, unitary representations of G and continuous positive definite functions on G. In order to obtain an analogue of the Lévy-Khinchine formula for all continuous, infinitely divisible, positive definite functions on G one should thus aim for a 'formula' for all 1-cocycles of continuous, unitary representations of G; furthermore, if G is locally compact, then the usual decomposition techniques allow one to restrict one's attention to *irreducible* representations of G. Such an analysis can indeed be carried out for those locally compact, second countable groups G whose irreducible, unitary representations are well understood and leads to explicit Lévy-Khinchine formulae for these groups (cf. [25] for some examples).

Let U be a continuous, irreducible representation of a Polish group G on a complex, separable Hilbert space $\mathcal{H}$, and let $Z^1(G,U,\mathcal{H})$ be the additive group of all 1-cocycles $a\colon G \longmapsto \mathcal{H}$ (i.e. of all continuous maps $a\colon G \longmapsto \mathcal{H}$ satisfying (i)). We write $B^1(G,U,\mathcal{H}) \subset Z^1(G,U,\mathcal{H})$ for the subgroup of coboundaries (cf. (2.5)), call two elements $a, a' \in Z^1(G,U,\mathcal{H})$ *cohomologous* if they differ by a coboundary, and denote by $H^1(G,U,\mathcal{H}) = Z^1(G,U,\mathcal{H})/B^1(G,U,\mathcal{H})$ the *first cohomology group* of G with values in the G-module $\mathcal{H}$. It turns out that, for many irreducible representations U, $H^1(G,U,\mathcal{H}) = \{0\}$. For example, if G is abelian, then every irreducible representation U of G is a continuous homomorphism from G into the unitary group of $\mathcal{H} = \mathbb{C}$ (i.e. the unit circle). If $U_h \neq 1$ for some $h \in G$ then $U_h - 1$ is invertible, so that every element $v \in \mathcal{H} = \mathbb{C}$ is of the form $v = U_h b - b$ for some $b \in \mathcal{H}$. In particular, any $a \in Z^1(G,U,\mathcal{H})$ satisfies that

$$a(h) = U_h b - b$$

for some $b \in \mathcal{H}$. If $h' \in G$ is a second element with $U_{h'} \neq 1$, then

$$a(h') = U_h' b' - b'$$

for some $b' \in \mathcal{H}$, and

$$a(hh') = U_h a(h') + a(h) = U_{hh'}b' - U_h b' + U_h b - b = U_{h'}a(h) + a(h') = U_{hh'}b - U_h b + U_h b' - b',$$

so that

$$(U_h - 1)(U_{h'} - 1)(b - b') = 0$$

and $b = b'$. The cocycle $a'\colon G \longmapsto \mathcal{H}$, defined by $a'(g) = a(g) - U_g b - b$ for every $g \in G$, is cohomologous to a and satisfies that $a'(g) = 0$ for every $g \in G$ with $U_g \neq 1$. If $G_0 = \ker(U) = \{g \in G : U_g = 1\}$, $g \in G_0$ and $h \in G \smallsetminus G_0$, then

$$0 = a'(gh) = U_g a'(h) + a'(g) = a'(h) + a'(g) = a'(g),$$

so that $a'(g) = 0$ for every $g \in G$. We have proved the following elementary result:

Proposition 2.1. *Let G be an abelian Polish group and U a continuous, irreducible representation of G on $\mathcal{H} = \mathbb{C}$. If U is nontrivial, then $H^1(G,U,\mathcal{H}) = \{0\}$. If U is trivial, then every element $a \in Z^1(G,U,\mathcal{H})$ is a continuous group homomorphism from G into $\mathcal{H}$.*

Proposition 2.1 shows that, if G is abelian, then $H^1(G,U,\mathcal{H}) \neq \{0\}$ only for the trivial representation. The next result implies that, if G is compact, $H^1(G,U,\mathcal{H}) = \{0\}$ for *every* continuous, unitary representation U on a complex, separable Hilbert space $\mathcal{H}$.

Proposition 2.2. *Let G be a Polish group, U a continuous, unitary representation of G on a complex Hilbert space $\mathcal{H}$, and $a \in Z^1(G,U,\mathcal{H})$ a cocycle which is* bounded*, i.e. which satisfies that $\sup_{g\in G} \|a(g)\| < \infty$. Then $a \in B^1(G,U,\mathcal{H})$.*

Proof. Let $C \subset \mathcal{H}$ be the closed, convex hull of the set $\{a(g) : g \in G\}$, and set

$$T_g v = U_g v + a(g)$$

for every $g \in G$ and $v \in \mathcal{H}$. Then C is invariant under the group $\{T_g : g \in G\}$ of affine transformations of $\mathcal{H}$, and the Ryll-Nardzewski fixed point theorem, applied to the weakly compact set C, shows that the T_g, $g \in G$, have a common fixed point $b \in C$. Hence

$$b = U_g b + a(g)$$

for every $g \in G$, and $a \in B^1(G,U,\mathcal{H})$. □

The Propositions 2.1–2.2 indicate that continuous, irreducible, unitary representations U with $H^1(G,U,\mathcal{H}) \neq \{0\}$ are rare. However, $H^1(G,U,\mathcal{H})$ may be nonzero for some reducible unitary representation U of G, although $H^1_V(G,\mathcal{H}_V) = \{0\}$ for each of the irreducible components V of U (where each V acts on a Hilbert space $\mathcal{H}_V$): there may exist, for a given $a \in Z^1(G,U,\mathcal{U})$, a bigger linear space $\mathcal{L} \supset \mathcal{H}$, to which the representation U can be extended, such that $a(g) = U_g b - b$ for some $b \in \mathcal{L} \smallsetminus \mathcal{H}$. For obvious reasons such cocycles are sometimes called *generalised coboundaries*. If G is locally compact and abelian, these generalised coboundaries give rise to the 'Poisson part' of the Lévy-Khinchine formula, whereas the nonzero cocycles (= homomorphisms) coming from the trivial representation lead to the 'Gaussian part' of the formula (cf. [25]).

The connection between infinitely divisible positive definite function and their associated 1-cocycles of unitary representations, with certain constructions in quantum field theory is the role they play in the construction of representations of *current groups*. In order to describe this construction we assume for simplicity that M is a compact manifold with a Borel probability measure μ, G a Polish group, U a continuous, unitary representation of G on a complex Hilbert space $\mathcal{H}$, and $a\colon G \longmapsto \mathcal{H}$ an element of $Z^1(G,U,\mathcal{H})$ for which there exists a continuous map $\psi\colon G \longmapsto \mathbb{C}$ with

$$L(g_1,g_2) = \langle a(g_2), a(g_1^{-1})\rangle = \psi(g_1g_2) - \psi(g_1) - \psi(g_2)$$

for every $g_1, g_2 \in G$ (cf. (2.6)–(2.7)).

Denote by $\Gamma = C(M,G)$ the group of continuous maps $\gamma\colon M \longmapsto G$ from M to G, furnished with pointwise multiplication and the topology of uniform convergence, and define a continuous, unitary representation $\mathbf{U}$ of Γ on Hilbert space $\boldsymbol{\mathcal{H}} = \int_M^{\oplus} \mathcal{H}\, d\mu = L^2_\mu(M,\mathcal{H})$ of square-integrable maps $f\colon M \longmapsto \mathcal{H}$ by setting

$$(\mathbf{U}_\gamma f)(x) = U_{\gamma(x)} f(x) \tag{2.8}$$

for every $\gamma \in \Gamma$, $x \in M$, and $f \in \boldsymbol{\mathcal{H}}$. The cocycle a gives rise to a cocycle $\mathbf{a}\colon \Gamma \longmapsto \boldsymbol{\mathcal{H}}$ for $\mathbf{U}$ with

$$\mathbf{a}(\gamma)(x) = a(\gamma(x)), \tag{2.9}$$

and the continuous map $\boldsymbol{\psi}\colon \Gamma \longmapsto \mathbb{C}$, where

$$\boldsymbol{\psi}(\gamma) = \int_M \psi(\gamma(x))\,d\mu(x) \tag{2.10}$$

for every $\gamma \in \Gamma$, satisfies that

$$\langle \mathbf{a}(\gamma_2), \mathbf{a}(\gamma_1^{-1})\rangle = \boldsymbol{\psi}(\gamma_1\gamma_2) - \boldsymbol{\psi}(\gamma_1) - \boldsymbol{\psi}(\gamma_2) \tag{2.11}$$

for every $\gamma_1, \gamma_2 \in \Gamma$.

The map

$$\boldsymbol{\phi} = \exp \boldsymbol{\psi}\colon \Gamma \longmapsto \mathbb{C} \tag{2.12}$$

is continuous, positive definite and infinitely divisible, and is *factorisable* in the following sense: let, for every nonempty, open subset $\mathcal{O} \subset M$,

$$\Gamma_{\mathcal{O}} = \{\gamma \in \Gamma : \gamma(x) = 1_G \ \text{ for every } \ x \in M \smallsetminus \mathcal{O}\};$$

if $\mathcal{O}_1, O_2$ are disjoint, open subsets of M then

$$\Gamma_{\mathcal{O}_1\cup\mathcal{O}_2} \cong \Gamma_{\mathcal{O}_1} \times \Gamma_{\mathcal{O}_2},$$

and (2.9)–(2.11) show that

$$\boldsymbol{\phi}(\gamma_1\gamma_2) = \boldsymbol{\phi}(\gamma_1)\boldsymbol{\phi}(\gamma_2) \tag{2.13}$$

whenever $\gamma_i \in \Gamma_{\mathcal{O}_i}$, $i = 1, 2$. If W is the cyclic representation of Γ defined by $\boldsymbol{\phi}$, then the 'factorisability' of $\boldsymbol{\phi}$ expressed by (2.13) implies that the restriction of W to $\Gamma_{\mathcal{O}_1\cup\mathcal{O}_2} \cong \Gamma_{\mathcal{O}_1} \times \Gamma_{\mathcal{O}_2}$ is unitarily equivalent to the representation $(\gamma_1, \gamma_2) \mapsto W_{\gamma_1} \otimes W_{\gamma_2}$, $(\gamma_1, \gamma_2) \in \Gamma_{\mathcal{O}_1} \times \Gamma_{\mathcal{O}_2}$.

This 'factorisability' of W means that the operator algebras generated by the sets $\{W_\gamma : \gamma \in \Gamma_{\mathcal{O}_i}\}$, $i = 1, 2$, are *independent*. For physical applications the local independence condition (2.13) only has to hold if the closures of the sets $\mathcal{O}_i$, $i = 1, 2$, are disjoint (physicists would also prefer the manifold M to be noncompact, but this is not a problem). In fact, for the factorisable representation W of Γ arising from the above construction the sets $\{W_\gamma : \gamma \in \Gamma_{\mathcal{O}_i}\}$, $i = 1, 2$ are *too* independent, and one would like to modify the above construction to obtain, say, representations of the group $C^\infty(M, \Gamma) \subset \Gamma$ of smooth maps from M to Γ which are still factorisable in a slightly weaker sense, and which cannot be extended to all of Γ. Partha and I made a further attempt in this direction in [27], but the general problem of constructing *irreducible* and physically relevant classes of factorisable representations still remains open (cf. also [35]).

3. Cohomology of ergodic transformation groups

In order to understand the nature of the representation W of Γ appearing at the end of the preceding section (cf. (2.1) and (2.12)), let us assume that G is a Polish group, U a continuous, orthogonal representation of G on a real, separable Hilbert space $\mathcal{H}'$, and $a\colon G \longmapsto \mathcal{H}'$ a 1-cocycle for U (cf. (i)). We consider the complex Hilbert space $\mathcal{H} = \mathcal{H}' + i\mathcal{H}'$, extend U linearly to a unitary representation of G on $\mathcal{H}$, choose an orthonormal basis $(\mathbf{e}_i,\ i = 0, 1, \dots)$, of $\mathcal{H}'$, and define a linear embedding $\eta'\colon \mathcal{H}' \longmapsto X = \mathbb{R}^{\mathbb{N}}$ by setting

$$\eta'(v) = (v_0, v_1, v_2, \dots) \in \mathbb{R}^{\mathbb{N}}$$

for every $v = \sum_{i\ge 0} v_i\mathbf{e}_i \in \mathcal{H}'$. Let ξ be the Gaussian probability measure on $\mathbb{R}$ with mean zero and variance one, and let $\nu = \xi^{\mathbb{N}}$ be the corresponding product measure on $X = \mathbb{R}^{\mathbb{N}}$. For every $i \ge 0$ we denote by $\pi_i\colon X \longmapsto \mathbb{R}$ the projection onto the i-th coordinate and observe that $\pi_i \in L^2(X, \nu)$, and that the map $\eta\colon \mathcal{H}' \longmapsto L^2(X, \nu)$ defined by

$$\eta(v) = \sum_{i\ge 0} v_i\pi_i,\ v = \sum_{i\ge 0} v_i\mathbf{e}_i \in \mathcal{H}',$$

is an orthogonal embedding of $\mathcal{H}'$ in $L^2(X,\nu)$. If V is an orthogonal operator on $\mathcal{H}'$ (or a unitary operator on $\mathcal{H}$ which preserves $\mathcal{H}'$), then V induces a unique measure preserving automorphism T_V of the probability space (X,ν) with

$$\eta(Vv) = \eta(v) \cdot T_V^{-1} \tag{3.1}$$

for every $v \in \mathcal{H}'$, and the unitary operator on $L^2(X,\nu)$ induced by the map $f \mapsto f \cdot T_V^{-1}$, $f \in L^2(X,\nu)$, is unitarily equivalent to the operator

$$1 \oplus V \oplus V \odot V \oplus V \odot V \odot V \oplus \cdots$$

on the Hilbert space

$$\mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H} \odot \mathcal{H} \oplus \mathcal{H} \odot \mathcal{H} \odot \mathcal{H} \oplus \cdots ,$$

where the symbol $\odot$ denotes the symmetric tensor product (cf. e.g. [11]). In particular, there exists a measure preserving action $T\colon g \mapsto T_g$ of G on (X,ν) such that

$$\eta(U_g v) = \eta(v) \cdot T_g^{-1}$$

for every $g \in G$, and T is ergodic if and only if there is no finite dimensional subspace of $\mathcal{H}'$ which is invariant under U (cf. [7]). For every $g \in G$ we set

$$a^*(g) = \eta(a(g^{-1})) \in L^2(X,\nu)$$

and observe that $a^*(g)$ is real valued, and that

$$a^*(g_1) \cdot T_{g_2} + a^*(g_2) = a^*(g_1 g_2)$$

for every $g_1, g_2 \in G$. If

$$c(g,x) = e^{2\pi i a^*(g)(x)} \tag{3.2}$$

for every $g \in G$ and $x \in X$ then we have, for every $g_1, g_2 \in G$,

$$\begin{gathered} T_{g_1} T_{g_2} x = T_{g_1 g_2} x, \\ c(g_1, T_{g_2} x) c(g_2, x) = c(g_1 g_2, x) \end{gathered} \tag{3.3}$$

for ν-a.e. $x \in X$. Moreover, if G is locally compact, then we can modify T and c on a null set so that (3.3) holds for every $g_1, g_2 \in G$ and $x \in X$ (cf. [37]). From (2.6) we know that

$$\langle a(g_2), a(g_1^{-1}) \rangle = -\frac{1}{2}(\|a(g_1 g_2)\| - \|a(g_1)\| - \|a(g_2)\|),$$

and that the map

$$\phi(g) = e^{-\|a(g)\|/2}$$

is positive definite. A calculation shows that the representation W arising from ϕ via (2.1) is (unitarily equivalent to) the representation of G on $L^2(X,\nu)$ defined by

$$W_g f = c(g^{-1}, \cdot)(f \cdot T_g^{-1}),\ f \in L^2(X,\nu), \tag{3.4}$$

restricted to the cyclic subspace of the constant function $1 \in L^2(X,\nu)$, i.e. that

$$\phi(g) = \langle c(g^{-1}, \cdot), 1 \rangle = \int c(g^{-1}, x)\, d\nu(x) \tag{3.5}$$

for every $g \in G$.

At this stage we have to introduce some terminology: let G, R be Polish groups, and let T be a measure preserving action of G on a probability space (X,ν) satisfying the first equation in (3.3). A measurable map $c\colon G \times X \longmapsto R$ is a (1-)*cocycle* for T if it satisfies the second equation in (3.3) ν-a.e., for every $g_1, g_2 \in G$. Two cocycles $c, c'\colon G \times X \longmapsto R$ are *cohomologous* if there exists a measurable map $b\colon X \longmapsto R$ such that, for every $g \in G$,

$$c'(g,x) = b(T_g x)^{-1} c(g,x) b(x) \tag{3.6}$$

for ν-a.e. $x \in X$; the map b is called the *transfer function* of (c, c'). A cocycle c is a *coboundary* if it is cohomologous to the constant cocycle $c'(g,x) = 1_R$ (in which case the map b in (3.6) is called the *cobounding function* of c), a *homomorphism* if $c(g,\cdot)$ is constant ν-*a.e.*, for every $g \in G$, and *trivial* if c is cohomologous to a homomorphism. We write $H^1(G,T,R)$ for the set of all cocycles for T with values in R and $B^1(G,T,R) \subset Z^1(G,T,R)$ for the subset of coboundaries. If R is abelian, $Z^1(G,T,R)$ is a group under pointwise addition, $B^1(G,T,R) \subset Z^1(G,T,R)$ is a subgroup, and the quotient group $H^1(G,T,R) = Z^1(G,T,R)/B^1(G,T,R)$ is called the *first (measurable) cohomology group* of T with values in R. If R is nonabelian, $H^1(G,T,R)$ is the space of equivalence classes in $Z^1(G,T,R)$, where two cocycles $c, c' \in Z^1(G,T,R)$ are equivalent if they are cohomologous. If the group G is locally compact we can furnish $Z^1(G,T,R)$ with the topology of convergence in measure on $G \times X$. It is easy to see that $Z^1(G,T,R)$ is a Polish group in this topology; however, $B^1(G,T,R)$ is usually not a closed subgroup of $Z^1(G,T,R)$, so that $H^1(G,T,R)$ is a complicated space.

The calculation above shows that infinitely divisible representations of a Polish group G are—modulo some assumptions resulting in a technical simplification—described by a measure preserving action T of G on a probability space (X,ν) and a cocycle $c \in Z^1(G,T,\mathbb{S})$, where $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$. In particular, the factorisable representation W of $\Gamma = C(M,G)$ constructed at the end of Section 2 is of the form (3.4)–(3.5) for a suitable cocycle $c\colon \Gamma \times X \longmapsto \mathbb{S}$. We note in passing that, if c is a coboundary, then the representation (3.4) is unitarily equivalent to the representation

$$W_g' f = f \cdot T_g^{-1},\ f \in L^2(X,\nu),$$

which is one of the many reasons why coboundaries are uninteresting.

If we start with an arbitrary cocycle $c \in Z^1(G,T,\mathbb{S})$ (or $c \in Z^1(\Gamma,T,\mathbb{S})$, where T is the action of Γ on X satisfying the analogue of (3.1) for the representation $\mathbf{U}$ in (2.8)), then the positive definite function ϕ in (3.5) will in general not be infinitely divisible (resp. factorisable). The cocycle c in (3.2) is obviously of a very special form, and one might hope for $Z^1(\Gamma,T,\mathbb{S})$ to contain some other cocycles which would lead to a representation W in (3.4) with physically relevant properties. This led me to become interested in cocycles for (ergodic) group actions with values in $\mathbb{S}$ and, more generally, in a Polish group R. Although this excursion into cohomology of ergodic transformation groups didn't teach me anything useful about representations of current groups it led to some other quite intriguing problems which have kept me busy—with interruptions—until today.

4. More about the cohomology of ergodic transformation groups

My serious interest in cocycles of the form (3.3) began in 1974 as a result of conversations with Bill Parry, who was also working on cohomological problems in ergodic theory. When I visited the Indian Statistical Institute in Delhi 1975/76 I wrote some lecture notes on cocycles of (countable) ergodic transformation groups ([29]), and during the following years I explored further the measurable cohomology of ergodic group actions and equivalence relations.

In order to avoid certain measurability problems, let us assume that G is a countably infinite group, T a measure preserving, ergodic action of G on a (Lebesgue) probability space (X,μ), and R a Polish group. If T is *free*, i.e. if $\mu(\{x \in X : T_g x = x\}) = 0$ whenever $g \neq 1_G$, then the sets $B^1(G,T,R) \subset Z^1(G,T,R)$ are determined completely by the *orbits* of T: in other words, if T' is a free, measure preserving action of another group G' on (X,μ) with the same orbits (up to null sets), then there is a canonical bijection of the sets $Z^1(G,T,R)$ and $Z^1(G',T',R)$ which carries $B^1(G,T,R)$ to $B^1(G',T',R)$.

In [5]–[6], H.A. Dye had shown that any two ergodic, measure preserving actions of countably infinite abelian groups on probability spaces generate the same orbits, and [3] shows that this holds even for countably infinite amenable groups. In particular, if we are interested in the cohomology of actions of an amenable group G, then we may as well assume that $G = \mathbb{Z}$, i.e. that the action T is given by the powers of a single ergodic, measure preserving transformation, which will again be denoted by T. In this case any cocycle $c \in Z^1(\mathbb{Z}, T, R)$ is completely determined by the measurable map $f = c(1, \cdot)\colon X \longmapsto R$, and

$$c(n,x) = \begin{cases} f(T^{n-1}x) \cdots f(x) & \text{for } n \geq 1, \\ 1_R & \text{for } n = 0, \\ c(-n, T^n x)^{-1} & \text{for } n < 0. \end{cases} \tag{4.1}$$

The cocycle c is trivial if and only if there exist a constant $r \in R$ and a measurable map $b\colon X \longmapsto R$ with

$$f(x) = c(1,x) = b(Tx)^{-1} r b(x) \tag{4.2}$$

for μ-a.e. $x \in X$. If $r = 1_R$ in (4.2), then c is a coboundary.

As an easy application of *Rokhlin's lemma* one obtains that $B^1(\mathbb{Z}, T, R)$ is dense, but not closed in $Z^1(\mathbb{Z}, T, R)$. In particular, $H^1(\mathbb{Z}, T, R)$ is large and messy, and by using orbit structures we see that, if T is a free, ergodic, measure preserving action of a countable, amenable group G, then $H^1(G, T, R)$ reflects nothing of the nature of G or T (cf. [28]). However, the richness of $H^1(G, T, R)$ for such group actions still makes it conceivable that one can construct interesting representations of the form (3.4)–(3.5) (cf. [10]). If G is not amenable, however, $H^1(\mathbb{Z}, T, R)$ may be very small: for example, if G is a Kazhdan group (such as $SL(n, \mathbb{Z})$, $n \geq 3$), then $H^1(G, T, \mathbb{R}) = \{0\}$ and $H^1(G, T, \mathbb{S})$ is countable (cf. [4], [30]). The cohomology of measure preserving actions of free groups tends to have an intermediate behaviour ([30]).

Since cocycles $c \in Z^1(G, T, R)$ are used in all sorts of constructions, many of which can be viewed as 'deformations' of the action T (cf. [19], [15]), the absence or scarcity of nontrivial cocycles in $Z^1(G, T, R)$ has a natural interpretation as a 'rigidity property' of the G-action T. The most celebrated result in this direction is *Zimmer's Rigidity Theorem* (cf. [38] for a full account): if T, T' are two free, ergodic, finite measure preserving actions of a group like $G = SL(3, \mathbb{Z})$ which give rise to the same orbits (modulo a null set), then there exists an automorphism α of G such that $T_g = T'_{\alpha(g)}$ for every $g \in G$.

Since T' has the same orbits as T there exists a measurable map $c\colon G \times X \longmapsto G$ with $T_g x = T'_c(g,x)x$ μ-a.e., and an elementary calculation shows that $c \in Z^1(G, T, G)$. Zimmer's theorem implies in particular that the cocycle c is trivial.

5. Cohomological rigidity of abelian group actions

Although it had become clear in the early 1980's that for measure preserving, ergodic actions T of 'exotic' groups like Kazhdan groups, the cohomology groups $H^1(G, T, R)$ become very small if R is, say, locally compact and abelian, there was no indication of any comparable phenomenon if G is abelian. It was known, however, that if one imposes stronger regularity conditions on the cocycles, the picture could change dramatically.

Assume, for example, that $X = \mathbb{R}/\mathbb{Z}$, $G = \mathbb{Z}$, and that T is the $\mathbb{Z}$-action on X defined by (the powers of) an irrational rotation R_α, where $R_\alpha x = x + \alpha \pmod 1$ for every $x \in X$. The Lebesgue measure λ on X is invariant and ergodic under R_α. As we saw in (4.1), every cocycle $c\colon \mathbb{Z} \times X \longmapsto R$ is determined completely by the measurable map $f = c(1, \cdot)\colon X \longmapsto R$, and we say that c is continuous or smooth if $f = c(1, \cdot)$ is continuous or smooth.

We know already that 'most' elements of $Z^1(\mathbb{Z}, R_\alpha, \mathbb{R})$ are nontrivial. However, if α is well behaved (e.g. an algebraic irrational), then every C^∞ cocycle $c\colon \mathbb{Z} \times X \longmapsto \mathbb{R}$ is trivial, and by looking at Fourier coefficients we see that the cobounding function b in (4.2) is in this case again C^∞.

For a second example, consider an irreducible, aperiodic 0-1-matrix $P = (P(i,j),\ 0 \leq i,j \leq n-1)$, and consider the shift space

$$X_P = \{x = (x_k) \in \{0,\dots,n-1\}^{\mathbb{Z}} : P(x_k, x_{k+1}) = 1 \text{ for every } k \in \mathbb{Z}\}$$

with the shift transformation $\sigma = \sigma_P$ given by

$$(\sigma(x))_k = x_{k+1}$$

for every $x = (x_k) \in X_P$. If μ is a nonatomic, shift-invariant and ergodic probability measure on X_P, then 'most' measurable functions $f\colon X_P \longmapsto \mathbb{R}$ again define nontrivial cocycles via (4.1), and $B^1(\mathbb{Z},\sigma,\mathbb{R})$ is dense in $Z^1(\mathbb{Z},\sigma,\mathbb{R})$. However, if we assume Hölder continuity, then something interesting happens: if we write $Z^1_H(\mathbb{Z},\sigma,\mathbb{R})$ for the group of real valued, Hölder continuous cocycles for the shift action of $\mathbb{Z}$ on X_P, then $B^1(\mathbb{Z},\sigma,\mathbb{R}) \cap Z^1_H(\mathbb{Z},\sigma,\mathbb{R})$ is a proper *closed subgroup* of $Z^1_H(\mathbb{Z},\sigma,\mathbb{R})$ in either the Hölder topology or the topology of uniform convergence, and

$$B^1(\mathbb{Z},\sigma,\mathbb{R}) \cap Z^1_H(\mathbb{Z},\sigma,\mathbb{R}) = B^1_H(\mathbb{Z},\sigma,\mathbb{R})$$

in the sense that every $c \in B^1(\mathbb{Z},\sigma,\mathbb{R}) \cap Z^1_H(\mathbb{Z},\sigma,\mathbb{R})$ is of the form

$$c(k,x) = b(\sigma^k(x)) - b(x)$$

for a Hölder continuous function $b\colon X_P \longmapsto \mathbb{R}$.

The reason for this is *Livshitz' Theorem* ([18]): a cocycle $c \in Z^1_H(\mathbb{Z},\sigma,\mathbb{R})$ is a coboundary if and only if, for every periodic point $x \in X_P$ with $\sigma^k(x) = x$, say, the 'weight'

$$w_c^{(k)}(x) = c(k,x) = \sum_{l=0}^{k-1} f(\sigma^l(x)) \tag{5.1}$$

is equal to zero, where $f = c(1,\cdot)$. Livshitz' theorem remains correct if $\mathbb{R}$ is replaced by any complete, separable, bi-invariant metric group (R,ρ) (*bi-invariant* means that $\rho(h_1h_3,h_2h_3) = \rho(h_1,h_2) = \rho(h_3h_1,h_3h_2)$ for all $h_1,h_2,h_3 \in H$). For any such group R it is easy to construct by hand nontrivial elements of $Z^1_H(\mathbb{Z},\sigma,H)$, so that $B^1_H(\mathbb{Z},\sigma,R)$ is indeed a closed, nontrivial subgroup of $Z^1_H(\mathbb{Z},\sigma,R)$.

Another class of closely related examples to which Livshitz' theorem can be applied are hyperbolic toral automorphisms. Let $X = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, and let $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, acting linearly on X. If (R,ρ) is a complete, separable, bi-invariant metric group, and if $Z^1_H(\mathbb{Z},A,R)$ is the set of R-valued, Hölder continuous 1-cocycles for the $\mathbb{Z}$-action on X defined by A, then $B^1_H(\mathbb{Z},A,R) = B^1(\mathbb{Z},A,R) \cap Z^1_H(\mathbb{Z},A,R)$ is a proper, closed subset of $Z^1_H(\mathbb{Z},A,R)$ in either the Hölder topology or the topology of uniform convergence, and the set of nontrivial elements is dense in $Z^1_H(\mathbb{Z},A,R)$. If R is a Lie group, Livshitz' theorem even implies the existence of nontrivial cocycles $c\colon \mathbb{Z} \times X \longmapsto R$ which are C^∞.

In Section 4 we saw that the measurable cohomology of $\mathbb{Z}$-actions looks exactly like the measurable cohomology of, say, $\mathbb{Z}^2$-actions. However, the Hölder cohomology of $\mathbb{Z}$-actions can behave quite differently from that of $\mathbb{Z}^d$-actions for $d \geq 2$. The first example of this phenomenon which came to my attention was due to J.W. Kammeyer ([12], [13]), who proved that every continuous cocycle for the shift-action of $\mathbb{Z}^d$ on the full d-dimensional k-shift with values in $\mathbb{Z}/2\mathbb{Z}$ is continuously cohomologous to a homomorphism from $\mathbb{Z}^d$ to $\mathbb{Z}/2\mathbb{Z}$; in contrast, the one-dimensional k-shift has many nontrivial cocycles with values in $\mathbb{Z}/2\mathbb{Z}$. A second instance of this phenomenon appeared in two papers by A. Katok and R.J. Spatzier ([16], [17]): every real-valued Hölder cocycle for an Anosov action of $\mathbb{Z}^d$ on a compact manifold is Hölder-cohomologous to a homomorphism, whereas a single Anosov map has a rich supply of nontrivial real-valued Hölder cocycles.

In order to describe the appropriate notion of Hölder continuity we assume that $d \geq 1$, write $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ for the Euclidean norm and inner product on $\mathbb{Z}^d \subset \mathbb{R}^d$, and assume that T is a continuous action of $\mathbb{Z}^d$ on a compact, metric space (X,δ). The action T is *expansive* if there exists an $\varepsilon>0$ such that $\sup_{\mathbf{m}\in\mathbb{Z}^d}\delta(T_{\mathbf{m}}x,T_{\mathbf{m}}x')>\varepsilon$ whenever x,x' are distinct elements of X, and *topologically mixing* if there exists, for every pair $\mathcal{O}_1,\mathcal{O}_2$ of nonempty, open subsets of X, an integer M with $T_{\mathbf{m}}(\mathcal{O}_1)\cap\mathcal{O}_2\neq\varnothing$ whenever $\mathbf{m}\in\mathbb{Z}^d$ and $\|\mathbf{m}\|>M$. We fix an expansive and topologically mixing continuous action T of $\mathbb{Z}^d$ on X and put $\mathbf{B}(r)=\{\mathbf{m}\in\mathbb{Z}^d:\|\mathbf{m}\|\le r\}$ for every $r\ge 0$. Let (R,ρ) be a complete, separable, bi-invariant metric group. For every continuous function $h\colon X\longmapsto R$ and every $\varepsilon,r\ge 0$, we set

$$\omega_r^{\delta,\rho}(h,T,\varepsilon)=\sup_{\{(x,x')\in X\times X:\delta(T_{\mathbf{m}}(x),T_{\mathbf{m}}(x'))<\varepsilon\ \text{ for every }\ \mathbf{m}\in\mathbf{B}(r)\}}\rho(h(x),h(x')). \tag{5.2}$$

The function h has *T-summable variation* if there exists an $\varepsilon>0$ such that

$$\omega^{\delta,\rho}(h,T,\varepsilon)=\sum_{r=1}^{\infty}\omega_r^{\delta,\rho}(h,T,\varepsilon)<\infty, \tag{5.3}$$

and h is *T-Hölder* if there are constants $\varepsilon,\omega'>0$ and $\omega\in(0,1)$ with

$$\omega_r^{\delta,\rho}(h,T,\varepsilon)<\omega'\omega^r \tag{5.4}$$

for every $r>0$. These notions obviously depend on ρ, but are independent of the metric δ on X. Furthermore, every T-Hölder function has T-summable variation. If the group R is discrete with the usual discrete metric, then a function $h\colon X\longmapsto R$ is T-Hölder if and only if it is continuous.

If there is no danger of confusion we suppress the prefix T- and simply speak of Hölder functions and functions with summable variation.

Note that the Hölder structure defined by an expansive $\mathbb{Z}^d$-action T is a purely topological notion: if T' is a second continuous, expansive $\mathbb{Z}^d$-action on a compact space Y which is topologically conjugate to T, then any homeomorphism $\psi\colon X\longmapsto Y$ implementing this topological conjugacy carries the set of T-Hölder functions on X to the set of T'-Hölder functions on Y. Furthermore, if X is a compact manifold, then the Hölder structore defined by an expansive $\mathbb{Z}^d$-action coincides with the familiar one.

When I became acquainted with the papers [12]–[13] and [16]–[17] I had been working on ergodic $\mathbb{Z}^d$-actions for some time (cf. [31]), and became very interested in exploring further this and other manifestations of rigidity of certain $\mathbb{Z}^d$-actions. The results due to Kammeyer and Katok-Spatzier show that expansive $\mathbb{Z}^d$-actions need no longer have nontrivial Hölder cocycles with values in a prescribed group G. Upon closer inspection a very intriguing picture begins to emerge: expansive $\mathbb{Z}^d$-actions appear to be very particular about the kinds of groups in which they can have nontrivial Hölder cohomology. Kammeyer's result can be extended to the statement that every Hölder cocycle on the d-dimensional k-shift with values in a complete, separable, bi-invariant metric group (R,ρ) is trivial (cf. [32]). On the other hand, if $X\subset(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$ is the closed, shift-invariant subgroup

$$\begin{aligned}X=\{x=(x_{(k,l)})\in(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}:x_{(k,l)}+x_{(k+1,l)}+x_{(k,l+1)}=0 \pmod 2\\ \text{for every }\ (k,l)\in\mathbb{Z}^2\}\end{aligned}$$

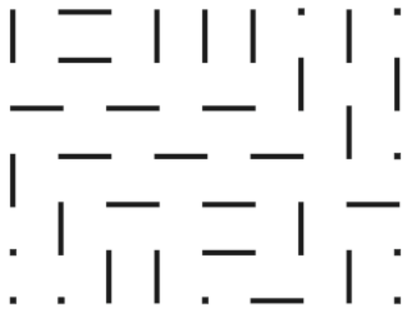
and σ' is the shift-action

$$(\sigma'_{\mathbf{m}}(x))_{\mathbf{n}}=x_{\mathbf{m}+\mathbf{n}} \tag{5.5}$$

of $\mathbb{Z}^2$ on X, then every Hölder cocycle $c\colon\mathbb{Z}^2\times X\longmapsto R$ with values in a complete, separable, abelian, bi-invariant metric group R is trivial, but σ' has nontrivial Hölder cocycles with values in certain finite groups ([32]–[33]). In other words, the higher dimensional full shifts dislike all bi-invariant metric groups, whereas σ' only dislikes abelian groups. There are many mysterious instances of this phenomenon, and a full account would exceed the space available here. For anybody interested in a bit of further reading I should mention the paper [8], in which a 'fundamental group' is associated with every expansive $\mathbb{Z}^2$-action on a compact, zero-dimensional space. These fundamental groups appear to be linked with the possible ranges of nontrivial cocycles of the $\mathbb{Z}^2$-action in a way yet to be determined precisely (cf. [34]).

Let me mention the 2-dimensional domino tilings as another example.

The *domino tilings* are a two-dimensional shift of finite type consisting of all coverings of $\mathbb{Z}^2$ by domino (or dimers), where each domino covers two horizontally or vertically adjacent points in $\mathbb{Z}^2$, and where each lattice point is covered by exactly one domino (cf. [14], [32]). Here is a typical partial domino tiling of $\mathbb{Z}^2$:



The space X of all domino tilings may be regarded as a closed, shift-invariant subset of the two-dimensional 4-shift $\{0,1,2,3\}^{\mathbb{Z}^2}$ by interpreting 0 and 2 as the right and left endpoints of a horizontal domino, and 1 and 3 as the top and bottom endpoints of a vertical domino. If σ is the restriction to $X \subset \{0,1,2,3\}^{\mathbb{Z}^2}$ of the shift-action (5.5), then one can show that every Hölder cocycle on X with values in a complete, separable, abelian, bi-invariant metric group is trivial. However, there exists a distinguished nontrivial, continuous cocycle c^* with values in a central extension $\bar{R}$ of $\mathbb{Z}^2$ by the discrete matrix group

$$R = \left\{ \left(\begin{smallmatrix} \pm 1 & k \\ 0 & \pm 1 \end{smallmatrix} \right) : k \in \mathbb{Z} \right\}$$

(cf. [32]) such that every continuous cocycle c with values in a discrete group G is cohomologous to the composition of c^* with some group homomorphism $\theta \colon \bar{R} \longmapsto G$ (cf. [34]). The fundamental group of the domino tilings is $\mathbb{Z}$ (cf. [8]), of which R is an extension by $(\mathbb{Z}/2\mathbb{Z})^2$.

6. Conclusion

Although the cohomology of ergodic transformation groups plays a role in many construction in ergodic theory (such as factor maps, skew products, quasi-invariant and invariant measures, velocity changes), the measurable first cohomology of a group action is an intriguing, but not particularly appealing object. For example, all finite measure preserving, ergodic actions of countable (or of locally compact, second countable, unimodular) amenable groups have a messy and essentially indistinguishable cohomology (cf. [28]). In order to find cohomological properties which are somehow 'canonical' and contain specific information about the action, one either has to go to groups like $SL(3,\mathbb{R})$, or impose certain regularity conditions on the cocycles under discussion. These conditions may arise from a variety of contexts: physical considerations lead to the study of factorisability (which can be viewed as a regularity condition), the investigation of finite-to-one topological factor maps leads to continuous cocycles with values in finite groups, and geometrical problems lead to smooth or Hölder cocycles. In some of these settings the cohomology becomes manageable and can be shown to carry specific and nontrivial information about the group action. In other problems (like that of factorisable representations) we simply don't know enough at this stage to make an educated guess about effective regularity conditions.

I haven't been able to include any technical details or proofs in this account; interested readers are referred to the short list of references. Maybe this little account can persuade someone to become interested in these problems and to start pursuing them with the kind of enjoyment and pleasurable curiosity which has always characterised Partha's work. If so, then I can regard myself lucky to have passed on a little bit of Partha's infectuous spirit!

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