ISOMORPHISM RIGIDITY OF IRREDUCIBLE ALGEBRAIC $\mathbb{Z}^d$-ACTIONS

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ABSTRACT. An *irreducible algebraic* $\mathbb{Z}^d$*-action* α on a compact abelian group X is a $\mathbb{Z}^d$-action by automorphisms of X such that every closed, α-invariant subgroup $Y \subsetneq X$ is finite. We prove the following result: if $d \geq 2$, then every measurable conjugacy between irreducible and mixing algebraic $\mathbb{Z}^d$-actions on compact zero-dimensional abelian groups is affine. For irreducible, expansive and mixing algebraic $\mathbb{Z}^d$-actions on compact connected abelian groups the analogous statement follows essentially from a result by Katok and Spatzier on invariant measures of such actions (cf. [4] and [3]). By combining these two theorems one obtains isomorphism rigidity of all irreducible, expansive and mixing algebraic $\mathbb{Z}^d$-actions with $d \geq 2$.

1. Introduction

Let $d \geq 1$. An *algebraic* $\mathbb{Z}^d$*-action* $\alpha\colon \mathbf{n} \mapsto \alpha^{\mathbf{n}}$ on a compact abelian group X is a $\mathbb{Z}^d$-action by continuous automorphisms of X. An algebraic $\mathbb{Z}^d$-action α on X is *expansive* if there exists an open set $\mathcal{O} \subset X$ with $\bigcap_{\mathbf{n}\in\mathbb{Z}^d} \alpha^{-\mathbf{n}}(\mathcal{O}) = \{0_X\}$, where 0_X is the identity element of X, and *irreducible* if every closed, α-invariant subgroup $Y \subsetneq X$ is finite. The action α is *ergodic* or *mixing* if the Haar measure λ_X of X is ergodic or mixing under α.

For every closed, α-invariant subgroup $Y \subset X$ we denote by α_Y and $\alpha_{X/Y}$ the $\mathbb{Z}^d$-action induced by α on Y and X/Y, respectively.

Perhaps the most familiar examples of expansive algebraic $\mathbb{Z}^d$-actions arise from commuting hyperbolic toral automorphisms. Another class of such actions are the *group shifts* appearing in coding theory: let A be a finite abelian group, and let $\Omega = A^{\mathbb{Z}^d}$ be the compact abelian group consisting of all maps $\omega\colon \mathbb{Z}^d \longrightarrow A$, furnished with the product topology and coordinate-wise addition. We write every $\omega \in \Omega$ as $\omega = (\omega_{\mathbf{n}})$ with $\omega_{\mathbf{n}} \in A$ for every $\mathbf{n} \in \mathbb{Z}^d$ and define the *shift-action* σ of $\mathbb{Z}^d$ on Ω by

$$(\sigma_{\mathbf{m}}\omega)_{\mathbf{n}} = \omega_{\mathbf{m}+\mathbf{n}} \tag{1.1}$$

for every $\omega \in \Omega$ and $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$. Clearly, σ is an expansive algebraic $\mathbb{Z}^d$-action on Ω. A *group shift* is the restriction of the shift-action σ in (1.1) to a

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closed, shift-invariant subgroup $X \subset \Omega$. Every group shift is a d-dimensional shift of finite type (cf. [6]–[8]).

In order to classify group shifts and, more generally, expansive algebraic $\mathbb{Z}^d$-actions, we introduce certain notions of conjugacy of such actions. For $i = 1, 2$, let α_i be an algebraic $\mathbb{Z}^d$-action on compact abelian group X_i with normalized Haar measure λ_{X_i}. A surjective Borel map $\phi\colon X_1 \longrightarrow X_2$ is a *measurable factor map* of α_1 and α_2 if

$$\lambda_{X_1}\phi^{-1} = \lambda_{X_2}, \tag{1.2}$$

and if

$$\phi \circ \alpha_1^{\mathbf{n}}(x) = \alpha_2^{\mathbf{n}} \circ \phi(x) \tag{1.3}$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and λ_{X_1}-*a.e.* $x \in X_1$. A bijective measurable factor map $\phi\colon X_1 \longrightarrow X_2$ is a *measurable conjugacy* of α_1 and α_2.

A continuous surjective group homomorphism $\phi\colon X_1 \longrightarrow X_2$ is an *algebraic factor map* of α_1 and α_2 if it satisfies (1.3) for every $\mathbf{n} \in \mathbb{Z}^d$ and $x \in X_1$. A bijective algebraic factor map $\phi\colon X_1 \longrightarrow X_2$ is an *algebraic conjugacy* of α_1 and α_2.

The action α_2 is a *measurable* (resp. *algebraic*) *factor* of α_1 if there exists a measurable (resp. algebraic) factor map $\phi\colon X_1 \longrightarrow X_2$ of α_1 and α_2. The actions α_1, α_2 are *measurably* (resp. *algebraically*) *conjugate* if there exists a measurable (resp. algebraic) conjugacy $\phi\colon X_1 \longrightarrow X_2$ of α_1 and α_2, and they are *weakly measurably* (resp. *weakly algebraically*) *conjugate* if each of them is a measurable (resp. algebraic) factor of the other.

Finally we call a map $\phi\colon X_1 \longrightarrow X_2$ *affine* if it is of the form

$$\phi(x) = \psi(x) + x' \tag{1.4}$$

for every $x \in X_1$, where $\psi\colon X_1 \longrightarrow X_2$ is a continuous surjective group homomorphism and $x' \in X_2$.

For $d = 1$, any algebraic $\mathbb{Z}$-action is determined by the powers of a single group automorphism α. If α is ergodic, then it is Bernoulli (cf. e.g. [1], [2], [5], [9], [11]), which implies that two such actions with equal entropy are measurably conjugate even if they are algebraically nonconjugate.

If $d > 1$ and α_1, α_2 have completely positive entropy with respect to Haar measure, then they are Bernoulli by [12], and can thus again be measurably conjugate without being algebraically conjugate. However, if these actions are irreducible, expansive and mixing, and if the groups X_1 and X_2 are connected, then [4, Theorem 5.1' and Corollary 5.2'], combined with an observation by J.-P. Thouvenot, implies that every measurable conjugacy is *a.e.* equal to an affine map (cf. [3]). The purpose of this note is to prove an analogous result for irreducible and mixing algebraic $\mathbb{Z}^d$-actions on compact, zero-dimensional abelian groups.

Theorem 1.1. *Let $d > 1$, and let α_1 and α_2 be mixing algebraic $\mathbb{Z}^d$-actions on compact zero-dimensional abelian groups X_1 and X_2, respectively. If α_1 is irreducible, and if $\phi\colon X_1 \longrightarrow X_2$ is a measurable conjugacy of α_1 and α_2, then α_2 is irreducible and ϕ is λ_{X_1}-a.e. equal to an affine map.*

Corollary 1.2. *Let $d > 1$, and let α_1 and α_2 be irreducible, mixing and expansive algebraic $\mathbb{Z}^d$-actions on compact abelian groups X_1 and X_2, respectively.*

Then every measurable conjugacy $\phi\colon X_1 \longrightarrow X_2$ *of* α_1 *and* α_2 *is* λ_{X_1}-a.e. *equal to an affine map.*

Corollary 1.3. *Let* $d > 1$*, and let* α_1 *and* α_2 *be measurably conjugate irreducible, mixing and expansive algebraic* $\mathbb{Z}^d$*-actions on compact abelian groups* X_1 *and* X_2*, respectively. Then* α_1 *and* α_2 *are algebraically conjugate.*

Proof of Corollary 1.2. If α is an irreducible algebraic $\mathbb{Z}^d$-action on a compact abelian group X, then the connected component of the identity $X^\circ \subset X$ is a closed α-invariant subgroup. By irreducibility, either $X = X^\circ$ or $X^\circ = \{0_X\}$. In the first case the result appears in [3], and in the second case it follows from Theorem 1.1 above. □

Proof of Corollary 1.3. If α_1 and α_2 are measurably conjugate then Corollary 1.2 shows that there exists an affine conjugacy $\phi\colon X_1 \longrightarrow X_2$ of α_1 and α_2 of the form (1.4). The group isomorphism $\psi\colon X_1 \longrightarrow X_2$ is an algebraic conjugacy of α_1 and α_2. □

Actions with completely positive entropy and irreducible actions lie — in a sense — at opposite ends of the spectrum of ergodic algebraic $\mathbb{Z}^d$-actions. As mentioned above, the kind of isomorphism rigidity described in Theorem 1.1 and its corollaries is impossible for actions with completely positive entropy. However, it may conceivably hold for all mixing algebraic $\mathbb{Z}^d$-actions with zero entropy (i.e. without Bernoulli factors — cf. [10]). The currently available techniques do not appear to shed any light on this question.

This paper is organized as follows: Section 2 provides background on irreducible algebraic $\mathbb{Z}^d$-actions, Section 3 contains the proof of Theorem 1.1, and Section 4 illustrates Theorem 1.1 with examples. In one of these examples (Example 4.4) we apply Theorem 1.1 to check measurable nonconjugacy of certain algebraic $\mathbb{Z}^2$-actions with positive (but not completely positive) entropy.

2. Irreducible $\mathbb{Z}^d$-actions

Following [7], [13] and [15] we denote by $\mathfrak{R}_d = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ the ring of Laurent polynomials with integral coefficients in the commuting variables $u_1, \dots, u_d$. Every $f \in \mathfrak{R}_d$ is written as

$$f = \sum_{\mathbf{m} \in \mathbb{Z}^d} c_f(\mathbf{m}) u^{\mathbf{m}} \tag{2.1}$$

with $u^{\mathbf{m}} = u_1^{m_1} \cdots u_d^{m_d}$ and $c_f(\mathbf{m}) \in \mathbb{Z}$ for every $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{Z}^d$, where $c_f(\mathbf{m}) = 0$ for all but finitely many $\mathbf{m}$.

Suppose that α is an algebraic $\mathbb{Z}^d$-action on a compact abelian group X. We denote by $\widehat{X}$ the additively written dual group of X and write $\langle a, x\rangle$ the value of a character $a \in \widehat{X}$ at a point $x \in X$. The *dual action* $\hat{\alpha}\colon \mathbf{n} \mapsto \hat{\alpha}^{\mathbf{n}}$ of $\mathbb{Z}^d$ on $\widehat{X}$ is defined by

$$\langle \hat{\alpha}^{\mathbf{n}} a, x\rangle = \langle a, \alpha^{\mathbf{n}} x\rangle$$

for every $\mathbf{n} \in \mathbb{Z}^d$, $x \in X$ and $a \in \widehat{X}$.

For every $f = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) u^{\mathbf{n}} \in \mathfrak{R}_d$, $x \in X$ and $a \in \widehat{X}$, we set

$$f(\alpha)(x) = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) \alpha^{\mathbf{n}} x, \qquad f(\hat{\alpha})(a) = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) \hat{\alpha}^{\mathbf{n}} a, \tag{2.2}$$

and note that $f(\alpha)\colon X \longrightarrow X$ is a group homomorphism with dual homomorphism

$$\widehat{f(\alpha)} = f(\hat{\alpha})\colon \widehat{X} \longrightarrow \widehat{X}. \tag{2.3}$$

The group $\widehat{X}$ is a module over the ring $\mathfrak{R}_d$ with operation

$$f \cdot a = f(\hat{\alpha})(a) \tag{2.4}$$

for $f \in \mathfrak{R}_d$ and $a \in \widehat{X}$. In particular,

$$u^{\mathbf{m}} \cdot a = \hat{\alpha}^{\mathbf{m}} a \tag{2.5}$$

for $\mathbf{m} \in \mathbb{Z}^d$ and $a \in \mathfrak{M}$. The module $\mathfrak{M} = \widehat{X}$ is called the *dual module* $\mathfrak{M}$ of α, and is Noetherian (and hence countable) whenever α is expansive (cf. [13, Proposition 5.4]).

Conversely, if $\mathfrak{M}$ is an $\mathfrak{R}_d$-module, we define an algebraic $\mathbb{Z}^d$-action $\alpha_{\mathfrak{M}}$ on the compact abelian group

$$X_{\mathfrak{M}} = \widehat{\mathfrak{M}} \tag{2.6}$$

by setting

$$\hat{\alpha}_{\mathfrak{M}}^{\mathbf{m}} a = u^{\mathbf{m}} \cdot a \tag{2.7}$$

for every $\mathbf{m} \in \mathbb{Z}^d$ and $a \in \mathfrak{M}$.

A prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$ is *associated* with an $\mathfrak{R}_d$-module $\mathfrak{M}$ if there exists an $a \in \mathfrak{M}$ with $\mathfrak{p} = \operatorname{ann}(a) = \{f \in \mathfrak{R}_d : f \cdot a = 0\}$. The set $\operatorname{asc}(\mathfrak{M})$ of all prime ideals associated with $\mathfrak{M}$ has the property that

$$\bigcup_{\mathfrak{p} \in \operatorname{asc}(\mathfrak{M})} \mathfrak{p} = \{f \in \mathfrak{R}_d : \text{multiplication by } f \text{ on } \mathfrak{M} \text{ is not injective}\}. \tag{2.8}$$

As was shown in [13], [10] and [15], many properties of an algebraic $\mathbb{Z}^d$-action α can be expressed in terms of the prime ideals associated with the dual module $\mathfrak{M} = \widehat{X}$ of α. As a starter we note that the group X is zero-dimensional if and only if every prime ideal $\mathfrak{p}$ associated with $\mathfrak{M}$ contains a nonzero constant ([15, Proposition 6.9]), and that α is ergodic (resp. mixing) if and only if $\alpha_{\mathfrak{R}_d/\mathfrak{p}}$ is ergodic (resp. mixing) for every $\mathfrak{p} \in \operatorname{asc}(\mathfrak{M})$ ([15, Proposition 6.6]).

The following result is contained in [15, Proposition 6.6 and Theorem 29.2]. We give a brief proof for the reader's convenience.

Proposition 2.1. *Let α be an irreducible and ergodic algebraic $\mathbb{Z}^d$-action on a compact zero-dimensional abelian group X. Then α is expansive and there exists a unique prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$ with the following properties.*

(1) *$\mathfrak{p}$ contains a rational prime $p > 1$;*
(2) *$\mathfrak{R}_d/\mathfrak{p}$ is infinite;*
(3) *For every ideal $I \supsetneq \mathfrak{p}$ in $\mathfrak{R}_d$, $\mathfrak{R}_d/I$ is finite;*
(4) *There exist continuous, surjective, finite-to-one group homomorphisms $\psi\colon X \longrightarrow X_{\mathfrak{R}_d/\mathfrak{p}}$ and $\psi'\colon X_{\mathfrak{R}_d/\mathfrak{p}} \longrightarrow X$ with*
$$\psi \circ \alpha^{\mathbf{n}} = \alpha^{\mathbf{n}}_{\mathfrak{R}_d/\mathfrak{p}} \circ \psi, \qquad \psi' \circ \alpha^{\mathbf{n}}_{\mathfrak{R}_d/\mathfrak{p}} = \alpha^{\mathbf{n}} \circ \psi', \tag{2.9}$$
for every $\mathbf{n} \in \mathbb{Z}^d$.
(5) *α is mixing if and only if $u^{\mathbf{m}} - 1 \notin \mathfrak{p}$ for every nonzero $\mathbf{m} \in \mathbb{Z}^d$;*

Conversely, if $\mathfrak{p} \subset \mathfrak{R}_d$ *is a prime ideal satisfying the conditions* (1)–(3) *above, then the* $\mathbb{Z}^d$*-action* $\alpha = \alpha_{\mathfrak{R}_d/\mathfrak{p}}$ *on the zero-dimensional group* $X_{\mathfrak{R}_d/\mathfrak{p}}$ *is irreducible, ergodic, and satisfies* (5) (*cf.* (2.6)–(2.7)).

Proof. Since X is zero-dimensional, every prime ideal associated with the dual module $\mathfrak{M} = \widehat{X}$ contains a nonzero constant. The ergodicity of α implies that every nonzero submodule $\mathfrak{N} \subset \mathfrak{M}$ is infinite: otherwise $Z = \widehat{\mathfrak{N}} = X/\mathfrak{N}^{\perp}$ would be a finite quotient of X by an α-invariant subgroup, contrary to ergodicity.

If $\mathfrak{p}$ is a prime ideal associated with $\mathfrak{M}$ then there exists, by definition, an element $a \in \mathfrak{M}$ with $\mathfrak{N} = \mathfrak{R}_d \cdot a \cong \mathfrak{R}_d/\mathfrak{p}$. The preceding paragraph shows that $\mathfrak{N}$ is infinite, and the irreducibility of α implies that the closed, α-invariant subgroup $Y = \mathfrak{N}^{\perp} \subset X$ is finite. Hence $\hat{Y} = \mathfrak{M}/\mathfrak{N}$ is finite.

If $I \supsetneq \mathfrak{p}$ is an ideal, then $\mathfrak{N}' = I \cdot a \cong I/\mathfrak{p}$ is a submodule of $\mathfrak{N}$ and hence — by irreducibility — of finite index in $\mathfrak{N}$. It follows that $\mathfrak{R}_d/I$ is finite, as claimed in (3).

If $\mathfrak{q} \neq \mathfrak{p}$ is a second prime ideal associated with $\mathfrak{M}$ then $\mathfrak{q} = \operatorname{ann}(b)$ for some $b \in \mathfrak{M} \smallsetminus \mathfrak{N}$. Every nonzero $b' \in \mathfrak{N}' = \mathfrak{R}_d \cdot b$ has $\mathfrak{q}$ as its annihilator. However, since $\mathfrak{R}_d/\mathfrak{q} \cong \mathfrak{N}'$ is infinite by ergodicity and $\mathfrak{N}'/\mathfrak{N} = \mathfrak{N}'/(\mathfrak{N}) \cap \mathfrak{N}')$ is finite by (3), there exists an $h \in \mathfrak{R}_d \smallsetminus \mathfrak{q}$ with $h \cdot b \in \mathfrak{N}$ and hence $\operatorname{ann}(h \cdot b) = \mathfrak{p}$. This contradiction implies that $\mathfrak{p}$ is the only prime ideal associated with $\mathfrak{M}$, and we denote by $p > 1$ the rational prime contained in $\mathfrak{p}$.

The surjective homomorphism $\psi \colon X \longrightarrow X_{\mathfrak{N}} = X_{\mathfrak{R}_d/\mathfrak{p}}$ dual to the inclusion $\mathfrak{N} \subset \mathfrak{M}$ satisfies the first equation in (2.9). For the definition of the homomorphism $\psi' \colon X_{\mathfrak{N}} \longrightarrow X$ we conclude as above that there exists, for every $b \in \mathfrak{M} \smallsetminus \mathfrak{N}$, an element $h_b \in \mathfrak{R}_d \smallsetminus \mathfrak{p}$ with $h \cdot b \in \mathfrak{N}$. The polynomial

$$h = \prod_{b \in \mathfrak{M} \smallsetminus \mathfrak{N}} h_b \in \mathfrak{R}_d \smallsetminus \mathfrak{p}$$

satisfies that $h \cdot \mathfrak{M} \subset \mathfrak{N}$. The map $m_h \colon \mathfrak{M} \longrightarrow \mathfrak{N}$ consisting of multiplication by h is injective by (2.8), and the surjective homomorphism $\psi' \colon X_{\mathfrak{N}} \longrightarrow X$ dual to m_h satisfies the second equation in (4).

Equation (2.14) shows that $\alpha_{\mathfrak{N}}$ is expansive, and the expansiveness of $\alpha_{\mathfrak{M}}$ is clear from the fact that the group homomorphisms ψ, ψ' in (4) are finite-to-one by irreducibility and (3).

The statement (5) follows from [15, Proposition 6.6], and the final assertion is a consequence of the properties of $\mathfrak{p}$ and [15, Proposition 6.6]. □

Remarks 2.2. (1) Let α be an irreducible and ergodic algebraic $\mathbb{Z}^d$-action on a compact zero-dimensional abelian group X, and let $\mathfrak{p} \subset \mathfrak{R}_d$ be the prime ideal satisfying the conditions (1)–(4) in Proposition 2.1. Condition (5) in Proposition 2.1 shows that α is mixing if and only if $\alpha^{\mathbf{m}} \neq \mathrm{id}_X$ whenever $\mathbf{0} \neq \mathbf{m} \in \mathbb{Z}^d$, where id_X is the identity automorphism of X.

(2) Condition (3) in Proposition 2.1 is equivalent to $\mathfrak{p}$ having Krull dimension (or depth) 1.

(3) If $\mathfrak{q} \subset \mathfrak{R}_d$ is a prime ideal satisfying the conditions (1)–(3) in Proposition 2.1, and if $\alpha = \alpha_{\mathfrak{R}_d/\mathfrak{q}}$, then the the prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$ satisfying the conditions (1)–(4) in Proposition 2.1 is equal to $\mathfrak{q}$.

Motivated by Proposition 2.1 we take a closer look at $\mathbb{Z}^d$-actions of the form $\alpha_{\mathfrak{R}_d/\mathfrak{p}}$, where $\mathfrak{p} \subset \mathfrak{R}_d$ is a prime ideal satisfying the conditions (1)–(3) described there. Denote by $\mathfrak{R}_d^{(p)} = F_p[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ the ring of Laurent polynomials in the variables $u_1, \dots, u_d$ with coefficients in the prime field $F_p = \mathbb{Z}/p\mathbb{Z}$, and define a ring homomorphism $f \mapsto f_{/p}$ from $\mathfrak{R}_d$ to $\mathfrak{R}_d^{(p)}$ by reducing each coefficient of f modulo p. Again we write every $h \in \mathfrak{R}_d^{(p)}$ as $h = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_h(\mathbf{n}) u^{\mathbf{n}}$ with $c_h(\mathbf{m}) \in F_p$ for every $\mathbf{m} \in \mathbb{Z}^d$. The set

$$\mathcal{S}(h) = \{\mathbf{n} \in \mathbb{Z}^d : c_h(\mathbf{n}) \neq 0\} \tag{2.10}$$

is called the *support* of $h \in \mathfrak{R}_d^{(p)}$.

If

$$\overline{\mathfrak{p}} = \{f_{/p} : f \in \mathfrak{p}\}, \tag{2.11}$$

then $\overline{\mathfrak{p}} \subset \mathfrak{R}_d^{(p)}$ is again a prime ideal, and the map $f \mapsto f_{/p}$ induces an $\mathfrak{R}_d$-module isomorphism

$$\mathfrak{R}_p/\mathfrak{p} \cong \mathfrak{R}_d^{(p)}/\overline{\mathfrak{p}}. \tag{2.12}$$

Let $\Omega = F_p^{\mathbb{Z}^d}$, furnished with the product topology and component-wise addition. We write every $\omega \in \Omega$ as $\omega = (\omega_{\mathbf{n}})$ with $\omega_{\mathbf{n}} \in F_p$ for every $\mathbf{n} \in \mathbb{Z}^d$ and define the shift-action σ of $\mathbb{Z}^d$ on Ω by (1.1). The additive group $\mathfrak{R}_d^{(p)}$ can be identified with the dual group $\widehat{\Omega}$ of Ω by setting

$$\langle h, \omega \rangle = e^{2\pi i \left(\sum_{\mathbf{n} \in \mathbb{Z}^d} c_h(\mathbf{n}) \omega_{\mathbf{n}}\right)/p} \tag{2.13}$$

for every $h \in \mathfrak{R}_d^{(p)}$ and $\omega \in \Omega$. With this identification the automorphism $\hat{\sigma}^{\mathbf{n}}$ of $\mathfrak{R}_d^{(p)}$ dual to the shift $\sigma^{\mathbf{n}}$ on Ω consists of multiplication by $u^{\mathbf{n}}$.

If $I \subset \mathfrak{R}_d^{(p)}$ is an ideal, then

$$I^{\perp} = \widehat{\mathfrak{R}_d^{(p)}/I} = X_{\mathfrak{R}_d^{(p)}/I} = \{\omega \in \Omega : \langle h, \omega \rangle = 1 \text{ for every } h \in I\} \tag{2.14}$$

is a closed, shift-invariant subgroup of Ω, and $\alpha_{\mathfrak{R}_d^{(p)}/I}$ is the restriction of σ to $X_{\mathfrak{R}_d^{(p)}/I}$. We conclude this section with a corollary of Proposition 2.1.

Corollary 2.3. *Let α be an irreducible and ergodic algebraic $\mathbb{Z}^d$-action on a compact zero-dimensional abelian group X. Then $h(\alpha^{\mathbf{n}}) < \infty$ for every $\mathbf{n} \in \mathbb{Z}^d$, where $h(\cdot)$ denotes topological entropy.*

Conversely, if α is an expansive, ergodic (and mixing) algebraic $\mathbb{Z}^d$-action on a compact zero-dimensional abelian group X with $h(\alpha^{\mathbf{n}}) < \infty$ for every $\mathbf{n} \in \mathbb{Z}^d$, and if $\mathfrak{p} \subset \mathfrak{R}_d$ is a prime ideal associated with the module $\mathfrak{M} = \widehat{X}$, then the $\mathbb{Z}^d$-action $\alpha_{\mathfrak{R}_d/\mathfrak{p}}$ on $X_{\mathfrak{R}_d/\mathfrak{p}}$ is irreducible, ergodic (and mixing).

Proof. If α is irreducible we may assume without loss in generality that $\alpha = \alpha_{\mathfrak{R}_d/\mathfrak{p}}$ and $X = X_{\mathfrak{R}_d/\mathfrak{p}}$ for some prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$ satisfying the conditions (1)–(3) in Proposition 2.1. We denote by $\overline{\mathfrak{p}} \subset \mathfrak{R}_d^{(p)}$ the prime ideal (2.11) and set $\mathfrak{M} = \mathfrak{R}_d/\mathfrak{p} = \mathfrak{R}_d^{(p)}/\overline{\mathfrak{p}}$.

If $d = 2$ our assertion is obvious (cf. [6]). Assume therefore that $d \geq 3$.

A nonzero element $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{Z}^d$ is *primitive* if $\gcd(m_1, \dots, m_d) = 1$. We claim that there exists, for every pair $\mathbf{m}, \mathbf{n}$ of linearly independent primitive elements in $\mathbb{Z}^d$, a nonzero element $h \in \mathfrak{R}_2^{(p)}$ with $h(u^{\mathbf{m}}, u^{\mathbf{n}}) \in \overline{\mathfrak{p}}$.

In order to prove this claim by contradiction we assume for notational simplicity that $\mathbf{m} = \mathbf{e^{(1)}} = (1,0,0,\dots,0)$ and $\mathbf{n} = \mathbf{e^{(2)}} = (0,1,0,\dots,0)$ (the proof in the general case is completely analogous). With Denote by β the $\mathbb{Z}^2$-action $(n_1,n_2) \mapsto \beta(n_1\widehat{,}n_2) = \alpha^{(n_1,n_2,0,\dots,0)}$ on X. The dual module $\mathfrak{N}$ of β is nothing but $\mathfrak{M}$, considered as a module over $\mathfrak{R}_2^{(p)} \subset \mathfrak{R}_d^{(p)}$. Our hypothesis that $\overline{\mathfrak{p}} \cap \mathfrak{R}_2^{(p)} = \{0\}$ implies that $\{0\}$ the only prime ideal in $\mathfrak{R}_2^{(p)}$ associated with $\mathfrak{N}$.

We represent X as the closed, shift-invariant subgroup (2.14) of $\Omega = F_p^{\mathbb{Z}^d}$ (with I replaced by $\overline{\mathfrak{p}}$) and denote by $\pi_E \colon X \longrightarrow F_p^E$ the projection of every $x \in X \subset F_p^{\mathbb{Z}^d}$ to its coordinates in a subset $E \subset \mathbb{Z}^d$. We write β_k for the $\mathbb{Z}^k$-action $(n_1,\dots,n_k) \mapsto \beta^{(n_1,\dots,n_k)} = \alpha^{(n_1,\dots,n_k,0,\dots,0)}$ on X, where $k = 1,\dots,d$.

We fix an integer $M \ge 0$ such that

$$Q(M) = \{\mathbf{n} = (n_1,\dots,n_d) \in \mathbb{Z}^d : |n_i| \le M \ \text{ for } \ i = 3,\dots,d\} \supset \mathcal{S}(g_i)$$

for $i = 1,\dots,L$, where $\{g_1,\dots,g_L\}$ is a set of generators of the ideal $\overline{\mathfrak{p}}$. For $k = 2,\dots,d$ we set

$$\begin{aligned}
E_k^+ &= \{\mathbf{n} = (n_1,\dots,n_d) \in \mathbb{Z}^d : n_{k+1} \ge 0,\ \ |n_i| \le M \ \text{ for } \ i = k+2,\dots,d\},\\
E_k^- &= \{\mathbf{n} = (n_1,\dots,n_d) \in \mathbb{Z}^d : n_{k+1} \le 0,\ \ |n_i| \le M \ \text{ for } \ i = k+2,\dots,d\},\\
E_k &= \{\mathbf{n} = (n_1,\dots,n_d) \in \mathbb{Z}^d : \ \ |n_i| \le M \ \text{ for } \ i = k+1,\dots,d\},\\
X_k &= \pi_{E_k}(X),\ \ Y_k^+ = \pi_{E_k}(\ker \pi_{E_k^+}),\ \ Y_k^- = \pi_{E_k}(\ker \pi_{E_k^-}).
\end{aligned}$$

The subgroups $Y_k^\pm \subset X$ are closed and β_k-invariant, and β_k induces an expansive $\mathbb{Z}^k$-action β_{X_k} on X_k. We denote by $\mathfrak{N}_k^\pm$ the dual $\mathfrak{R}_{k-1}^{(p)}$-modules defined by the $\mathbb{Z}^{k-1}$-actions $(\beta_k)_{Y_k^\pm}$ induced by β_k on $Y_k^\pm 1$. Although these actions are not necessarily expansive and their dual modules need not be Noetherian, it is easy to check that each of these modules has only finitely many associated prime ideals in $\mathfrak{R}_k^{(p)}$. We view all these prime ideals as prime ideals in $\mathfrak{R}_d^{(p)}$ and choose an $f \in \mathfrak{R}_2^{(p)}$ which does not lie in any of these prime ideals, and whose support $\mathcal{S}(f)$ contains at least two elements. We claim that the group homomorphism $f(\alpha) = f(\beta_2) \colon X \longrightarrow X$ defined by (2.2) has the following properties:

(a) $f(\alpha) \colon X \longrightarrow X$ is surjective,
(b) $\ker f(\alpha)$ is uncountable.

Since $Y_f = \ker f(\alpha)$ is a closed, α-invariant subgroup of X this will violate irreducibility and prove that $\overline{\mathfrak{p}} \cap \mathfrak{R}_2^{(p)}$ cannot be equal to $\{0\}$.

Since $\overline{\mathfrak{p}} \cap \mathfrak{R}_2^{(p)} = \{0\}$, Condition (a) is satisfied for every nonzero $f \in \mathfrak{R}_2^{(p)}$. Furthermore, since the restriction of β_2 to Y_2 is expansive and its dual module is therefore Noetherian, it is easy to check that the kernel $\ker(f(\alpha)_{X_2})$ of the surjective automorphism $f(\alpha)_{X_2} \colon X_2 \longrightarrow X_2$ is uncountable, where $f(\alpha)_{X_k}$ is the homomorphism of X_k induced by $f(\alpha)$. By using an induction argument involving duality and our choice of f one can check that $\pi_{X_k}(\ker(f(\alpha)_{X_{k+1}})) = \ker(f(\alpha)_{X_k})$ for $k = 2,\dots,d$, which proves (b) and completes the proof of our assertion that there exists, for every pair $\mathbf{m},\mathbf{n}$ of linearly independent primitive elements in $\mathbb{Z}^d$, a nonzero element $h \in \mathfrak{R}_2^{(p)}$ with $h(u^{\mathbf{m}}, u^{\mathbf{n}}) \in \overline{\mathfrak{p}}$.

Next we assert that $h(\alpha^{\mathbf{m}}) < \infty$ for every primitive $\mathbf{m} \in \mathbb{Z}^d$. In order to verify this we represent α as the shift-action on $X = \overline{\mathfrak{p}}^{\perp}$ (cf. (1.1) and (2.14)) and assume for simplicity that $\mathbf{m} = \mathbf{e}^{(1)} = (1, 0, \dots, 0)$. The last paragraph shows that there exist nonzero elements $h_i \in \mathfrak{R}_2^{(p)}$ with $h_i(u_1, u_i) \in \overline{\mathfrak{p}}$ for $i = 2, \dots, d$. After multiplying each h_i by a power of u_i we can write it as

$$h_i^{(0)}(u_1) + h_i^{(1)}(u_1)u_i + \cdots + h_i^{(L_i)}(u_1)u_i^{L_i}$$

with $h_i^{(j)} \in \mathfrak{R}_1^{(p)}$ and $h_i^{(0)} h_i^{(L_i)} \neq 0$. Then we claim that

$$h(\alpha^{\mathbf{e}^{(1)}}) \leq \left(\prod_{i=2}^{d} (L_i + 1) \right) \cdot \log 2. \tag{2.15}$$

In order to prove (2.15) we set, for every $\mathbf{N} = (N_2, \dots, N_d) \in \mathbb{N}^{d-1}$,

$$Q(\mathbf{N}) = \{\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d : 0 \leq n_i \leq N_i \text{ for } i = 2, \dots, d\} \tag{2.16}$$

and write $\pi_{Q(\mathbf{N})} \colon X \longrightarrow F_p^{Q(\mathbf{N})}$ for the projection which restricts each $x \in X \subset F_p^{\mathbb{Z}^d}$ to its coordinates in $Q(\mathbf{N})$. The map $\pi_{Q(\mathbf{N})}$ is a continuous group homomorphism, and $\ker(\pi_{Q(\mathbf{N})})$ is an $\alpha^{\mathbf{e}^{(1)}}$-invariant subgroup of X. If $N_i \geq L_i$ for $i = 2, \dots, d$, then

$$|\ker(\pi_{Q(\mathbf{N})}) / \ker(\pi_{Q(N_2, \dots, N_{j-1}, N_j+1, N_{j+1}, \dots, N_d)})| < \infty$$

for $j = 2, \dots, d$, and hence

$$|\ker(\pi_{Q(\mathbf{N})}) / \ker(\pi_{Q(\mathbf{N}')})| < \infty$$

whenever $N_i' \geq N_i \geq L_i$ for $i = 2, \dots, d$. It follows that group automorphism $\sigma_{\mathbf{N}} = \alpha^{\mathbf{e}^{(1)}}_{X/\ker(\pi_{Q(\mathbf{N})})}$ induced by $\alpha^{\mathbf{e}^{(1)}}$ on $X/\ker(\pi_{Q(\mathbf{N})})$ satisfies that

$$h(\alpha^{\mathbf{e}^{(1)}}) = \lim_{N_2 \to \infty, \dots, N_d \to \infty} h(\sigma_{\mathbf{N}}) = h(\sigma_{\mathbf{L}}),$$

where $\mathbf{L} = (L_2, \dots, L_d)$. This proves (2.15).

Since $h(\alpha^{\mathbf{m}}) < \infty$ and $h(\alpha^{k\mathbf{m}}) = |k| h(\alpha^{\mathbf{m}})$ for every primitive $\mathbf{m} \in \mathbb{Z}^d$ and every $k \in \mathbb{Z}$, $h(\alpha^{\mathbf{n}}) < \infty$ for every $\mathbf{n} \in \mathbb{Z}^d$.

Conversely, let $p > 1$ be a rational prime and $\mathfrak{p} \subset \mathfrak{R}_d$ a prime ideal with $p \in \mathfrak{p}$ and $h(\alpha^{\mathbf{n}}_{\mathfrak{R}_d/\mathfrak{p}}) < \infty$ for every $\mathbf{n} \in \mathbb{Z}^d$. We set $\alpha = \alpha_{\mathfrak{R}_d/\mathfrak{p}}$, $X = X_{\mathfrak{R}_d/\mathfrak{p}}$ and define $g(\alpha)$ by (2.2) for every $g \in \mathfrak{R}_d$. Since $g(\alpha)$ is dual to multiplication by g on $\widehat{X} = \mathfrak{R}_d/\mathfrak{p}$, $g(\alpha)$ is surjective if and only if $g \in \mathfrak{R}_d \smallsetminus \mathfrak{p}$, and $g(\alpha)(x) = 0_X$ for every $x \in X$ otherwise.

For every closed α-invariant subgroup $Y \subset X$ we have that

$$h(\alpha^{\mathbf{n}}) = h(\alpha^{\mathbf{n}}_Y) + h(\alpha^{\mathbf{n}}_{X/Y})$$

for every $\mathbf{n} \in \mathbb{Z}^d$ (cf. e.g. [10], [17]). Fix $g \in \mathfrak{R}_d \smallsetminus \mathfrak{p}$ and set $Y = \ker(g(\alpha))$. Then $h(\alpha^{\mathbf{n}}) = h(\alpha^{\mathbf{n}}_{X/Y})$ and hence $h(\alpha^{\mathbf{n}}_Y) = 0$ for every $\mathbf{n} \in \mathbb{Z}^d$.

Let $\mathbf{n} \in \mathbb{Z}^d$ be a primitive element. We view the $\mathfrak{R}_d$-module $\mathfrak{N} = \widehat{Y} = \widehat{\ker(g(\alpha))}$ as a module over the ring $\mathbb{Z}[u^{\pm \mathbf{n}}] \cong \mathfrak{R}_1$. Since $h(\alpha^{\mathbf{n}}_Y) = 0$, the entropy formula in [10] shows that every prime ideal $\mathfrak{q}' \subset \mathbb{Z}[u^{\pm \mathbf{n}}]$ associated with $\mathfrak{N}$ is nonprincipal: as $p \in \mathfrak{q}'$ this means that $\mathfrak{q}' \supsetneq p\mathbb{Z}[u^{\pm \mathbf{n}}]$, and hence that $u^{k\mathbf{n}} - 1 \in \mathfrak{q}'$ for some $k \geq 1$. From the definition of an associated prime we obtain that every prime ideal $\mathfrak{q} \subset \mathfrak{R}_d$ associated with the $\mathfrak{R}_d$-module $\mathfrak{N}$ contains p and $u^{k\mathbf{n}} - 1$ for some $k \geq 1$. Since this is true for every primitive $\mathbf{n} \in \mathbb{Z}^d$, $\mathfrak{R}_d/\mathfrak{q}$ is finite for every prime ideal $\mathfrak{q} \subset \mathfrak{R}_d$ associated with $\mathfrak{N}$. As $\mathfrak{N}$ is Noetherian, $\mathfrak{N}$ — and hence $Y = \ker(g(\alpha))$ — is finite for every $g \in \mathfrak{R}_d \smallsetminus \mathfrak{p}$.

Duality and the Noetherian property of $\mathfrak{M} = \widehat{X}$ allow us to write every closed, α-invariant subgroup $Y \subsetneq X$ as an intersection of finitely many subgroups $\ker(g(\alpha))$ with $g \in \mathfrak{R}_d \smallsetminus \mathfrak{p}$. Hence every such subgroup is finite and $\alpha = \alpha_{\mathfrak{R}_d/\mathfrak{p}}$ is irreducible.

Finally, if α is an expansive and ergodic algebraic $\mathbb{Z}^d$-action on a zero-dimensional compact abelian group X with $h(\alpha^{\mathbf{n}}) < \infty$ for every $\mathbf{n} \in \mathbb{Z}^d$, then $h(\alpha^{\mathbf{n}}_{\mathfrak{R}_d/\mathfrak{p}}) < \infty$ for every $\mathbf{n} \in \mathbb{Z}^d$ and every prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$ associated with $\mathfrak{M} = \widehat{X}$, and the argument above implies that $\alpha_{\mathfrak{R}_d/\mathfrak{p}}$ is irreducible and ergodic. If α is also mixing, then $\alpha_{\mathfrak{R}_d/\mathfrak{p}}$ is mixing by [15, Proposition 6.6]. □

3. The proof of Theorem 1.1

The proof of Theorem 1.1 relies on the notion of a *mixing set* introduced in [6]–[8], and on the application of mixing sets to invariant measures in [14].

Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X, and let μ be an α-invariant probability measure on the Borel field $\mathcal{B}_X$ of X. A nonempty finite set $S \subset \mathbb{Z}^d$ is *μ-mixing* under α if

$$\lim_{k\to\infty} \mu\biggl(\bigcap_{\mathbf{n}\in S} \alpha^{-k\mathbf{n}}(B_{\mathbf{n}})\biggr) = \prod_{\mathbf{n}\in S} \mu(B_{\mathbf{n}})$$

for every collection $(B_{\mathbf{n}}, \mathbf{n} \in S)$ in $\mathcal{B}_X$, and *μ-nonmixing* otherwise. A nonempty set $S \subset \mathbb{Z}^d$ is *minimal μ-nonmixing* if S is μ-nonmixing, but every nonempty subset $S' \subsetneq S$ is μ-mixing. A λ_X-(non-)mixing set is called a (non-)mixing set of α.

If the group X is connected and λ_X is mixing under α, then every nonempty set $S \subset \mathbb{Z}^d$ is mixing under α by [16]. If X is disconnected, then a mixing algebraic $\mathbb{Z}^d$-action α on X has nonmixing sets if and only if λ_X does not have completely positive entropy (cf. [8] and [15, Section 27]). In particular, if α is an irreducible and mixing algebraic $\mathbb{Z}^d$-action on a compact zero-dimensional abelian group X, then α has nonmixing sets.

Lemma 3.1. *Let $d > 1$, α an irreducible and ergodic algebraic $\mathbb{Z}^d$-action on a compact zero-dimensional abelian group X, and let $\mathfrak{p} \subset R_d$ be the prime ideal described in Proposition* 2.1.

1. *The $\mathbb{Z}^d$-actions α and $\alpha_{\mathfrak{R}_d/\mathfrak{p}} = \alpha_{\mathfrak{R}_d^{(p)}/\overline{\mathfrak{p}}}$ have the same nonmixing sets* (*cf.* (2.11)–(2.12));
2. *For every nonzero element $h \in \overline{\mathfrak{p}} \subset \mathfrak{R}_d^{(p)}$, the support $\mathcal{S}(h)$ of h is a nonmixing set of $\alpha_{\mathfrak{R}_d/\mathfrak{p}}$ and hence of α* (*cf.* (2.10)).

Proof. The first assertion is an immediate consequence of (2.9). For the second statement we note that

$$h^{p^n} = \sum_{\mathbf{n}\in\mathbb{Z}^d} c_h(\mathbf{n}) u^{p^n \mathbf{n}} \in \overline{\mathfrak{p}}$$

for every $h \in \overline{\mathfrak{p}}$ and $n \geq 1$. By setting $I = \overline{\mathfrak{p}}$ in (2.14) and putting $B_{\mathbf{n}} = \{\omega \in X_{\mathfrak{R}_d/\mathfrak{p}} : \omega_{\mathbf{0}} = 0\}$ for every $\mathbf{n} \in \mathcal{S}(h)$ we see that that $\mathcal{S}(h)$ is nonmixing for $\alpha_{\mathfrak{R}_d/\mathfrak{p}}$ and hence for α. □

Lemma 3.2. *Let α_1 and α_2 be algebraic $\mathbb{Z}^d$-actions on compact abelian groups X_1 and X_2, respectively.*

(1) *If α_1 and α_2 are measurably conjugate then they have the same nonmixing sets;*
(2) *If α_2 is a measurable factor of α_1, and if $S \subset \mathbb{Z}^d$ is a nonempty finite set which is minimal nonmixing for both α_1 and α_2, then there exist nonzero elements $a_i(\mathbf{n}) \in \widehat{X}_i$, $\mathbf{n} \in S$, $i = 1, 2$, such that*

$$\sum_{\mathbf{n} \in S} \hat{\alpha}_1^{k\mathbf{n}}(a_1(\mathbf{n})) = \sum_{\mathbf{n} \in S} \hat{\alpha}_2^{k\mathbf{n}}(a_2(\mathbf{n})) = 0 \tag{3.1}$$

for every k in an infinite subset $K \subset \mathbb{N}$.

Proof. It is clear that α_1 and α_2 have the same nonmixing sets if they are measurably conjugate. If $\phi \colon X_1 \longrightarrow X_2$ is a measurable factor map of α_1 and α_2, and if S is minimal nonmixing for α_1 and α_2, there exist an $\varepsilon > 0$ and Borel sets $B_{\mathbf{n}}$, $\mathbf{n} \in S$, in X_2 such that

$$0 < \lambda_{X_2}(B_{\mathbf{n}}) < 1 \ \text{ for every } \ \mathbf{n} \in S,$$
$$\left| \lambda_{X_2}\Bigl(\bigcap_{\mathbf{n} \in S} \alpha_2^{-k\mathbf{n}}(B_{\mathbf{n}}) \Bigr) - \prod_{\mathbf{n} \in S} \lambda_{X_2}(B_{\mathbf{n}}) \right| > \varepsilon$$

for every k in an infinite subset $L \subset \mathbb{N}$. We write $1_{B_{\mathbf{n}}}$ for the indicator function of $B_{\mathbf{n}}$, set $f_{\mathbf{n}} = 1_{B_{\mathbf{n}}} - \lambda_{X_2}(B_{\mathbf{n}}) \in L^2(X_2, \mathcal{B}_{X_2}, \lambda_{X_2})$, and obtain that $\|f_{\mathbf{n}}\|_\infty \leq 1$ and $\int f_{\mathbf{n}} \, d\lambda_{X_2} = 0$ for every $\mathbf{n} \in S$, and that

$$\left| \int \Bigl(\prod_{\mathbf{n} \in S} f_{\mathbf{n}} \cdot \alpha_2^{k\mathbf{n}} \Bigr) \, d\lambda_{X_2} \right| > \varepsilon'$$

for some $\varepsilon' > 0$ and every $k \in L$.

Fix an enumeration $S = \{\mathbf{n}_1, \dots, \mathbf{n}_M\}$ and define inductively trigonometric polynomials $f'_{\mathbf{n}_i} \colon X_2 \longrightarrow \mathbb{C}$, $i = 1, \dots, M$, with

$$\int f'_{\mathbf{n}_i} \, d\lambda_{X_2} = 0 \quad \text{and} \quad \|f_{\mathbf{n}_i} - f'_{\mathbf{n}_i}\|_2 < \varepsilon' \Big/ 2M \cdot \Bigl(\prod_{j=1}^{i-1} \|f'_{\mathbf{n}_j}\|_\infty \Bigr)$$

for $i = 1, \dots, M$. Then

$$\left| \int \Bigl(\prod_{\mathbf{n} \in S} f'_{\mathbf{n}} \cdot \alpha_2^{k\mathbf{n}} \Bigr) \, d\lambda_{X_2} \right| > \varepsilon'/2M$$

for every $k \in L$. Since each $f'_{\mathbf{n}}$ is a trigonometric polynomial with zero constant term there exist nontrivial characters $a_2(\mathbf{n}) \in \widehat{X}_2$, $\mathbf{n} \in S$, and an infinite sequence $L' \subset L \subset \mathbb{N}$ such that $\sum_{\mathbf{n} \in S} \hat{\alpha}_2^{k\mathbf{n}}(a_2(\mathbf{n})) = 0$ for every $k \in L'$.

Define $g_{\mathbf{n}} \colon X_1 \longrightarrow \mathbb{C}$ by $g_{\mathbf{n}}(x) = \langle a_2(\mathbf{n}), \phi(x) \rangle$ for every $x \in X_1$ and $\mathbf{n} \in S$, and observe that $\|g_{\mathbf{n}}\|_\infty = \|g_{\mathbf{n}}\|_2 = 1$ and $\int g_{\mathbf{n}} \, d\lambda_{X_1} = 0$ for every

$\mathbf{n} \in S$, and that

$$1 = \int \Bigl(\prod_{\mathbf{n}\in S} g_{\mathbf{n}} \cdot \alpha_1^{k\mathbf{n}}\Bigr)\, d\lambda_{X_1} \neq \prod_{\mathbf{n}\in S}\Bigl(\int g_{\mathbf{n}}\, d\lambda_{X_1}\Bigr) = 0$$

for every $k \in L'$. By approximating each $g_{\mathbf{n}}$ by a trigonometric polynomial as above we obtain nontrivial characters $a_1(\mathbf{n}) \in \widehat{X}_1$, $\mathbf{n} \in S$, and an infinite subsequence $K \subset L'$ such that (3.1) holds for every $k \in K$. □

Lemma 3.3. *Let α_1, α_2 be irreducible and mixing algebraic $\mathbb{Z}^d$-actions on compact zero-dimensional abelian groups X_1 and X_2, respectively, and let $\phi\colon X_1 \longrightarrow X_2$ be a measurable factor map of α_1 and α_2 such that there exists a nonempty finite set $S \subset \mathbb{Z}^d$ which is minimal nonmixing for α_1 and α_2. Then ϕ coincides λ_{X_1}-a.e. with an affine map $\psi\colon X_1 \longrightarrow X_2$.*

Proof. Let $\phi\colon X_1 \longrightarrow X_2$ be a measurable factor map of α_1 and α_2, and let $S \subset \mathbb{Z}^d$ be minimal nonmixing for α_1 and α_2. Put $X = X_1 \times X_2$, denote by $\alpha = \alpha_1 \times \alpha_2\colon \mathbf{n} \mapsto \alpha_1^{\mathbf{n}} \times \alpha_2^{\mathbf{n}}$ the product-action of $\mathbb{Z}^d$ on X, and let

$$\Gamma(\phi) = \{(x, \phi(x)) : x \in X_1\} \subset X$$

be the graph of ϕ. We denote by μ the unique α-invariant probability measure on $\Gamma(\phi)$ with $\mu\pi_i^{-1} = \lambda_{X_i}$ for $i = 1,2$, where $\pi_i\colon X \longrightarrow X_i$ are the coordinate projections. Since π_1 is a measurable conjugacy of the $\mathbb{Z}^d$-actions α on $(X, \mathcal{B}_X, \mu)$ and α_1 on X_1, S is minimal μ-nonmixing.

If we can show that μ is a translate of the Haar measure of a closed, α-invariant subgroup $Y \subset Z$, then ϕ is affine (mod λ_{X_1}), and the lemma is proved.

In order to verify that μ is a translate of a Haar measure we use Lemma 3.2 to find nonzero elements $a_i(\mathbf{n}) \in \widehat{X}_i$, $\mathbf{n} \in S$, $i = 1,2$, such that (3.1) holds for every k in an infinite subset $K \subset \mathbb{N}$. It follows that

$$\sum_{\mathbf{n}\in S} \hat\alpha_1^{k\mathbf{n}}\bigl(f_1(\hat\alpha_1)(a_1(\mathbf{n}))\bigr) = \sum_{\mathbf{n}\in S} \hat\alpha_2^{k\mathbf{n}}\bigl(f_2(\hat\alpha_2)(a_2(\mathbf{n}))\bigr) = 0 \tag{3.2}$$

for every $k \in K$ and $f_1, f_2 \in \mathfrak{R}_d$.

As $\widehat{X} = \widehat{X}_1 \times \widehat{X}_2$ and every proper subset of S is μ-mixing, the Fourier transform $\hat\mu\colon \widehat{X} \longrightarrow \mathbb{C}$ satisfies that

$$\begin{aligned}
&\overline{\hat\mu\bigl(f_1(\hat\alpha_1)(a_1(\mathbf{m})), f_2(\hat\alpha_2)(a_2(\mathbf{m}))\bigr)} \\
&\qquad = \hat\mu\bigl(-\hat\alpha_1^{k\mathbf{m}}(f_1(\hat\alpha_1)(a_1(\mathbf{m}))), -\hat\alpha_2^{k\mathbf{m}}(f_2(\hat\alpha_2)(a_2(\mathbf{m})))\bigr) \\
&\qquad = \lim_{\substack{k\to\infty\\ k\in K}} \hat\mu\Bigl(\sum_{\mathbf{n}\in S\smallsetminus\{\mathbf{m}\}} \bigl(\hat\alpha_1^{k\mathbf{n}}(f_1(\hat\alpha_1)(a_1(\mathbf{n}))), \hat\alpha_2^{k\mathbf{n}}(f_2(\hat\alpha_2)(a_2(\mathbf{n})))\bigr)\Bigr) \\
&\qquad = \prod_{\mathbf{n}\in S\smallsetminus\{\mathbf{m}\}} \hat\mu\bigl(f_1(\hat\alpha_1)(a_1(\mathbf{n})), f_2(\hat\alpha_2)(a_2(\mathbf{n}))\bigr)
\end{aligned}$$

for every $\mathbf{m} \in S$. By varying $\mathbf{m} \in S$ we see that

$$\bigl|\hat\mu\bigl(f_1(\hat\alpha_1)(a_1(\mathbf{m})), f_2(\hat\alpha_2)(a_2(\mathbf{m}))\bigr)\bigr| \in \{0,1\}$$

for every $\mathbf{m} \in S$ and $f_1, f_2 \in \mathfrak{R}_d$.

We fix an element $\mathbf{n} \in S$, consider the $\hat\alpha$-invariant subgroup

$$\mathfrak{N} = \bigl\{\bigl(f_1(\hat\alpha_1)(a_1(\mathbf{n})), f_2(\hat\alpha_2)(a_2(\mathbf{n}))\bigr) : f_1, f_2 \in \mathfrak{R}_d\bigr\} \subset \widehat{X},$$

and set

$$\mathfrak{N}' = \{(a,b) \in \mathfrak{N} : |\hat{\mu}(a,b)| = 1\}.$$

Since $|\hat{\mu}|^2 \colon \mathfrak{N} \longrightarrow \mathbb{C}$ is positive definite, $\mathfrak{N}'$ is a subgroup of $\mathfrak{N}$, and the α-invariance of μ implies that $\mathfrak{N}'$ is $\hat{\alpha}$-invariant. We also note that $\mathfrak{N} \cong \mathfrak{N}_1 \times \mathfrak{N}_2$, where $\mathfrak{N}_i \subset \widehat{X}_i$ is an infinite $\hat{\alpha}_i$-invariant subgroup, and the irreducibility of α_i implies that

$$\widehat{X}/\mathfrak{N} \cong (\mathfrak{M}_1/\mathfrak{N}_1) \times (\mathfrak{M}_2/\mathfrak{N}_2)$$

is finite. Let

$$Z = \widehat{\mathfrak{N}} = X/\mathfrak{N}^{\perp},$$

denote by α_Z the $\mathbb{Z}^d$-action induced by α on Z, and write $Z' = \mathfrak{N}'^{\perp} \subset Z$ for the annihilator of $\mathfrak{N}'$ in Z. If $\pi \colon X \longrightarrow Z$ is the quotient map and $\nu = \mu\pi^{-1}$, then $\hat{\nu} \colon \mathfrak{N} \longrightarrow \mathbb{C}$ is the restriction of $\hat{\mu} \colon \widehat{X} \longrightarrow \mathbb{C}$ to $\mathfrak{N}$, and the further restriction of $\hat{\nu}$ to $\mathfrak{N}'$ is a positive definite function of absolute value 1. Hence $\hat{\nu}$ is a character on $\mathfrak{N}'$. We extend this character to an element $z \in Z = \widehat{\mathfrak{N}}$ and write p_{-z} for the point-mass concentrated in $-z$. Then the convolution $\nu' = \nu * \mathsf{p}_{-z}$ satisfies that

$$\widehat{\nu'}(a,b) = \begin{cases} 1 & \text{if } (a,b) \in \mathfrak{N}', \\ 0 & \text{if } (a,b) \in \mathfrak{N} \smallsetminus \mathfrak{N}'. \end{cases}$$

In other words, $\nu' = \lambda_{Z'}$ is the Haar measure of the α_Z-invariant subgroup $Z' \subset Z$, and ν is a translate of $\lambda_{Z'}$.

Since the map $\pi \colon X \longrightarrow Z$ is finite-to-one and μ is ergodic, an elementary skew-product argument shows that μ is also a translate of the Haar measure of a closed, α-invariant subgroup $Y \subset X$. As explained above, this completes the proof of the lemma. □

Proof of Theorem 1.1. Let $\phi \colon X_1 \longrightarrow X_2$ be a measurable conjugacy of α_1 and α_2. We fix a minimal nonmixing set $S \subset \mathbb{Z}^d$ for α_1 and α_2 and apply [8, Theorem 3.3] to find a prime ideal $\mathfrak{q} \subset \mathfrak{R}_d$ associated with $\mathfrak{M}_2 = \widehat{X}_2$ such that S is minimally nonmixing for $\alpha_{\mathfrak{R}_d/\mathfrak{q}}$. Choose an element $a \in \mathfrak{M}_2$ with $\mathfrak{q} = \operatorname{ann}(a)$ and $\mathfrak{N} = \mathfrak{R}_d \cdot a \cong \mathfrak{R}_d/\mathfrak{q}$. The inclusion $\mathfrak{N} \subset \mathfrak{M}_2$ determines a dual factor map $\psi \colon X_2 \longrightarrow X_2' = \widehat{\mathfrak{N}}$ of α_2 and $\alpha_2' = \alpha_{\mathfrak{N}}$. We set $\phi' = \psi \circ \phi \colon X_1 \longrightarrow X_2'$, $X' = X_1 \times X_2'$, write $\alpha' = \alpha_1 \times \alpha_2'$ for the product $\mathbb{Z}^d$-action on X', and denote by μ' the α'-invariant probability measure on $\Gamma(\phi') = \{(x, \phi'(x)) : x \in X_1\} \subset X'$ which satisfies that $\mu'\pi_1^{-1} = \lambda_{X_1}$ and $\mu'\pi_2'^{-1} = \lambda_{X_2'}$, where $\pi_1 \colon X \longrightarrow X_1$ and $\pi_2' \colon X \longrightarrow X_2'$ are the coordinate projections.

Since α_1 is irreducible, $h(\alpha_2^{\mathbf{n}}) = h(\alpha_1^{\mathbf{n}}) < \infty$ for every $\mathbf{n} \in \mathbb{Z}^d$ by Corollary 2.3. Hence α_2' is irreducible and mixing by Corollary 2.3, S is minimal nonmixing for μ, α_1 and α_2', and Lemma 3.3 implies that μ' is a translate of the Haar measure of a closed, α-invariant subgroup $Y \subset X$. It follows that ϕ' coincides λ_{X_1}-*a.e.* with an affine map.

We denote by $\psi' \colon X_1 \longrightarrow X_2'$ the homomorphism part of ϕ' (cf. (1.4)). As α_1 is irreducible, ψ' is finite-to-one. Hence $\phi' = \psi \circ \phi$ and ψ are both finite-to-one, and α_2 is again irreducible by Proposition 2.1. We repeat the first part of the proof with α_2' and X_2' replaced by α_2 and X_2 and obtain that ϕ coincides λ_{X_1}-*a.e.* with an affine map. □

4. Examples

Let α be a mixing algebraic $\mathbb{Z}^d$-action on a compact abelian group X. We denote by $\mathrm{Aut}(X)$ the group of continuous group automorphisms of X and write $\mathrm{Aff}(X)$ for the group of bijective affine maps $\phi\colon X \longrightarrow X$ (cf. (1.4)).

Let

$$A(\alpha) = \{\alpha^{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\} \subset \mathrm{Aut}(X) \tag{4.1}$$

and denote by

$$C(\alpha) \supset A(\alpha) \cong \mathbb{Z}^d \tag{4.2}$$

the *measurable centralizer* of α, i.e. the group of all λ_X-preserving Borel automorphisms of X which commute with $\alpha^{\mathbf{n}}$ λ_X-a.e., for every $\mathbf{n} \in \mathbb{Z}^d$. If X is zero-dimensional and α is irreducible and mixing, then Corollary 1.2 implies that every $\beta \in C(\alpha)$ is λ_X-a.e. equal to an affine map, i.e. that $C(\alpha) \subset \mathrm{Aff}(X)$. If α has the single fixed point 0_X then every $\phi \in \mathrm{Aff}(X)$ which commutes with α must send 0_X to 0_X, so that

$$C(\alpha) \subset \mathrm{Aut}(X). \tag{4.3}$$

The measurable centralizer is obviously a measurable conjugacy invariant. In the following examples we apply Theorem 1.1 to distinguish between, and calculate the measurable centralizer of, irreducible and mixing algebraic $\mathbb{Z}^d$-actions which look indistinguishable to other dynamical invariants.

For the first two examples we assume that $d = 2$. If $0 \neq f \in \mathfrak{R}_2^{(2)}$, then the principal ideal $\overline{\mathfrak{p}} = (f) = f\mathfrak{R}_2^{(2)}$ is prime, and the $\mathbb{Z}^2$-action $\alpha = \alpha_{\mathfrak{R}_2^{(2)}/\overline{\mathfrak{p}}}$ is irreducible, if and only if f is irreducible.

Example 4.1. Let

$$\begin{aligned} f_1 &= 1 + u_1 + u_1^2 + u_1u_2 + u_2^2, \\ f_2 &= 1 + u_1^2 + u_2 + u_1u_2 + u_2^2, \\ f_3 &= 1 + u_1 + u_1^2 + u_2 + u_2^2, \\ f_4 &= 1 + u_1 + u_1^2 + u_2 + u_1u_2 + u_2^2, \\ f_5 &= 1 + u_1 + u_1^2 + u_2^2, \end{aligned}$$

in $\mathfrak{R}_2^{(2)}$. The prime ideals $\overline{\mathfrak{p}}_i = (f_i) = f_i\mathfrak{R}_2^{(2)}$ define irreducible and mixing $\mathbb{Z}^2$-actions $\alpha_i = \alpha_{\mathfrak{R}_2^{(2)}/\overline{\mathfrak{p}}_i}$ on $X_i = X_{\mathfrak{R}_2^{(2)}/\overline{\mathfrak{p}}_i}$.

Since $\overline{\mathfrak{p}}_i \neq \overline{\mathfrak{p}}_j$ for $1 \leq i < j \leq 5$, the $\mathfrak{R}_2$-modules $\mathfrak{M}_i = \mathfrak{R}_2^{(2)}/\overline{\mathfrak{p}}_i$ are nonisomorphic. As every algebraic conjugacy $\phi\colon X_i \longrightarrow X_j$ of α_i and α_j would induce a module isomorphism $\hat{\phi}\colon \mathfrak{M}_j \longrightarrow \mathfrak{M}_i$, Corollary 1.3 implies that α_i and α_j cannot be measurably conjugate for $1 \leq i < j \leq 5$. However, all directional entropies of these actions are the same, and the set $S = \{(0,0),(1,0),(0,1)\}$ is minimal nonmixing for α_i with $i = 1,\dots,4$, but not for α_5 (cf. [6], [8]).

We shall calculate the centralizers $C(\alpha_i)$ for $i = 1,\dots,5$. Clearly, $C(\alpha_1) \subset \mathrm{Aut}(X_1)$, since 0_X is the only fixed point of α_1. Every $\beta \in \mathrm{Aut}(X_1)$ which commutes with α_1 induces a module automorphism $\hat{\beta}\colon \mathfrak{M}_1 \longrightarrow \mathfrak{M}_1$ of the form

$$\hat{\beta}(a) = g_\beta \cdot a, \ \ a \in \mathfrak{M}_1, \tag{4.4}$$

for some $g_\beta \in \mathfrak{R}_2^{(2)}$.

Although the polynomial f_1 is irreducible in $\mathfrak{R}_2^{(2)}$, it is not *absolutely irreducible*: let $F_4 = \{0, 1, \omega, \omega^2\}$ be the field with 4 elements, $\mathfrak{R}_2^{(4)} = F_4[u_1^{\pm 1}, u_2^{\pm 1}]$ and $f_1' = 1 + \omega u_1 + u_2, f_1'' = 1 + \omega^2 u_1 + u_2 \in \mathfrak{R}_2^{(4)}$. Then $f_1 = f_1' f_1''$.

The inclusion $\mathfrak{R}_2^{(2)} \subset \mathfrak{R}_2^{(4)}$ induces an $\mathfrak{R}_2$-module homomorphism $\mathfrak{R}_2^{(2)} \longrightarrow \mathfrak{R}_2^{(4)}/f_1'\mathfrak{R}_2^{(4)}$ with kernel $\overline{\mathfrak{p}}_1 \subset \mathfrak{R}_2^{(2)}$, and hence an injective $\mathfrak{R}_2$-module homomorphism $\jmath \colon \mathfrak{M}_1 \longrightarrow \mathfrak{R}_2^{(4)}/\overline{\mathfrak{q}}_1$, where $\overline{\mathfrak{q}}_1 = f_1'\mathfrak{R}_2^{(4)}$.

Let $h \in F_4[u_2^{\pm 1}] \subset \mathfrak{R}_2^{(4)}$ be the Laurent polynomial obtained by replacing every occurrence of u_1 in g_β by $\omega^2(1 + u_2)$. Then $h - g_\beta \in \overline{\mathfrak{q}}_1$.

Suppose that h vanishes at a point $\omega' \in \overline{F}_2 \smallsetminus \{0, 1\}$, where $\overline{F}_2$ is the algebraic closure of F_2. As $f_1(\omega^2(1+\omega'), \omega') = g_\beta(\omega^2(1+\omega'), \omega') = 0$, every element in the ideal $J = (f_1, g_\beta) \subset \mathfrak{R}_2^{(2)}$ generated by f_1 and g_β vanishes at the point $(\omega^2(1+\omega'), \omega')$, i.e. $J \neq \mathfrak{R}_2^{(2)}$. Since

$$g_\beta \cdot \mathfrak{M}_1 = J/\mathfrak{p} \subsetneq \mathfrak{M}_1, \tag{4.5}$$

this shows that $\hat{\beta} \colon \mathfrak{M}_1 \longrightarrow \mathfrak{M}_1$ is not surjective and $\beta \colon X_1 \longrightarrow X_1$ is not injective. This contradiction implies that

$$h(u_2) = \omega^m (1 + u_2)^{n_1} u_2^{n_2}$$

for some $(n_1, n_2) \in \mathbb{Z}^2$ and $m \in \mathbb{Z}$. By remembering that $h + \overline{\mathfrak{q}}_1 = \jmath(g_\beta + \overline{\mathfrak{p}}_1)$ we obtain that

$$g_\beta = u^{\mathbf{n}}$$

for some $\mathbf{n} \in \mathbb{Z}^2$.

We have proved that $C(\alpha_1) = A(\alpha_1)$ (cf. (4.1)), and the same kind of argument shows that $C(\alpha_i) = A(\alpha_i)$ for $i = 2, 3$.

Since α_4 and α_5 have nonzero fixed points, $C(\alpha_j) \neq A(\alpha_j)$ for $j = 4, 5$.

In order to calculate $C(\alpha_4) \cap \mathrm{Aut}(X_4)$ we proceed exactly as for α_1, assume that $\beta \in C(\alpha_4) \cap \mathrm{Aut}(X_4)$, and define $g_\beta \in \mathfrak{R}_2^{(2)}$ by (4.4). We set $f_4' = 1 + \omega u_1 + \omega^2 u_2$, $f_4'' = 1 + \omega^2 u_1 + \omega u_2 \in \mathfrak{R}_2^{(4)}$, note that $f_4 = f_4' f_4''$, and obtain an injective $\mathfrak{R}_2$-module homomorphism $\jmath \colon \mathfrak{M}_4 \longrightarrow \mathfrak{R}_2^{(4)}/\overline{\mathfrak{q}}_4$, where $\overline{\mathfrak{q}}_4 = f_4'\mathfrak{R}_2^{(4)}$.

Let $h \in F_4[u_1^{\pm 1}] \subset \mathfrak{R}_2^{(4)}$ be the Laurent polynomial obtained by replacing every occurrence of u_2 in g_β by $\omega + \omega^2 u_1$. As above we find that h can only vanish at 0 and ω^2, and hence that $h(u_1) = \omega^m u_1^{n_1}(1 + \omega u_1)^{n_2}$ for some $m, n_1, n_2 \in \mathbb{Z}$. It follows that g_β is of the form

$$g_\beta = u^{\mathbf{n}}$$

for some $\mathbf{n} \in \mathbb{Z}^2$, and hence that

$$C(\alpha_4) \cap \mathrm{Aut}(X_4) = A(\alpha_4) \cong \mathbb{Z}^2.$$

Finally we calculate $C(\alpha_5) \cap \mathrm{Aut}(X_5)$. Assume that $\beta \in C(\alpha_5) \cap \mathrm{Aut}(X_5)$, define $g_\beta \in \mathfrak{R}_2^{(2)}$ by (4.4), and consider the Laurent polynomial $h \in F_2[u_1^{\pm 1}]$

obtained by replacing every occurrence of u_2^2 in $g_\beta^2 = g_{\beta^2}$ by $1+u_1+u_1^2$. As above we find that

$$h(u_1) = u_1^{n_1}(1+u_1+u_1^2)^{n_2},$$

and hence that

$$g_{\beta^2} = u_1^{n_1} u_2^{2n_2}$$

for some $(n_1, n_2) \in \mathbb{Z}^2$. This shows that $\beta^2 \in A(\alpha_5)$ for every $\beta \in C(\alpha_5) \cap \mathrm{Aut}(X_5)$.

Put $\beta = 1 + \alpha_5^{(1,0)} + \alpha_5^{(0,1)}$. Then $g_\beta = 1 + u_1 + u_2$, $\beta^2 = \alpha^{(1,0)}$, and

$$C(\alpha_5) \cap \mathrm{Aut}(X_5) \cong \mathbb{Z}^2,$$
$$|(C(\alpha_5) \cap \mathrm{Aut}(X_5))/A(\alpha_5)| = 2.$$

Example 4.2. We use the notation of Example 4.1 and set $f = 1 + \omega u_1 + \omega^2 u_2 \in \mathfrak{R}_2^{(4)}$. The inclusion $\mathfrak{R}_2^{(2)} \subset \mathfrak{R}_2^{(4)}$ induces an injective $\mathfrak{R}_2$-module homomorphism $\jmath\colon \mathfrak{M}_4 = \mathfrak{R}_2^{(2)}/\overline{\mathfrak{p}} \longrightarrow \mathfrak{M} = \mathfrak{R}_2^{(4)}/\overline{\mathfrak{q}}$, where $\overline{\mathfrak{p}} = f_4 \mathfrak{R}_2^{(2)} \subset \mathfrak{R}_2^{(2)}$ (cf. Example 4.1), and $\overline{\mathfrak{q}} = f\mathfrak{R}_2^{(4)}$. Since $\jmath(\mathfrak{M}_4)$ has index 2 in $\mathfrak{M}$, there exists a two-to-one surjective dual homomorphism $\psi\colon X_{\mathfrak{M}} = \widehat{\mathfrak{M}} \longrightarrow X_4 = \widehat{\mathfrak{M}}_4$ with $\alpha_4^{\mathbf{n}} \cdot \psi = \psi \cdot \alpha_{\mathfrak{M}}^{\mathbf{n}}$ for every $\mathbf{n} \in \mathbb{Z}^2$.

Since the $\mathfrak{R}_2$-modules $\mathfrak{M}$ and $\mathfrak{M}_4$ are nonisomorphic, Corollary 1.3 shows that the irreducible and mixing $\mathbb{Z}^2$-actions $\alpha_{\mathfrak{M}}$ and α_4 are not measurably conjugate, although they are weakly algebraically conjugate.

Example 4.3. Let $f = 1 + u_1 + u_2 \in \mathfrak{R}_2^{(2)}$, $\overline{\mathfrak{p}} = f\mathfrak{R}_2^{(2)} \subset \mathfrak{R}_2^{(2)}$ and $\mathfrak{M} = \mathfrak{R}_2^{(2)}/\overline{\mathfrak{p}}$. As in Example 4.1 one can show that

$$C(\alpha_{\mathbf{M}}) = A(\alpha_{\mathbf{M}}).$$

A proof of this can also be found in [15, Corollary 31.3].

Example 4.4 (Conjugacy of $\mathbb{Z}^2$-actions with positive entropy)**.** As in Example 4.1 we set

$$\begin{aligned} f_1 =& 1 + u_1 + u_1^2 + u_1u_2 + u_2^2, \\ f_2 =& 1 + u_1^2 + u_2 + u_1u_2 + u_2^2, \\ f_3 =& 1 + u_1 + u_1^2 + u_2 + u_2^2, \\ f_4 =& 1 + u_1 + u_1^2 + u_2 + u_1u_2 + u_2^2, \end{aligned}$$

but this time we view these polynomials as elements of the form (2.1) in $\mathfrak{R}_2$ and set

$$X_i = \Bigl\{ (x_{\mathbf{n}}) \in (\mathbb{Z}/4\mathbb{Z})^{\mathbb{Z}^2} : \sum_{\mathbf{m} \in \mathbb{Z}^2} c_{f_i}(\mathbf{m}) x_{\mathbf{m}+\mathbf{n}} = 0 \,(\mathrm{mod}\, 2) \text{ for every } \mathbf{n} \in \mathbb{Z}^2 \Bigr\}.$$

Denote by α_i the restriction to X_i of the shift-action (1.1) of $\mathbb{Z}^2$ on $\Omega = (\mathbb{Z}/4\mathbb{Z})^{\mathbb{Z}^2}$. The dual module $\mathfrak{M}_i = \widehat{X}_i$ of α_i is of the form

$$\mathfrak{M}_i = \mathfrak{R}_2/I_i,$$

where $I_i = (4, f_i) = 4\mathfrak{R}_2 + f_i\mathfrak{R}_2$. The prime ideals associated with $\mathfrak{M}_i$ are $\mathfrak{q} = (2)$ and $\mathfrak{p}_i = (2, f_i)$. Since $h(\alpha_{\mathfrak{R}_2/\mathfrak{q}}) = \log 2$ and $h(\alpha_{\mathfrak{R}_2/\mathfrak{p}_i}) = 0$, Theorem

6.5 in [10] implies that the Pinsker algebra $\pi(\alpha_i)$ of α_i is the sigma-algebra $\mathcal{B}_{X_i/Y_i}$ of Y_i-invariant Borel sets in X_i, where $Y_i = \mathfrak{N}_i^{\perp}$ and

$$\mathfrak{N}_i = \{a \in \mathfrak{M}_i : \mathfrak{p}_i \cdot a = 0\} = 2\mathfrak{M}_i \cong \mathfrak{R}_2/\mathfrak{p}_i.$$

In other words, the $\mathbb{Z}^2$-action β_i induced by α_i on the Pinsker algebra $\pi(\alpha_i)$ is isomorphic to $\alpha_{\mathfrak{R}_2/\mathfrak{p}_i}$.

Since any measurable conjugacy of α_i and α_j would map $\pi(\alpha_i)$ to $\pi(\alpha_j)$ and induce a conjugacy of β_i and β_j, Example 4.1 implies that α_i and α_j are measurably nonconjugate for $1 \leq i < j \leq 4$.

Examples 4.5. We assume that $d = 3$.

(1) Let $f_1 = 1 + u_1 + u_2 + u_3$, $f_2 = 1 + u_1 + u_2^2 \in \mathfrak{R}_3^{(2)}$, $\overline{\mathfrak{p}} = (f_1, f_2) = f_1\mathfrak{R}_3^{(2)} + f_2\mathfrak{R}_3^{(2)}$, $\mathfrak{M} = \mathfrak{R}_3^{(2)}/\overline{\mathfrak{p}}$, $\alpha = \alpha_{\mathfrak{M}}$ and $X = X_{\mathfrak{M}}$. We denote by

$$\begin{aligned} V(\overline{\mathfrak{p}}) &= \{c \in (\overline{F}_2 \smallsetminus \{0\})^3 : f(c) = 0 \text{ for every } f \in \overline{\mathfrak{p}}\} \\ &= \{(1 + a^2, a, a + a^2) : a \in \overline{F}_2 \smallsetminus \{0\}\} \end{aligned} \tag{4.6}$$

the *variety* of $\overline{\mathfrak{p}}$. Since $\overline{\mathfrak{p}}$ is *radical*, i.e. since

$$\overline{\mathfrak{p}} = \{f \in \mathfrak{R}_3^{(2)} : f(c) = 0 \text{ for every } c \in V(\overline{\mathfrak{p}})\},$$

the ideal $\overline{\mathfrak{p}}$ is easily seen to be prime. From Proposition 2.1 we conclude that α is irreducible and ergodic. However, α is not mixing by Proposition 2.1, since $u_1u_2^2u_3^{-2} - 1 \in \overline{\mathfrak{p}}$.

Although α is not mixing, the $\mathbb{Z}^2$-action $\beta\colon \mathbf{n} = (n_1, n_2) \mapsto \beta^{\mathbf{n}} = \alpha^{(0,n_1,n_2)}$ on X is of the form $\beta = \alpha_{\mathfrak{R}_2^{(2)}/\overline{\mathfrak{q}}}$ with $\overline{\mathfrak{q}} = (1+u_1+u_1^{-1}u_2)\mathfrak{R}_2^{(2)}$. By Proposition 2.1, β is irreducible and mixing, and Theorem 1.1 allows us to prove as in Example 4.3 that $C(\alpha) = C(\beta) = A(\beta) = A(\alpha)$.

(2) Let Let $f_1 = 1 + u_1 + u_2$, $f_2 = 1 + u_1 + u_3 + u_3^2 \in \mathfrak{R}_3^{(2)}$, $\overline{\mathfrak{p}} = (f_1, f_2) = f_1\mathfrak{R}_3^{(2)} + f_2\mathfrak{R}_3^{(2)}$, $\mathfrak{M} = \mathfrak{R}_3^{(2)}/\overline{\mathfrak{p}}$, $\alpha = \alpha_{\mathfrak{M}}$ and $X = X_{\mathfrak{M}}$. Then

$$V(\overline{\mathfrak{p}}) = \{(1 + a + a^2, a + a^2, a) : a \in \overline{F}_2 \smallsetminus \{0\}\},$$

$\overline{\mathfrak{p}}$ is prime, α is irreducible and mixing, and $C(\alpha) = A(\alpha)$.

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