ALGEBRAIC CODING OF EXPANSIVE GROUP AUTOMORPHISMS AND TWO-SIDED BETA-SHIFTS

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Abstract. Let α be an expansive automorphisms of compact connected abelian group X whose dual group $\hat{X}$ is cyclic w.r.t. α (i.e. $\hat{X}$ is generated by $\{\chi \cdot \alpha^n : n \in \mathbb{Z}\}$ for some $\chi \in \hat{X}$). Then there exists a canonical group homomorphism ξ from the space $\ell^\infty(\mathbb{Z}, \mathbb{Z})$ of all bounded two-sided sequences of integers onto X such that $\xi \cdot \sigma = \alpha \cdot \xi$, where σ is the shift on $\ell^\infty(\mathbb{Z}, \mathbb{Z})$. We prove that there exists a sofic subshift $V \subset \ell^\infty(\mathbb{Z}, \mathbb{Z})$ such that the restriction of ξ to V is surjective and almost one-to-one. In the special case where α is a hyperbolic toral automorphism with a single eigenvalue $\beta > 1$ and all other eigenvalues of absolute value < 1 we show that, under certain technical and possibly unnecessary conditions, the restriction of ξ to the two-sided beta-shift $V_\beta \subset \ell^\infty(\mathbb{Z}, \mathbb{Z})$ is surjective and almost one-to-one.

The proofs are based on the study of homoclinic points and an associated algebraic construction of symbolic representations in [13] and [7]. Earlier results in this direction were obtained by Vershik ([24]–[26]), Kenyon and Vershik ([10]), and Sidorov and Vershik ([20]–[21]).

1. Introduction

The classical constructions of symbolic representations of hyperbolic toral automorphisms are based on their geometrical properties and make no significant use of algebra (cf. [1], [3], [6], [22]). In [24] a different approach was proposed, based on arithmetical ideas, leading to a symbolic representation of the automorphism $\alpha = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ of the two-torus $\mathbb{T}^2$ in terms of the two-sided golden mean shift. The paper [10] describes a much more general, but also less canonical, algebraic method for finding finite-to-one sofic and Markov covers of arbitrary hyperbolic toral automorphisms by using an alphabet consisting of a suitable finite set of integers in the number field generated by the characteristic polynomial of the automorphism (for terminology we refer to Section 4). As was pointed out in [10], these constructions also give rise to certain self-similar tilings (cf. [9], [16] and [23]).

The symbolic representation of $\alpha = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ on $\mathbb{T}^2$ in [24] was defined in terms of homoclinic points of the automorphism α (cf. also [25], [10], [20]–[21]). In [7], a systematic approach to the algebraic construction of symbolic

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covers of expansive group automorphisms (and, more generally, of expansive $\mathbb{Z}^d$-actions by automorphisms of compact abelian groups) was developed, based on the analysis of the 'homoclinic group' of $\mathbb{Z}^d$-actions in [13]. In order to explain this approach in the special case of a single expansive automorphism α of a compact connected abelian group X (the relevant definitions can be found in Section 2) we follow [13] and introduce the notion of a fundamental homoclinic point of α (Definition 3.1). Proposition 3.2 shows that α has a fundamental homoclinic point $x^\Delta \in X$ if and only if the dual group $\hat{X}$ of X is *cyclic* with respect to α, i.e. if there exists a character $a \in \hat{X}$ such that the group $\hat{X}$ is generated by $\{\hat{\alpha}^n(a) : n \in \mathbb{Z}\}$, where $\hat{\alpha}$ is the automorphism of $\hat{X}$ dual to α. For a hyperbolic toral automorphism $\alpha \in \mathrm{GL}(n, \mathbb{Z})$ of $\mathbb{T}^n$ the latter condition is equivalent to the requirement that α is conjugate within $\mathrm{GL}(n, \mathbb{Z})$ to the companion matrix of its characteristic polynomial (Remark 2.4).

Due to its exponential decay properties the fundamental homoclinic point x^Δ defines a surjective group homomorphism $\xi \colon \ell^\infty(\mathbb{Z}, \mathbb{Z}) \longmapsto X$ from the space $\ell^\infty(\mathbb{Z}, \mathbb{Z})$ of all bounded two-sided sequences of integers to X given by

$$\xi(v) = \sum_{n \in \mathbb{Z}} v_n \alpha^n x^\Delta \tag{1.1}$$

for every $v = (v_n) \in \ell^\infty(\mathbb{Z}, \mathbb{Z})$. We refer to [13], [7] and Section 3 for background and details.

The map ξ in (1.1) is, of course, not injective, and the algebraic construction of symbolic representations of the expansive automorphism α consists of finding 'nice' weak*-closed, bounded, shift-invariant subsets $W \subset \ell^\infty(\mathbb{Z}, \mathbb{Z})$ such that $\xi(W) = X$ and ξ is finite-to-one or (preferably) almost one-to-one on W. Here 'nice' means that W is sofic or of finite type, and 'almost one-to-one' means that the restriction of ξ to W is injective on the set of doubly transitive points in W — cf. Section 4.

In this paper we prove the following extension of the main result in [10].

Theorem 1.1. *Let α be an expansive automorphism of a compact connected abelian group X such that the dual group $\hat{X}$ is cyclic w.r.t. α. Then there exists a mixing sofic shift $V \subset \ell^\infty(\mathbb{Z}, \mathbb{Z})$ with the following properties.*

(1) *$\xi(V) = X$, where $\xi \colon \ell^\infty(\mathbb{Z}, \mathbb{Z}) \longmapsto X$ is given by* (1.1)*;*
(2) *The restriction of ξ to V is almost one-to-one.*

We remark in passing that the restriction to connected groups in Theorem 1.1 is made only for convenience: if the group X is totally disconnected, α is topologically conjugate to a full shift by [11], and in the general situation of a disconnected, but not zero-dimensional compact abelian group is a fairly straightforward combination of the connected and zero-dimensional cases.

Theorem 1.1 is not constructive: it only asserts the *existence* of a sofic shift $V \subset \ell^\infty(\mathbb{Z}, \mathbb{Z})$ with the properties described there. The papers [24] and [21] deal with specific choices of such subshifts $V \subset \ell^\infty(\mathbb{Z}, \mathbb{Z})$ in some very special cases. Although this is not stated explicitly in [24], the symbolic representation of the automorphism $\alpha = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ by the 'golden mean shift' $V \subset \{0,1\}^{\mathbb{Z}} \subset \ell^\infty(\mathbb{Z}, \mathbb{Z})$ constructed there is based on the fundamental homoclinic point of α. In [21], the authors write down the map (1.1) explicitly and prove the following more general result: *Suppose that* $\alpha \in \mathrm{GL}(2,\mathbb{Z})$ *is* (*conjugate to*) *the companion matrix of its characteristic polynomial, and let* β *be the larger eigenvalue of* α. *Then the restriction of* ξ *to the two-sided beta-shift* $V_\beta \subset \ell^\infty(\mathbb{Z},\mathbb{Z})$ *is surjective and almost one-to-one.*

This result raises an interesting question. Let $\alpha \in \mathrm{GL}(n,\mathbb{Z})$ be an automorphism of $X = \mathbb{T}^n$ with the following properties:

(1) α is conjugate (in $\mathrm{GL}(n,\mathbb{Z})$) to the companion matrix of its characteristic polynomial,

(2) α has a single eigenvalue $\beta > 1$, and all other eigenvalues of α have absolute value < 1.

If $V_\beta \subset \ell^\infty(\mathbb{Z},\mathbb{Z})$ is the two-sided beta-shift (cf. (6.10)), then we prove in Proposition 6.1 that $\xi(V_\beta) = X$ and the restriction of ξ to V_β is bounded-to-one.

Question 1.2. Is the restriction of ξ to V_β almost one-to-one?

If this is the case then the two-sided beta-shift V_β could be viewed as a 'natural' sofic representation of the automorphism α. In Theorem 6.3 we provide further support for the conjecture that Question 1.2 always has a positive answer: if β is 'simple' (i.e. if 1 has a strictly periodic beta-expansion), and if the set of nonzero elements in $\xi^{-1}(\{0\}) \cap V_\beta$ consists of a single orbit under the shift, then the restriction of ξ to V_β is almost one-to-one (it should be noted that the first of these assumptions also implies that V_β is a shift of finite type).

The exposition is organised as follows. Section 2 contains the characterisation of automorphisms α of compact abelian groups whose duals are cyclic w.r.t. α (Proposition 2.2). Section 3 discusses homoclinic and fundamental homoclinic points (Definition 3.1), characterises those expansive and ergodic automorphisms of compact abelian groups which possess a fundamental homoclinic point (Proposition 3.2), and introduces the map ξ in (1.1). In Section 4 we investigate the construction of almost one-to-one sofic covers of expansive and ergodic group automorphisms, following the approach in [14], [10] and [7] (Proposition 4.2). Section 5 is devoted to proving Theorem 1.1 in an equivalent form (Theorem 5.1), and Section 6 discusses and provides partial answers to Question 1.2.

2. Expansive automorphisms of compact abelian groups

Definition 2.1. Let α be a continuous automorphism of a compact abelian additive group X with identity element $0 = 0_X$. Then α is *expansive* if there exists an open set $\mathcal{O} \subset X$ such that

$$\bigcap_{n\in\mathbb{Z}} \alpha^{-n}(\mathcal{O}) = \{0\}.$$

The dual group $\hat{X}$ is *cyclic with respect to* α if there exists a character $a \in \hat{X}$ such that $\hat{X}$ is generated by the set $\{\hat{\alpha}^n(a) : n \in \mathbb{Z}\}$, where $\hat{\alpha}$ is the automorphism of $\hat{X}$ dual to α.

In order to describe all automorphisms α of compact connected abelian groups whose dual is cyclic w.r.t. α we denote by $R = \mathbb{Z}(u^{\pm 1})$ the ring of

Laurent polynomials with integer coefficients in the variable u and write every $h \in R$ as

$$h = \sum_{n\in\mathbb{Z}} h_n u^n, \tag{2.1}$$

with $h_n \in \mathbb{Z}$ for every $n \in \mathbb{Z}$ and $h_n \neq 0$ for only finitely many $n \in \mathbb{Z}$. An element $h \in R$ is *primitive* if the highest common factor of the coefficients $\{h_n : n \in \mathbb{Z}\}$ is equal to 1.

Every primitive $h \in R$ defines an automorphism α_h of a compact connected abelian group X_h as follows. Denote by $\sigma\colon \mathbb{T}^{\mathbb{Z}} \longmapsto \mathbb{T}^{\mathbb{Z}}$ the shift

$$(\sigma x)_n = x_{n+1} \tag{2.2}$$

for every $x = (x_n) \in \mathbb{T}^{\mathbb{Z}}$, and put

$$h(\sigma)(x) = \sum_{n\in\mathbb{Z}} h_n \sigma^n x \tag{2.3}$$

for every $x \in \mathbb{T}^{\mathbb{Z}}$ and $h \in R$. Then $\ker(h(\sigma))$ is a closed, connected, shift-invariant subgroup of $\mathbb{T}^{\mathbb{Z}}$. The following proposition is a special case of much more general results (cf. [18] and [19]).

Proposition 2.2. *Let $h \in R$ be a Laurent polynomial, and let α_h be the restriction to*

$$X_h = \ker(h(\sigma)) \subset \mathbb{T}^{\mathbb{Z}}$$

of the shift-action σ of $\mathbb{Z}$ on $\mathbb{T}^{\mathbb{Z}}$. The following conditions are equivalent.

(1) *α_h is expansive;*
(2) *h has no roots of absolute value* 1.

If α_h is expansive then it is ergodic with respect to the normalised Haar measure λ_{X_h} of X_h.

Finally, if α is an arbitrary automorphism of a compact connected abelian group X, then the dual group $\hat{X}$ of X is cyclic w.r.t. α if and only if there exists a primitive Laurent polynomial $h \in R$ and a continuous group isomorphism $\phi\colon X \longmapsto X_h$ such that $\alpha_h \cdot \phi = \phi \cdot \alpha$.

Motivated by Proposition 2.2 we adopt the following terminology.

Definition 2.3. A Laurent polynomial $h \in R$ is *hyperbolic* if it is primitive and has no roots of absolute value 1 or, equivalently, if X_h is connected and the automorphism α_h is expansive.

Remarks 2.4. (1) Let $\alpha \in \mathrm{GL}(n, \mathbb{Z})$ be an automorphism of $X = \mathbb{T}^n$ with characteristic polynomial h. Then α is (algebraically) conjugate to α_h if and only if it is conjugate in $\mathrm{GL}(n, \mathbb{Z})$ to the companion matrix

$$\begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -h_0 & -h_1 & \dots & -h_{n-2} & -h_{n-1} \end{pmatrix}$$

of $h = h_0 + h_1 u + \cdots + u^n$.

For example, the matrices $\alpha = \left(\begin{smallmatrix} 3 & 4 \\ 1 & 1 \end{smallmatrix}\right)$ and $\alpha' = \left(\begin{smallmatrix} 3 & 2 \\ 2 & 1 \end{smallmatrix}\right)$ in $\mathrm{GL}(2, \mathbb{Z})$ have the same characteristic polynomial $h = -1 - 4u + u^2$, α is conjugate to the companion matrix $\beta = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 4 \end{smallmatrix}\right)$ of h, but there is no $M \in \mathrm{GL}(2, \mathbb{Z})$ with

$\alpha' M = M\beta$. For references concerning conjugacy of matrices in $\mathrm{GL}(2,\mathbb{Z})$ and related problems we refer to [2].

(2) Every $f = \sum_{n\in\mathbb{Z}} f_n u^n \in R$ determines a character $\langle f,\cdot\rangle$ of $\mathbb{T}^{\mathbb{Z}}$ by

$$\langle f,x\rangle = \prod_{n\in\mathbb{Z}} e^{2\pi i f_n x_n} \tag{2.4}$$

for every $x = (x_n) \in \mathbb{T}^{\mathbb{Z}}$. Furthermore, if $h \in R$, and if $X_h \subset \mathbb{T}^{\mathbb{Z}}$ is defined as in Proposition 2.2, then (2.4) allows us to identify $\widehat{\mathbb{T}^{\mathbb{Z}}}$ with R and $\widehat{X_h} = \mathbb{T}^{\mathbb{Z}}/X_h^{\perp}$ with the quotient ring $R/(h)$, where $(h) = hR = \{hf : f \in R\} \subset R$ is the principal ideal generated by h. Under this identification the automorphism α_h of X_h is dual to multiplication by u on $R/(h)$.

(3) If $I \subset R$ is an arbitrary ideal, then

$$X_I = \bigcap_{h\in I} X_h = \widehat{R/I} \tag{2.5}$$

is a closed, shift-invariant subgroup of $\mathbb{T}^{\mathbb{Z}}$, and the restriction α_I of σ to X_I is dual to multiplication by u on R/I (cf. Example (2) above). However, we claim that α_I is nonergodic if I is not principal.

Indeed, let $I \subset R$ be nonprincipal, and let $h = \gcd(I)$ be the highest common factor of all elements of I (this highest common factor is well defined up to multiplication by $\pm u^n$, $n \in \mathbb{Z}$; in particular, the principal ideal $(h) = hR$ is uniquely defined). Then $(h) \supset I$, $(h)/I$ is finite, and we denote by $\hat{\zeta}\colon R/I \longmapsto R/(h)$ the quotient map. The dual group homomorphism $\zeta\colon X_h \longmapsto X_I$ is shift-commuting and injective, and $X_I/\zeta(X_h)$ is finite. This shows that σ_I must be nonergodic, since X_I has a closed, shift-invariant subgroup of finite index. For details we refer to [19].

3. Homoclinic points

Definition 3.1. Let α be an automorphism of a compact abelian group X. A point $x \in X$ is *homoclinic* if $\lim_{|n|\to\infty} \alpha^n x = 0$. The set of homoclinic points of α is a subgroup of X denoted by $\Delta_\alpha(X)$. A homoclinic point $x \in X$ is *fundamental* if $\Delta_\alpha(X)$ is generated by $\{\alpha^n x : n \in \mathbb{Z}\}$ or, equivalently, if every $y \in \Delta_\alpha(X)$ is of the form

$$y = h(\alpha)(x) = \sum_{n\in\mathbb{Z}} h_n \alpha^n x$$

for some $h = \sum_{n\in\mathbb{Z}} h_n u^n \in R$.

Proposition 3.2. *Let α be an expansive automorphism of a compact connected abelian group X. Then the group $\Delta_\alpha(X)$ of homoclinic points of α is countable and dense in X. The following conditions are equivalent.*

1. *α has a fundamental homoclinic point;*
2. *$\hat{X}$ is cyclic with respect to α (Definition 2.1);*
3. *There exists a hyperbolic polynomial $h \in R$ and a continuous group isomorphism $\phi\colon X \longmapsto X_h$ such that $\phi \cdot \alpha = \alpha_h \cdot \phi$.*

Proof. The first assertion is clear from Lemma 3.2 and Theorem 4.2 in [13], and the equivalence of (2) and (3) follows from Proposition 2.2. If $\alpha = \alpha_h$

and $X = X_h$ for some hyperbolic $h \in R$ then the proof of Lemma 4.5 in [13] shows that there exists a fundamental homoclinic point of α (cf. (3.16)).

In order to prove the last remaining implication (1) $\Rightarrow$ (3) we view $\Delta_\alpha(X)$ as a discrete group, denote by $\hat{\beta}$ the restriction of α to $\Delta_\alpha(X)$, write $Y = \widehat{\Delta_\alpha(X)}$ for the dual group of $\Delta_\alpha(X)$, and consider the automorphism β of Y dual to $\hat{\beta}$. Since $\alpha^n x \neq x$ whenever $n \neq 0$ and $0 \neq x \in \Delta_\alpha(X)$, β is ergodic on Y.

Let $\imath\colon \Delta_\alpha(X) \longmapsto X$ be the inclusion map. As $\imath$ is injective and $\Delta_\alpha(X)$ is dense in X, the dual homomorphism $\hat{\imath}\colon \hat{X} \longmapsto Y$ is injective and $\hat{\imath}(\hat{X})$ is dense in Y. Furthermore, since

$$\imath \cdot \hat{\beta} = \alpha \cdot \imath, \tag{3.1}$$

we obtain that

$$\hat{\imath} \cdot \hat{\alpha} = \beta \cdot \hat{\imath}. \tag{3.2}$$

We write $\Delta_\beta(Y)$ for the homoclinic group of β and claim that $\hat{\imath}(\hat{X}) \subset \Delta_\beta(Y)$.

In order to prove this claim we fix $\chi \in \hat{X}$ for the moment. Since $\lim_{|n|\to\infty} \alpha^n x = 0$ for every $x \in \Delta_\alpha(X)$,

$$\begin{aligned}\lim_{|n|\to\infty} \chi(\alpha^n \cdot \imath(x)) &= \lim_{|n|\to\infty} \chi(\imath \cdot \hat{\beta}^n(x)) \\ &= \lim_{|n|\to\infty} \hat{\imath}(\chi)(\hat{\beta}^n x) = \lim_{|n|\to\infty} \beta^n \cdot \hat{\imath}(\chi)(x) = 1\end{aligned}$$

for every $x \in \Delta_\alpha(X)$, which implies that $\hat{\imath}(\chi) \in \Delta_\beta(Y)$. As $\chi \in \hat{X}$ was arbitrary we conclude that $\hat{\imath}(\hat{X}) \subset \Delta_\beta(Y)$, as claimed.

This allows us to view $\hat{\imath}$ as a map $\hat{\imath}'\colon \hat{X} \longmapsto \Delta_\beta(Y)$ with $\hat{\imath}'(b) = \hat{\imath}(b)$ for every $b \in \hat{X}$. We write $\jmath\colon \Delta_\beta(Y) \longmapsto Y$ for the inclusion and observe that the injective maps

$$\hat{X} \xrightarrow{\hat{\imath}'} \Delta_\beta(Y) \xrightarrow{\jmath} Y$$

dualise to

$$\Delta_\alpha(X) \xrightarrow{\hat{\jmath}} \widehat{\Delta_\beta(Y)} \xrightarrow{\imath'} X,$$

where the homomorphism $\imath'$ dual to $\hat{\imath}'$ is surjective. Furthermore, $\imath'$ is injective, since $\imath' \cdot \hat{\jmath} = \imath$ is injective. Similarly one sees that $\hat{\jmath}$ is a bijection. This allows us to make the following identifications:

$$\begin{aligned}\hat{X} &= \Delta_\beta(Y) \subset Y, & \hat{Y} &= \Delta_\alpha(X) \subset X, \\ \hat{\alpha} &= \beta_{\Delta_\beta(Y)}, & \hat{\beta} &= \alpha_{\Delta_\alpha(X)},\end{aligned} \tag{3.3}$$

where $\alpha_{\Delta_\alpha(X)}$ and $\beta_{\Delta_\beta(Y)}$ are the restrictions of α and β to $\Delta_\alpha(X)$ and $\Delta_\beta(Y)$, respectively (cf. (3.1)–(3.2)).

If α has a fundamental homoclinic point, then there exist an ideal $I \subset R$ and a group isomorphism $\hat{\zeta}\colon \Delta_\alpha(X) \longmapsto R/I$ such that $\hat{\zeta} \cdot \hat{\zeta}(x) = u\hat{\zeta}(x)$ for every $x \in \Delta_\alpha(X)$. We denote by $\zeta\colon \widehat{R/I} \longmapsto Y = \widehat{\Delta_\alpha(X)}$ the dual isomorphism and obtain that $\zeta \cdot \alpha_I = \beta \cdot \zeta$, where α_I is defined as in Remark 2.4 (3). The ergodicity of β implies that I is principal (cf. Remark 2.4 (3)). We choose a Laurent polynomial $h \in R$ with $I = (h) = hR$ and apply Lemma 4.5 in [13] to obtain that $\Delta_\beta(Y) = \hat{X} \cong R/(h)$. According to

Remark 2.4 (2) there exists an isomorphism $\phi\colon X \longmapsto X_h$ with $\phi\cdot\alpha = \alpha_h\cdot\phi$, as claimed. □

For the remainder of this section we fix a hyperbolic polynomial

$$f = f_0 + \cdots + f_m u^m \in R$$

with $m \geq 1$, $f_0 \neq 0$, $f_m > 0$, and write $\alpha = \alpha_f$ for the corresponding expansive automorphism of the compact connected abelian group $X = X_f$ (cf. Proposition 2.2 and Definition 2.3). Put

$$\tilde{f} = f_0 u^m + f_1 u^{m-1} + \cdots + f_m. \tag{3.4}$$

We denote by $\|\cdot\|_1$ and $\|\cdot\|_\infty$ the norms on the Banach spaces $\ell^1(\mathbb{Z},\mathbb{R})$ and $\ell^\infty(\mathbb{Z},\mathbb{R})$ and write $\ell^1(\mathbb{Z},\mathbb{Z}) \subset \ell^1(\mathbb{Z},\mathbb{R})$ and $\ell^\infty(\mathbb{Z},\mathbb{Z}) \subset \ell^\infty(\mathbb{Z},\mathbb{R})$ for the subgroups of integer-valued functions. By viewing every $h = \sum_{n\in\mathbb{Z}} h_n u^n \in R$ as the element $(h_n) \in \ell^1(\mathbb{Z},\mathbb{Z})$ we can identify R with $\ell^1(\mathbb{Z},\mathbb{Z})$.

Consider the surjective map $\eta\colon \ell^\infty(\mathbb{Z},\mathbb{R}) \longmapsto \mathbb{T}^\mathbb{Z}$ given by

$$\eta(v)_n = v_n \pmod 1 \tag{3.5}$$

for every $v = (v_n) \in \ell^\infty(\mathbb{Z},\mathbb{R})$, and denote by $\bar{\sigma}$ the shift

$$(\bar{\sigma}v)_n = v_{n+1} \tag{3.6}$$

on $\ell^\infty(\mathbb{Z},\mathbb{R})$. As in (2.3) we set

$$h(\bar{\sigma}) = \sum_{n\in\mathbb{Z}} h_n \bar{\sigma}^n$$

for every $h \in R$ and note that the expansiveness of α is equivalent to the condition that

$$\ker(f(\bar{\sigma})) = \{0\} \subset \ell^\infty(\mathbb{Z},\mathbb{R}) \tag{3.7}$$

(cf. [18], Theorem 6.5 in [19], or Proposition 2.2 in [7]).

According to the proof of Lemma 4.5 in [13] there exists a unique point $w^\Delta \in \ell^\infty(\mathbb{Z},\mathbb{R})$ with

$$f(\bar{\sigma})(w^\Delta) = v^\Delta, \tag{3.8}$$

where

$$v_n^\Delta = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.9}$$

The point w^Δ also has the properties that there exist constants $c_1 > 0$, $0 < c_2 < 1$ with

$$|w_n^\Delta| \leq c_1 c_2^{|n|} \tag{3.10}$$

for every $n \in \mathbb{Z}$. It follows that

$$\|w^\Delta\|_1 = \sum_{n\in\mathbb{Z}} |w_n^\Delta| < \infty, \tag{3.11}$$

and that

$$\bar{\xi}(v) = \sum_{n\in\mathbb{Z}} v_n \bar{\sigma}^n w^\Delta \tag{3.12}$$

is a well-defined element of $\ell^\infty(\mathbb{Z},\mathbb{R})$ for every $v \in \ell^\infty(\mathbb{Z},\mathbb{Z})$. As in [7] we denote by

$$\bar{\xi}\colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longmapsto \ell^\infty(\mathbb{Z},\mathbb{R}), \qquad \xi = \eta\cdot\bar{\xi}\colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longmapsto X \tag{3.13}$$

the resulting group homomorphisms. The following proposition was proved in [7].

Proposition 3.3. *For every $v \in \ell^\infty(\mathbb{Z},\mathbb{Z})$,*

$$\begin{aligned} f(\bar\sigma)(\bar\xi(v)) &= \bar\xi(f(\bar\sigma)(v)) = v, \\ \|\bar\xi(v)\|_\infty &\le \|w^\Delta\|_1 \|v\|_\infty, \\ \|v\|_\infty &\le \|f\|_1 \|\bar\xi(v)\|_\infty. \end{aligned} \tag{3.14}$$

Furthermore, $\xi\colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longmapsto X$ is a surjective group homomorphism and

$$\begin{aligned} \xi \cdot \bar\sigma^n &= \alpha^n \cdot \xi \ \text{ for every } \ n \in \mathbb{Z}, \\ \ker(\xi) &= f(\bar\sigma)(\ell^\infty(\mathbb{Z},\mathbb{Z})), \\ \ker(\xi) \cap \ell^1(\mathbb{Z},\mathbb{Z}) &= f(\bar\sigma)(\ell^1(\mathbb{Z},\mathbb{Z})) = \tilde f R. \end{aligned} \tag{3.15}$$

If we denote by

$$x^\Delta = \eta(w^\Delta) = \xi(x^\Delta) \tag{3.16}$$

the fundamental homoclinic point of $\alpha = \alpha_f$ (cf. Lemma 4.5 in [13]), then the map $\xi\colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longmapsto X_f$ is given by (1.1). From (3.10)–(3.13) it is clear that the restrictions of $\bar\xi$ and ξ to every bounded subset of $\ell^\infty(\mathbb{Z},\mathbb{Z})$ are continuous in the weak*-topology.

It is not difficult to see that there exist closed, bounded, shift-invariant subsets $V \subset \ell^\infty(\mathbb{Z},\mathbb{Z})$ with $\xi(V) = X$. A convenient set V with this property is described in Corollary 2.1 in [7]:

Proposition 3.4. *For every $h = \sum_{n\in\mathbb{Z}} h_n u^n \in R$ we set*

$$\begin{aligned} h^+ &= \sum_{n\in\mathbb{Z}} \max(0,h_n)u^n, & h^- &= -\sum_{n\in\mathbb{Z}} \min(0,h_n)u^n, \\ \|h^+\|_1' &= \max(\|h^+\|_1 - 1, 0), & \|h^-\|_1' &= \max(\|h^-\|_1 - 1, 0), \end{aligned}$$
$$\|h\|_1^* = \|h^+\|_1' + \|h^-\|_1'.$$

Then the set

$$V = \{v \in \ell^\infty(\mathbb{Z},\mathbb{Z}) : 0 \le v_n \le \|f\|_1^* \ \text{ for every } \ n \in \mathbb{Z}\} \tag{3.17}$$

satisfies that $\xi(V) = X$.

Examples 3.5. *Examples of fundamental homoclinic points.* Let $f = f_0 + \dots + f_m u^m \in R$ be a hyperbolic polynomial with $f_0 f_m \ne 0$, and define X_f and α_f as in Proposition 2.2. We arrange the roots $c_1, \dots, c_m$ of f such that $|c_1| \le \dots \le |c_l| < 1 < |c_{l+1}| \le \dots \le |c_m|$ and set

$$M_f = \begin{pmatrix} 1 & \dots & 1 \\ c_1 & \dots & c_m \\ \vdots & & \vdots \\ c_1^{m-1} & \dots & c_m^{m-1} \end{pmatrix}.$$

Then the fundamental homoclinic point x^Δ of α_f is of the form $x^\Delta = \eta(w^\Delta)$ with

$$w_k^\Delta = \begin{cases} b_1 c_1^k + \dots + b_l c_l^k & \text{for } \ k \ge 1, \\ 1 + b_1 + \dots + b_l & \text{for } \ k = 0, \\ b_{l+1} c_{l+1}^k + \dots + b_m c_m^k & \text{for } \ k \le m-1, \end{cases} \tag{3.18}$$

where

$$M_f \begin{pmatrix} b_1 \\ \vdots \\ b_l \\ -b_{l+1} \\ \vdots \\ -b_m \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.19}$$

(1) Let $f = 1 + u - u^2$. If $\pi\colon X_f \longmapsto \mathbb{T}^2$ is the coordinate projection $\pi(x) = \left(\begin{smallmatrix} x_0 \\ x_1 \end{smallmatrix}\right)$, then π is a group isomorphism and $\pi \cdot \alpha_f = \alpha \cdot \pi$ with $\alpha = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{Z})$. Then $|c_1| < 1 < |c_2|$, w^Δ is of the form

$$w_k^\Delta = \begin{cases} -\frac{1}{\sqrt{5}} c_1^{k-1} & \text{if } k \geq 1, \\ -\frac{1}{\sqrt{5}} c_2^{k-1} & \text{if } k \leq 0, \end{cases}$$

and $x^\Delta = \eta(w^\Delta)$ (cf. Example 4.7 in [13]). The fundamental homoclinic point of $\alpha = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ is thus given by

$$\pi(x^\Delta) = \begin{pmatrix} -\frac{2}{5+\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix} \pmod 1.$$

(2) Let $f = 2 - u$. Then $c_1 > 1$ and $x^\Delta = \eta(w^\Delta)$ with

$$w_k^\Delta = \begin{cases} 0 & \text{if } k \geq 1, \\ 2^{k-1} = \frac{1}{2} \cdot c_1^k & \text{if } k \leq 0. \end{cases}$$

Note that the automorphism α_f is the canonical extension of the endomorphism of $\mathbb{T}$ given by multiplication by 2.

(3) Let $f = 3 - 2u$. Then $c_1 > 1$ and $x^\Delta = \eta(w^\Delta)$ with

$$w_k^\Delta = \begin{cases} 0 & \text{if } k \geq 1, \\ \frac{3^{k-1}}{2^k} = \frac{1}{3} c_1^k & \text{if } k \leq 0. \end{cases}$$

The automorphism α_f is the canonical extension of 'multiplication by 3/2' on $\mathbb{T}$.

4. Sofic covers

Let A be a finite set (the *alphabet*), and let $V \subset A^{\mathbb{Z}}$ be a closed, shift-invariant subset, where the shift σ on $A^{\mathbb{Z}}$ is defined as in (2.2). A point $v \in V$ is *doubly transitive* if the sets $\{\sigma^n v : n \geq k\}$ and $\{\sigma^n v : n \leq -k\}$ are dense in V for every $k \geq 0$. If V contains a doubly transitive point then V is called *transitive*, and V is (*topologically*) *mixing* if, for all nonempty open sets $\mathcal{O}_1, \mathcal{O}_2 \subset V$, $\mathcal{O}_1 \cap \sigma^n(\mathcal{O}_2) \neq \varnothing$ for all sufficiently large $|n|$.

The set V is a *shift of finite type* (*SFT*) if there exists an integer $N \geq 0$ and a subset $P \subset A^N = A^{\{0,\dots,N-1\}}$ with

$$V = \{v = (v_n) \in A^{\mathbb{Z}} : (v_n, \dots, v_{n+N-1}) \in P \text{ for every } n \in \mathbb{Z}\}. \tag{4.1}$$

Note that V is a *SFT* if and only if there exists an integer $N \geq 0$ such that

$$V = \{v \in A^{\mathbb{Z}} : \pi_{\{0,\dots,N-1\}}(\sigma^n v) \in \pi_{\{0,\dots,N-1\}}(V) \text{ for every } n \in \mathbb{Z}\},$$

where $\pi_{\{0,\dots,N-1\}}\colon A^{\mathbb{Z}} \longmapsto A^{\{0,\dots,N-1\}}$ is the coordinate projection.

The set V is a *sofic shift* if there exists a finite set B, a *SFT* $W \subset B^{\mathbb{Z}}$ and a continuous, surjective, shift-equivariant map $\chi\colon W \longmapsto V$ (cf. [27]).

We return to our study of expansive automorphisms of compact groups. Let $f \in R$ be hyperbolic, and let $\alpha = \alpha_f$ and $X = X_f$ be given as in Section 3. We define $\bar{\xi}\colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longmapsto \ell^\infty(\mathbb{Z},\mathbb{R})$ and $\xi\colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longmapsto X$ by (3.13) and Proposition 3.4.

Following [7] we introduce, for every closed, bounded, shift-invariant subset $W \subset \ell^\infty(\mathbb{Z},\mathbb{Z})$ with $\xi(W) = X$, the equivalence relations

$$\begin{aligned} \mathbf{R}_W &= \{(v,v') \in W \times W : \xi(v) = \xi(v')\}, \\ \Delta_W &= \{(v,v') \in W \times W : v_n \neq v'_n \text{ for only finitely many } n \in \mathbb{Z}\}, \\ \Delta'_W &= \{(v,v') \in W \times W : v - v' \in f(\bar{\sigma})(\ell^1(\mathbb{Z},\mathbb{Z})) = \tilde{f}R\} \\ &= \mathbf{R}_W \cap \Delta_W \subset \Delta_W. \end{aligned} \tag{4.2}$$

Consider the lexicographic order $\prec$ on $R = \ell^1(\mathbb{Z},\mathbb{Z})$ defined by setting $0 \prec h$ if and only if $h_m > 0$ for the smallest $m \in \mathbb{Z}$ with $h_m \neq 0$, and by saying that $h \prec h'$ whenever $h - h' \prec 0$. The order $\prec$ on R induces the lexicographic order (again denoted by $\prec$) on each equivalence class of Δ_W: if $(v,v') \in \Delta_W$ then $v - v' \in \ell^1(\mathbb{Z},\mathbb{Z}) = R$, and $v' \prec v''$ if and only if $v' - v'' \prec 0$.

We put $R^+ = \{h \in R : 0 \prec h\}$ and set

$$\begin{aligned} W^* &= \bigcap_{h \in R^+} (W \smallsetminus (W + \tilde{f}h)) = W \smallsetminus \bigcup_{h \in R^+} (W + \tilde{f}h) \\ &= \{w \in W : w' \preceq w \text{ for every } w' \in W \text{ with } (w,w') \in \Delta'_W\}. \end{aligned} \tag{4.3}$$

Proposition 4.1. *Let $L \geq 1$, $W \subset Z = \{-L,\dots,L\}^{\mathbb{Z}}$ a* SFT *with $\xi(W) = X$, and let $W^* \subset W$ be defined by* (4.3)*. Then $\xi(W^*) = X$, W^* intersects each equivalence class $\Delta'_W(v)$, $v \in W$, in at most one point, and $h(\alpha) = h(\bar{\sigma}_{W^*})$, where $\bar{\sigma}_{W^*}$ is the restriction of $\bar{\sigma}$ to W^* and $h(\cdot)$ is topological entropy. Furthermore, the restriction of ξ to W^* is bounded-to-one.*

Proof. This is — in essence — a simplified version of the proofs of Theorem 3.1, Corollary 3.1 and Corollary 3.2 in [7]. ☐

Proposition 4.2. *Let $f \in R$ be a hyperbolic Laurent polynomial* (*cf. Definition* 2.3)*, and let $\alpha = \alpha_f$ be the expansive automorphism of the compact connected abelian group $X = X_f$ described in Proposition* 2.2*.*

Suppose furthermore that $L \geq 1$, that $W \subset Z = \{-L,\dots,L\}^{\mathbb{Z}}$ is a transitive SFT *with $\xi(W) = X$, and that there exists a fixed point $\mathbf{c} \in W$ of $\bar{\sigma}$ with*

$$\xi^{-1}(\{\xi(\mathbf{c})\}) \cap W = \{\mathbf{c}\}. \tag{4.4}$$

Then the subshift $W^* \subset W$ defined by (4.3) *is sofic and mixing, and the restriction to W^* of the group homomorphism $\xi\colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longmapsto X$ in* (3.13) *is surjective and almost one-to-one.*

We begin the proof of Proposition 4.2 with two lemmas.

Lemma 4.3. *Let $K \geq 1$, $W = \{-K,\dots,K\}^{\mathbb{Z}}$ and*

$$W' = W \cap f(\bar{\sigma})(\ell^\infty(\mathbb{Z},\mathbb{Z})).$$

Then W' is a SFT.

Proof. The expansiveness of α implies the existence of a neighbourhood $\mathcal{U}$ of $0 = 0_X$ in X such that $\bigcap_{n\in\mathbb{Z}} \alpha^n(\mathcal{U}) = \{0\}$. From (3.13) we conclude that there exists an $\varepsilon > 0$ such that every $v \in \bar{\xi}(\ell^\infty(\mathbb{Z},\mathbb{Z}))$ with

$$\sup_{n\in\mathbb{Z}} \min_{k\in\mathbb{Z}} |v_n - k| < \varepsilon \tag{4.5}$$

lies in $\ell^\infty(\mathbb{Z},\mathbb{Z})$.

According to (3.15),

$$W' = \{w \in W : \xi(w) = 0_X\} = \{w \in W : \bar{\xi}(w) \in \ell^\infty(\mathbb{Z},\mathbb{Z})\}.$$

The inequality (3.11) allows us to choose an integer $P(K) \geq 1$ with the following property: if $w, w' \in W$ satisfy that

$$w_n = w'_n \ \text{ for } \ -P(K) \leq n \leq P(K), \tag{4.6}$$

and if

$$w''_n = \begin{cases} w_n & \text{if } n \geq 0, \\ w'_n & \text{if } n < 0, \end{cases} \tag{4.7}$$

then $w'' \in W$ and

$$\begin{aligned} |(\bar{\xi}(w''))_n - (\bar{\xi}(w))_n| < \varepsilon \ \text{ for every } \ n \geq 0, \\ |(\bar{\xi}(w''))_n - (\bar{\xi}(w'))_n| < \varepsilon \ \text{ for every } \ n \leq 0. \end{aligned} \tag{4.8}$$

If the points w, w' in (4.6) lie in W', then (4.8) shows that the point w'' in (4.7) satisfies (4.5), since $(\bar{\xi}(w))_n \in \mathbb{Z}$ and $(\bar{\xi}(w'))_n \in \mathbb{Z}$ for every $n \in \mathbb{Z}$. We conclude that $\bar{\xi}(w'') \in \ell^\infty(\mathbb{Z},\mathbb{Z})$, and hence that $w'' \in W'$. This proves that W' is a *SFT*, since it satisfies (4.1) with $2P(K) + 1$ replacing N. □

Lemma 4.4. *Let Ω^* be a transitive* SFT *and $\psi\colon \Omega^* \longmapsto X$ a continuous, surjective and bounded-to-one map with $\psi \cdot \tau = \alpha \cdot \psi$, where τ is the shift on Ω^*. If there exists an element $x \in X$ with $|\psi^{-1}(\{x\})| = 1$ then ψ is injective on the set of doubly transitive points of Ω^*.*

Proof. This is the usual 'no diamonds' argument. Let $\omega^* = (\omega^*_n) \in \Omega^*$ be the unique pre-image of x under ψ. Since Ω^* is compact and ψ is continuous there exists, for every $k \geq 0$, a neighbourhood $N_k(x)$ with

$$\begin{aligned} \psi^{-1}(N_k(x)) \subset N_k(\omega^*) &= [\omega^*_{-k}, \dots \omega^*_k] \\ &= \{\omega \in \Omega^* : \omega_j = \omega^*_j \ \text{ for } \ j = -k, \dots, k\}. \end{aligned}$$

As Ω^* is a *SFT* we can choose the integer k such that, for all $\omega, \omega' \in N_k(\omega^*)$, the point $\omega'' = (\omega''_n)$ with

$$\omega''_n = \begin{cases} \omega_n & \text{if } n < 0 \\ \omega'_n & \text{if } n \geq 0 \end{cases}$$

lies in Ω^*.

With this choice of k we obtain that, for all $l \geq 1$ and $\omega, \omega' \in N_k(\omega^*) \cap \tau^{-l}(N_m(\omega^*))$ with $\psi(\omega) = \psi(\omega')$, $\omega_j = \omega'_j$ for $j = 0, \dots, l$ (otherwise we could easily find a point $y \in X$ with uncountably many pre-images under ψ). It follows that ψ is injective on

$$\bigcap_{l\geq 0}\Bigl(\bigcup_{j\geq l} \tau^{-j} N_k(\omega^*) \cap \bigcup_{j\geq l} \tau^j N_k(\omega^*)\Bigr),$$

and hence on the set of doubly transitive points in Ω^*. □

The following proof of Proposition 4.2 is based on the proof of Proposition 3.1 in [14] and Theorem 4.1 in [7] (cf. also Theorem 3 in [10]).

Proof of Proposition 4.2. By assumption, $f = f_0 + \cdots + f_m u^m$ with $f_0 \neq 0$ and $f_m > 0$. According to (3.15), any two points $w, w' \in W$ with $\xi(w) = \xi(w')$ differ by an element in $f(\bar{\sigma})(\ell^\infty(\mathbb{Z},\mathbb{Z}))$. Furthermore, if $w - w' = f(\bar{\sigma})(h)$ for some $h \in \ell^\infty(\mathbb{Z},\mathbb{Z})$, then (3.14) implies that

$$\|h\|_\infty = \|\bar{\xi}(f(\bar{\sigma})(h))\| \leq \|w^\Delta\|_1 \|f(\bar{\sigma})(w - w')\|_\infty \leq 2L\|w^\Delta\|_1\|f\|_1. \tag{4.9}$$

We write the fixed point $\mathbf{c} \in W$ in (4.4) as

$$\mathbf{c} = (\ldots, c, c, c, \ldots)$$

and conclude from (4.4) that there exists a neighbourhood $\mathcal{U}$ of the identity element $0 = 0_X \in X$ with

$$w_n = c \tag{4.10}$$

for any $n = -N, \ldots, N$ and $w \in \xi^{-1}(\mathcal{U} + \xi(\mathbf{c}))$. By increasing N, if necessary, we may assume that W satisfies (4.1) and that $N \geq P(K)$ for

$$K = (2L+1)\|w^\Delta\|_1\|f\|_1. \tag{4.11}$$

Finally we fix an integer $M \geq N$ with

$$\xi([w_{-M}, \ldots, w_M]) \subset \mathcal{U} + \xi(w) \tag{4.12}$$

for every $w \in W$, where

$$[w_{-M}, \ldots, w_M] = \{w' \in W : w'_i = w_i \text{ for } i = -M, \ldots, M\}.$$

The SFT Ξ. For $P = M, N$ we set

$$W^{(P)} = \pi_{\{-P,\ldots,P\}}(W) \subset \{-L, \ldots, L\}^{2P+1}$$

and call an element $v' = (v'_{-P}, \ldots, v'_P) \in W^{(P)}$ a *follower* of $v = (v_{-P}, \ldots, v_P) \in W^{(P)}$ if

$$v'_i = v_{i+1} \text{ for } i = -P, \ldots, P-1. \tag{4.13}$$

The set of followers of $v \in W^{(P)}$ will be written as $\mathsf{f}_v \subset W^{(P)}$. By $\bar{\pi} \colon W^{(M)} \longmapsto W^{(N)}$ we denote the projection

$$\bar{\pi}(v_{-M}, \ldots, v_M) = (v_{-N}, \ldots, v_N)$$

for every $v = (v_{-M}, \ldots, v_M) \in W^{(M)}$.

Next we set $H = \{-K, \ldots, K\}^{2M+m+1}$ (cf. (4.9) and (4.11)) and put, for every $h = (h_{-M-m}, \ldots, h_M) \in H$,

$$\begin{aligned} h_i^* &= \sum_{j=0}^m h_{i-j} f_{m-j}, \ i = -M, \ldots, M, \\ h^* &= (h^*_{-M}, \ldots, h^*_M) \in H^* = \{-K\|f\|_1, \ldots, K\|f\|_1\}^{2M+1}. \end{aligned} \tag{4.14}$$

Put

$$\begin{aligned} A = \{a = (p(a), q(a), s(a)) \in W^{(M)} \times W^{(N)} \times \{0,1\} : \text{for some } h \in H, \\ p(a) + h^* \in W^{(M)} \text{ and } q(a) = \bar{\pi}(p(a) + h^*)\}. \end{aligned} \tag{4.15}$$

Our choice of the integers $M \geq N$ implies the following:

$$\begin{aligned} &\text{if } a \in A \text{ and } p(a) = \mathbf{c}^{(M)} = (c, \dots, c) \in W^{(M)}, \\ &\text{then } q(a) = \bar{\pi}(\mathbf{c}^{(N)}) = c^{(N)} = (c, \dots, c) \in W^{(N)}. \end{aligned} \tag{4.16}$$

We call an element $a' \in A$ a *follower* of $a \in A$ if $p(a') \in \mathsf{f}_{p(a)}$, $q(a') \in \mathsf{f}_{q(a)}$, and if one of the following conditions is satisfied:

$$\begin{aligned} &(1)\ q(a) = \bar{\pi}(p(a)),\ q(a') = \bar{\pi}(p(a')),\ s(a) = s(a') = 0, \\ &(2)\ q(a) = \bar{\pi}(p(a)),\ q(a')_N > p(a')_N,\ s(a) = 0,\ s(a') = 1, \\ &(3)\ s(a) = s(a') = 1. \end{aligned} \tag{4.17}$$

For every $a \in A$ we write $\mathsf{f}_a \subset A$ for the set of followers of a and say that an element $a' \in A$ can be *reached* from a if there exists a sequence $a = a_0, \dots, a_l = a'$ in A with $a_{i+1} \in \mathsf{f}_{a_i}$ for every $i = 0, \dots, l-1$. Since W is transitive and

$$(p', \bar{\pi}(p'), 0) \in \mathsf{f}_{(p, \bar{\pi}(p), 0)}$$

for every $p' \in \mathsf{f}_p$, $(q, \bar{\pi}(q), 0)$ can be reached from $(p, \bar{\pi}(p), 0)$, for every $p, q \in P$. Let

$$\begin{aligned} A_0 = \{a \in A : a &\text{ can be reached from some} \\ &\text{(and hence any) } (p, \bar{\pi}(p), 0) \text{ with } p \in P\}, \\ \Xi = \{z = (z_n) &\in A_0^{\mathbb{Z}} : z_{n+1} \in \mathsf{f}_{z_n} \text{ for every } n \in \mathbb{Z}\}, \end{aligned} \tag{4.18}$$

and note that $\Xi \subset A_0^{\mathbb{Z}}$ is a shift of finite type.

For every $z = (z_n) \in \Xi$ and $n \in \mathbb{Z}$ we write $p(z_n) \in W^{(M)}, q(z_n) \in W^{(N)}$ as

$$p(z_n) = (p(z_n)_{-M}, \dots, p(z_n)_M), \qquad q(z_n) = (q(z_n)_{-N}, \dots, q(z_n)_N),$$

with $p(z_n)_i, q(z_n)_i \in \{-L, \dots, L\}$ for every i. Consider the maps

$$\theta_1, \theta_2 \colon \Xi \longmapsto W,$$

given by

$$(\theta_1(z))_n = p(z_n)_0, \qquad (\theta_2(z))_n = q(z_n)_0, \tag{4.19}$$

for every $z = (z_n) \in \Xi$ and $n \in \mathbb{Z}$. We claim that these maps have the following properties:

$$\begin{gathered} \theta_1(\Xi) = \theta_2(\Xi) = W, \\ \xi \cdot \theta_1 = \xi \cdot \theta_2. \end{gathered} \tag{4.20}$$

Indeed, let $w = (w_n) \in W$ and put

$$w_n^{(M)} = (w_{n-M}, \dots, w_{n+M}), \qquad w_n^{(N)} = (w_{n-N}, \dots, w_{n+N}) \tag{4.21}$$

for every $n \in \mathbb{Z}$. Then the sequence $z = (z_n)$ with $p(z_n) = w_n^{(M)}$, $q(z_n) = w_n^{(N)}$ and $s(z_n) = 0$ for every $n \in \mathbb{Z}$ lies in Ξ and $\theta_1(z) = \theta_2(z) = w$. This proves the first equation in (4.20).

For the second equation in (4.20) we fix $w \in W$ and $z \in \Xi$ and note that there exists, for every $m \in \mathbb{Z}$, an element $z(m) \in \Xi$ with $z_n = z(m)_n$ for $n \geq m$, and with

$$z(m)_n = (w_n^{(M)}, w_n^{(N)}, 0)$$

for all but finitely many $n < m$ (cf. (4.21)). From (4.17), our choice of $N \geq P(K)$ and Lemma 4.3 it is clear that $\theta_2(z(m)) - \theta_1(z(m)) \in f(\bar{\sigma})(\ell^\infty(\mathbb{Z},\mathbb{Z}))$. The second equation in (3.15) guarantees that

$$\xi \cdot \theta_1(z(m)) = \xi \cdot \theta_2(z(m))$$

for every $m \in \mathbb{Z}$, and by letting $m \to -\infty$ we obtain that $\xi \cdot \theta_1(z) = \xi \cdot \theta_2(z)$, as claimed in (4.20).

The SFT*'s* Ω *and* Ω^**.* For every $p \in W^{(M)}$ we set

$$S_p = \{(q,s) \in W^{(N)} \times \{0,1\} : (p,q,s) \in A_0\}. \tag{4.22}$$

Let

$$\mathbf{A} = \{\mathbf{a} = (p(\mathbf{a}), S(\mathbf{a})) : p(\mathbf{a}) \in W^{(M)}, (\bar{\pi}(p(\mathbf{a})), 0) \in S(\mathbf{a}) \subset S_{p(\mathbf{a})}\}.$$

We call $\mathbf{a}' \in \mathbf{A}$ a *follower* of $\mathbf{a} \in \mathbf{A}$ if

$$\begin{aligned} p(\mathbf{a}') &\in \mathsf{f}_{p(\mathbf{a})}, \\ S(\mathbf{a}') &= \bigcup_{(q,s)\in S(\mathbf{a})} \{(q',s') : (p(\mathbf{a}'), q', s') \in \mathsf{f}_{(p(\mathbf{a}),q,s)}\}. \end{aligned} \tag{4.23}$$

Again we denote by $\mathsf{f}_{\mathbf{a}}$ the set of followers of $\mathbf{a}$.

From (4.23) and (4.16) the following properties are clear for every $\mathbf{a} \in \mathbf{A}$:

$$\begin{aligned} &(1)\ \text{if } p(\mathbf{a}) = \mathbf{c}^{(M)} \text{ then } S(\mathbf{a}) \subset \{(\mathbf{c}^{(N)},0),(\mathbf{c}^{(N)},1)\}. \\ &(2)\ \text{for every } p' \in \mathsf{f}_{p(\mathbf{a})} \text{ there is a unique } \mathbf{a}' \in \mathsf{f}_{\mathbf{a}} \text{ with } p(\mathbf{a}') = p', \\ &(3)\ \text{if } \mathbf{b} \in \mathbf{A} \text{ with } p(\mathbf{a}) = p(\mathbf{b}) \text{ and } S(\mathbf{a}) \subset S(\mathbf{b}), \text{ and if} \\ &\qquad \mathbf{a}' \in \mathsf{f}_{\mathbf{a}}, \mathbf{b}' \in \mathsf{f}_{\mathbf{b}} \text{ and } p(\mathbf{a}') = p(\mathbf{b}'), \text{ then } S(\mathbf{a}') \subset S(\mathbf{b}'). \end{aligned} \tag{4.24}$$

Finally we set

$$\begin{aligned} \Omega &= \{\omega = (\omega_n) \in \mathbf{A}^{\mathbb{Z}} : \omega_{n+1} \in \mathsf{f}_{\omega_n} \text{ for every } n \in \mathbb{Z}\} \\ \mathbf{A}^* &= \{\mathbf{a} \in \mathbf{A} : (\bar{\pi}(p(\mathbf{a})), 1) \notin S(\mathbf{a})\}, \\ \Omega^* &= \Omega \cap (\mathbf{A}^*)^{\mathbb{Z}}. \end{aligned} \tag{4.25}$$

Again we note that $\Omega^* \subset \Omega \subset \mathbf{A}^{\mathbb{Z}}$ are *SFT*'s.

The map $\theta \colon \Omega \longmapsto W$*.* For every $\omega = (\omega_n) \in \Omega$ and $n \in \mathbb{Z}$ we set $\omega_n = (p(n), S(n))$ with $p(n) = p(\omega_n)$ and $S(n) = S(\omega_n)$ and write

$$p(n) = (p(n)_{-M}, \dots, p(n)_M)$$

with $p(n)_i \in \{-L, \dots, L\}$ for every $i = -M, \dots, M$. As in (4.19) we define $\theta \colon \Omega \longmapsto W$ by setting

$$(\theta(\omega))_n = p(n)_0$$

for every $\omega \in \Omega$ and $n \in \mathbb{Z}$ and claim that

$$\theta(\Omega) = W, \qquad \theta(\Omega^*) = W^*. \tag{4.26}$$

In order to prove (4.26) we fix $w \in W$ for the moment and define $w_n^{(M)}, w_n^{(N)}$ by (4.21) for every $n \in \mathbb{Z}$. Consider, for every $l \in \mathbb{Z}$, the sequence $\omega(w,l) = (\omega(w,l)_n) \in \mathbf{A}^{\mathbb{Z}}$ defined recursively by

$$\omega(w,l)_n = \begin{cases} (w_n^{(M)}, \{(w_n^{(N)}, 0)\}) & \text{if } n \leq l, \\ \mathsf{f}_{\omega(w,l)_{n-1}} & \text{if } n > l. \end{cases}$$

The inclusion (4.24) (3) shows that

$$S(\omega(w,l)_n) \subset S(\omega(w,l-1)_n)$$

for every $l,n \in \mathbb{Z}$, and we set

$$\omega(w)_n = \Bigl(w_n^{(M)}, \bigcup_{l\le 0} S(\omega(v,l)_n)\Bigr)$$

for every $n \in \mathbb{Z}$. The resulting point $\omega(w) \in \Omega$ obviously satisfies that $\theta(\omega(w)) = w$. Since $w \in W$ was arbitrary this shows that $\theta\colon \Omega \longmapsto W$ is surjective.

If $\omega(w) \notin \Omega^*$ there exist an integer $n \in \mathbb{Z}$ with $\omega(w)_n \notin \mathbf{A}^*$ and a largest integer $l < n$ with $(v^{(N)},1) \in S(\omega(w,l)_n)$. From the definition of $\mathsf{f}_{\mathbf{a}}$ in (4.23) it is clear that there exist finite sequences $(q(l),\dots,q(k)) \in (W^{(N)})^{k-l+1}$ and $(s(l),\dots,s(k)) \in \{0,1\}^{k-l+1}$ with the following properties:

$$\begin{aligned}
&(1)\ \ (q(i),s(i)) \in S(\omega(w,l)_i) \subset S_{w_i^{(M)}} \ \text{ for } i=l,\dots,n,\\
&(2)\ \ q(l)=w_l^{(N)},\ \ q(l+1)\neq w_{l+1}^{(N)},\ \ q(n)=w_n^{(N)},\\
&(3)\ \ s(l)=0,\ \ s(n)=1,\\
&(4)\ \ (w_{i+1}^{(M)},q(i+1),s(i+1)) \in \mathsf{f}_{(w_i^{(M)},q(i),s(i))} \ \text{ for } i=l+1,\dots,n.
\end{aligned} \tag{4.27}$$

From (2), (4) and (4.17) it follows in particular that $q(l+1)_N > w_{l+N+1}$.

Define $z \in \Xi$ by

$$z_k = \begin{cases} (w_k^{(M)}, w_k^{(N)}, 0) & \text{for } k \le l,\\ (w_k^{(M)}, q(k), s(k)) & \text{for } l < k < n,\\ (w_k^{(M)}, w_k^{(N)}, 1) & \text{for } k \ge n, \end{cases}$$

and set $w' = \theta_2(z)$ (cf. (4.19)). According to (4.20),

$$\xi \cdot \theta_1(z) = \xi(w) = \xi(w') = \xi \cdot \theta_2(z),$$

and the definition of z guarantees that

$$\begin{aligned} &w_k = w'_k \ \text{ for } k \le l+N \ \text{ and } k \ge n,\\ &w_k < w'_k \ \text{ for } k = l+N+1. \end{aligned}$$

Since $w \prec w'$ and $\xi(w) = \xi(w')$, $w \notin W^*$ by (4.3).

Conversely, if $w \notin W^*$, then (4.3) shows that there exists an element $w' \in W$ with $(w,w') \in \Delta'_W$ and $w \prec w'$. We set

$$z_n = (w_n^{(M)}, w'^{(N)}_n, s(n))$$

for every $n \in \mathbb{Z}$, where

$$s(n) = \begin{cases} 0 & \text{if } n < l+N, \ \text{ where } l = \min\{k \in \mathbb{Z} : w_n^{(N)} \neq w'^{(N)}_n\},\\ 1 & \text{otherwise.} \end{cases}$$

Then $z = (z_n) \in \Xi$ and $(w'^{(N)}_n, s(n)) \in S(\omega_n)$ for every $\omega \in \theta^{-1}(\{w\})$ and $n \in \mathbb{Z}$. It follows that $w \notin \theta(\Omega^*)$, which completes the proof of (4.26).

Completion of the proof of Proposition 4.2. According to (4.24) (2), the shift-covariant surjective map θ is right-resolving (cf. [12]), and hence $|\theta^{-1}(v)| < |\mathcal{P}(A \times H)|$ for every $v \in W$, where $|S|$ denotes the cardinality of a set S.

In particular, the restriction θ^* of θ to Ω^* is a continuous, bounded-to-one, shift-covariant map of the *SFT* Ω^* onto W^*, and W^* is sofic.

Proposition 4.1 shows that the restriction of ξ to W^* is bounded-to-one. Hence $\psi = \xi \cdot \theta^*\xi \colon \Omega^* \longmapsto X$ is bounded-to-one.

We set $\bar{x} = \psi(\mathbf{c}) \in X$ and claim that the pre-image $\psi^{-1}(\{\bar{x}\}) \in \Omega^*$ of $\bar{x}$ under ψ consists of a single point.

Indeed, (4.16) shows that every $a \in A$ with $p(a) = c^{(M)}$ is of the form $(c^{(M)}, c^{(N)}, s)$ with $s \in \{0,1\}$. In the notation (4.22) this implies that $S_{c^{(M)}} = \{(c^{(N)}, 0), (c^{(N)}, 1)\}$, and hence that $\bar{\mathbf{a}} = ((c^{(M)}, \{(c^{(N)}, 0)\})$ is the only element in $\mathbf{A}^*$ with $p(\mathbf{a}) = c^{(M)}$. From (4.10) it follows that $\psi^{-1}(\bar{x})$ consists of the single fixed point $\bar{\omega} = (\dots, \bar{\mathbf{a}}, \bar{\mathbf{a}}, \bar{\mathbf{a}}, \dots)$. If Ω^* is not transitive it must contain a transitive component Ω^{**} with $\psi(\Omega^{**}) = X$ and hence with $\bar{\omega} \in \Omega^{**}$. As Ω^{**} is transitive and contains a fixed point, it is mixing, hence the sofic shift $W^* = \theta(\Omega^{**})$ is mixing, and Lemma 4.4 implies that the maps $\psi \colon \Omega^{**} \longmapsto X$ and $\xi \colon \theta(\Omega^{**}) = W^* \longmapsto X$ are almost one-to-one. $\square$

5. Sofic partitions

Theorem 1.1 is a consequence of Proposition 3.2 and the following result.

Theorem 5.1. *Let $f \in R$ be a hyperbolic Laurent polynomial* (*Definition* 2.3)*, and let $\alpha = \alpha_f$ be the expansive automorphism of the compact connected abelian group $X = X_f$ described in Proposition* 2.2*. Then there exists a mixing sofic shift $V \subset \ell^\infty(\mathbb{Z}, \mathbb{Z})$ with the following properties.*

(1) $\xi(V) = X$*, where $\xi \colon \ell^\infty(\mathbb{Z}, \mathbb{Z}) \longmapsto X$ is the group homomorphism* (3.13)*;*
(2) *The restriction of ξ to V is injective on the set of doubly transitive points in V.*

The remainder of this section will be devoted to the proof of Theorem 5.1. Since α is expansive on X, its fixed point group

$$\mathrm{Fix}(\alpha) = \{x \in X : \alpha x = x\}$$

is finite. It follows that the set

$$\mathrm{Fix}(\bar{\sigma}) = \{v \in \bar{\xi}(\ell^\infty(\mathbb{Z}, \mathbb{Z})) : \bar{\sigma} v = v\} \subset \ell^\infty(\mathbb{Z}, \mathbb{R})$$

has the property that

$$\{t \in \mathbb{R} : (\dots, t, t, t, \dots) \in \mathrm{Fix}(\bar{\sigma})\} = \frac{1}{\kappa} \cdot \mathbb{Z} \tag{5.1}$$

for some integer $\kappa \geq 1$.

Proposition 5.2. *There exist an integer $L \geq 1$ and a transitive* SFT

$$W \subset Z = \{-L, \dots, L\}^{\mathbb{Z}} \subset \ell^\infty(\mathbb{Z}, \mathbb{Z})$$

satisfying the following conditions.

(1) $\bar{\xi}(W) \subset (-3/4\kappa, 1 - 1/4\kappa)^{\mathbb{Z}}$ (*cf.* (5.1))*;*
(2) $\xi(W) = X$.

Proof. Let $I = [-1/2\kappa, 1-1/2\kappa] \subset \mathbb{R}$, $J = [-1,2] \subset \mathbb{R}$, $V' = \eta^{-1}(X) \cap I^{\mathbb{Z}} \subset \ell^\infty(\mathbb{Z},\mathbb{R})$ and $W' = \eta^{-1}(X) \cap J^{\mathbb{Z}} \subset \ell^\infty(\mathbb{Z},\mathbb{R})$. Clearly, $\eta(V') = \eta(W') = X$ (cf. (3.13)), and

$$\bar{\xi}^{-1}(W') = f(\bar{\sigma})(W') \subset Z = \{-L, \dots, L\}^{\mathbb{Z}} \subset \ell^\infty(\mathbb{Z},\mathbb{Z}) \tag{5.2}$$

for some $L \geq 1$. Choose an integer $K \geq 0$ with

$$L \sum_{|k|>K} |w_k^\Delta| < 1/8\kappa, \tag{5.3}$$

put

$$w'_k = \begin{cases} w_k^\Delta & \text{if } |k| \leq K, \\ 0 & \text{otherwise,} \end{cases}$$

and define a map $\bar{\xi}' \colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longmapsto \ell^\infty(\mathbb{Z},\mathbb{R})$ by setting

$$\bar{\xi}'(w) = \sum_{n\in\mathbb{Z}} w_n \bar{\sigma}^n w'$$

for every $w = (w_n) \in \ell^\infty(\mathbb{Z},\mathbb{Z})$. From (5.3) it follows that $|\bar{\xi}(w)_n - \bar{\xi}'(w)_n| < 1/8\kappa$ for every $n \in \mathbb{Z}$ and $w = (w_n) \in W$, so that the set

$$W = \{w \in W : -5/8\kappa \leq \bar{\xi}'(w)_n < 1 - 3/8\kappa \text{ for every } n \in \mathbb{Z}\} \tag{5.4}$$

contains $V = f(\bar{\sigma})(V')$. Hence $\xi(W) = X$, and the definitions of w' and $\bar{\xi}'$ guarantee that W is a *SFT*. If W is not transitive, then the ergodicity of α guarantees that one of the finitely many transitive components of W will also cover X, and we replace W by such an transitive component. □

Proof of Theorem 5.1. Let $L \geq 1$ and $W \subset Z = \{-L, \dots, L\}^{\mathbb{Z}}$ be chosen as in Proposition 5.2. From the definition of W it is clear that $|\xi^{-1}(\{x\}) \cap W| = 1$ for every $x \in \mathrm{Fix}(\alpha)$: indeed, if $v \neq w \in W$ and $\xi(w) = x$, then there exists a $j \in \{0, \dots, \kappa - 1\}$ with $\bar{v}_n = \bar{w}_n = j/\kappa \pmod 1$ for every $n \in \mathbb{Z}$, and condition (1) in Proposition 5.2 implies that $\bar{v}_n = \bar{w}_n$ for every $n \in \mathbb{Z}$. A glance at Proposition 3.3 shows that $v = w$.

We define $W^* \subset W$ by (4.3) and obtain from Proposition 4.2 that W^* is a mixing sofic shift, and that the restriction of ξ to W^* is almost one-to-one. □

6. Beta-expansions

An algebraic integer $\beta > 1$ is a *Pisot number* if its conjugates $c_2, \dots, c_m$ satisfy that $|c_i| < 1$ for $i = 2, \dots, m$. We call an irreducible element $f = f_0 + \dots + f_{m-1}u^{m-1} + u^m \in R$ with $f_0 \neq 0$ a *Pisot polynomial* if one of its roots is a Pisot number.

For the remainder of this section we fix a Pisot polynomial $f \in R$ and write β for the unique root of f with $\beta > 1$. Following [15] we consider the map

$$T_\beta x = \beta x \pmod 1 \tag{6.1}$$

from the unit interval $I = [0,1]$ to itself and define, for every $x \in I$, the *beta-expansion* $\omega_\beta(x) = (\omega_\beta(x)_n)$ of x by setting

$$\omega_\beta(x)_n = \beta T_\beta^{n-1} x - T_\beta^n x \tag{6.2}$$

for every $n \geq 1$. Note that $\omega_\beta(x)_n \in \{0, \dots, \mathrm{Int}(\beta)\}$ for every $n \geq 1$, where $\mathrm{Int}(\beta)$ is the integral part of β, and that

$$x = \sum_{n \geq 1} \omega_\beta(x)_n \beta^{-n} \tag{6.3}$$

for every $x \in I$.

Since β is a Pisot number, the orbit $\{T_\beta^n 1 : n \geq 0\}$ is finite (cf. [4], [5], [17]), and the sequence $\omega_\beta(1)$ is pre-periodic (i.e. $(v_{n+k}, n \geq 1)$ is periodic for some $k \geq 0$). If $T_\beta^n 1 = 0$ for some (smallest) $n \geq 1$, then β is called *simple* (cf. [15]), $\omega_\beta(1)$ is of the form $(\omega_\beta(1)_1, \dots, \omega_\beta(1)_n, 0, \dots)$ with $\omega_\beta(1)_n > 0$, and we write

$$\omega_\beta^*(1) = (\omega_\beta(1)_1, \dots, \omega_\beta(1)_n - 1, \omega_\beta(1)_1, \dots, \omega_\beta(1)_n - 1, \dots) \tag{6.4}$$

for the *periodic* β-expansion of 1. If $T_\beta^n 1 \neq 0$ for every $n \geq 1$ we set $\omega_\beta^*(1) = \omega_\beta(1)$. In either case,

$$1 = \sum_{n \geq 1} \omega_\beta^*(1)_n \beta^{-n}. \tag{6.5}$$

We set $\mathbb{N} = \{1, 2, \dots\}$, denote by $\prec$ the lexicographic order on $\Sigma_\beta^+ = \{0, \dots, \mathrm{Int}(\beta)\}^\mathbb{N}$, write σ_+ for the one-sided shift (2.2) on Σ_β^+, and recall that

$$\sigma_+^k \omega_\beta^*(1) \preceq \omega_\beta^*(1) \tag{6.6}$$

for every $k \geq 1$ (cf. [15]). The restriction of σ_+ to the closed, shift-invariant set

$$V_\beta^+ = \{v \in \Sigma_\beta^+ : \sigma_+^n v \preceq \omega_\beta^*(1) \text{ for every } n \geq 0\} \tag{6.7}$$

is called the *β-shift*.

Define a map $\rho_\beta \colon V_\beta^+ \longmapsto I$ by

$$\rho_\beta(v) = \sum_{n \geq 1} v_n \beta^{-n} \tag{6.8}$$

for every $v = (v_n) \in V_\beta^+$. Then ρ_β is continuous, surjective, bounded-to-one, and

$$\omega_\beta(\rho_\beta(v)) = v \tag{6.9}$$

for all v in the complement of a countable subset of V_β^+ (cf. [15] and (6.3)).

Here we are interested in two-sided versions of the beta-expansion and the beta-shift. Denote by σ the shift (2.2) on $\Sigma_\beta = \{0, \dots, \mathrm{Int}(\beta)\}^\mathbb{Z}$, write $v^+ = (v_1, v_2, \dots) \in \Sigma_\beta^+$ for every $v = (v_n) \in \Sigma_\beta$, and put

$$V_\beta = \{v \in \Sigma_\beta : (\sigma^n v)^+ \in V_\beta^+ \text{ for every } n \in \mathbb{Z}\}. \tag{6.10}$$

From (6.10) and the eventual periodicity of $\omega_\beta^*(1)$ it is not difficult to see that $V_\beta \subset \Sigma_\beta \subset \ell^\infty(\mathbb{Z}, \mathbb{Z})$ is sofic. If β is simple then V_β is, in fact, a *SFT*.

Proposition 6.1. *Let $\beta > 1$ be a Pisot number, $f \in R$ an irreducible polynomial with $f(\beta) = 0$, and let $\alpha = \alpha_f$ be the expansive automorphism of the compact abelian group $X = X_f$ described in Proposition 2.2. We write $\xi \colon \ell^\infty(\mathbb{Z}, \mathbb{Z}) \longmapsto X$ for the group homomorphism (3.13) and define $V_\beta \subset \ell^\infty(\mathbb{Z}, \mathbb{Z})$ by (6.10). Then $\xi(V_\beta) = X$, and the restriction of ξ to V_β is bounded-to-one.*

Proof. The main argument in following proof is due to B. Solomyak. We denote by $c_1 = \beta, c_2, \dots, c_m$ the roots of f, set

$$A = \begin{pmatrix} c_1^{-1} & \dots & c_m^{-1} \\ \vdots & \ddots & \vdots \\ c_1^{-m} & \dots & c_m^{-m} \end{pmatrix},$$

and write

$$\|A^{-1}\| = \max_{\mathbf{0} \neq \mathbf{z} \in \mathbb{C}^m} \|A^{-1}\mathbf{z}\| / \|\mathbf{z}\|$$

for the norm of the inverse matrix A^{-1} with respect to the maximum norm $\|\cdot\|$ on $\mathbb{C}^m$. Let $\|f\|_1^*$ and $V \subset \ell^\infty(\mathbb{Z}, \mathbb{Z})$ be defined as in (3.17), choose $K \geq 1$ with

$$\|f\|_1^* \cdot \sum_{k \geq K} \beta^{-k} < 1$$

and put

$$L' = \|A^{-1}\| \cdot \biggl(1 + (\beta + \|f\|_1^*) \cdot \sum_{j=2}^{m} \sum_{k \geq -K} |c_j|^k\biggr),$$
$$L = L' + \|f\|_1^*,$$
$$W = \{-L, \dots L\}^{\mathbb{Z}}.$$

For every $N \geq 1$ we denote by $\pi_N = \pi_{\{1,\dots,N\}}$ the projection onto the coordinates $1, \dots, N$ and consider the closed set

$$W^{(N)} = \{w \in W : \pi_N(w) \in \pi_N(V_\beta)\}.$$

Suppose that we can prove the following:

$$\text{For every } N \geq 1, \ \xi(W^{(N)}) = X. \tag{6.11}$$

Then $\xi^{-1}(x) \cap W^{(N)} \neq \varnothing$ for every $x \in X$ and $N \geq 1$. As the sequence $(\xi^{-1}(x) \cap W^{(N)}, N \geq 1)$ is nonincreasing in W and W is compact, there exists, for every $x \in X$, a point $w \in \bigcap_{N \geq 1} W^{(N)} = W^+$ with $\xi(w) = x$. By shift-invariance, $\xi(\sigma^n(W^+)) = X$ for every $n \geq 0$, and by repeating the above argument for the nonincreasing sequence $(\sigma^n(W^+), n \geq 0)$ we obtain that

$$\xi\biggl(\bigcap_{n \geq 0} \sigma^n(W^+)\biggr) = \xi(V_\beta) = X.$$

In order to verify (6.11) we fix $v \in \Sigma_\beta$ and $N \geq 1$. Choose $K \geq 1$ with

$$y = \beta^{-K} \sum_{n=1}^{N} v_n \beta^{-n} < 1,$$

and let $\omega_\beta(y) = (\omega_1, \omega_2, \dots) \in V_\beta^+$ be the beta-expansion (6.2) of y. An elementary induction argument shows that there exist, for every $s \geq 0$, integers $\gamma_s^{(1)}, \dots, \gamma_s^{(m)}$ with

$$T_\beta^s y = \beta^s y - \sum_{j=1}^{s} \omega_j \beta^{s-j} = \gamma_s^{(1)} \beta^{-1} + \dots + \gamma_s^{(m)} \beta^{-m}. \tag{6.12}$$

Since all terms of this equation lie in the number field $\mathbb{Q}(\beta)$ we obtain that, for every $s \geq 0$,

$$\Gamma_i(s) = c_i^{s-K} \cdot \sum_{n=1}^{N} v_n c_i^{-n} - \sum_{j=1}^{s} \omega_j c_i^{s-j} = \gamma_s^{(1)} c_i^{-1} + \cdots + \gamma_s^{(m)} c_i^{-m} \tag{6.13}$$

for every root $c_1 = \beta, c_2, \ldots, c_m$ of f.

For $s = K + N$, (6.12)–(6.13) imply that $0 \leq \Gamma_1(K+N) \leq 1$ and

$$|\Gamma_i(K+N)| < (\beta + \|f\|_1^*) \cdot \sum_{k \geq -K} |c_i|^k$$

for $i = 2, \ldots, m$. A glance at the definition of L' yields that

$$\|(\gamma_1^{(K+N)}, \ldots, \gamma_m^{(K+N)})\| = \|(\Gamma_1^{(K+N)}, \ldots, \Gamma_m^{(K+N)}) A^{-1}\| < L'.$$

Set

$$w_n = \begin{cases} v_n & \text{if } n \leq -K \text{ or } n > N+m, \\ v_n + \omega_\beta(y)_{K+n} & \text{if } -K+1 \leq n \leq 0, \\ \omega_\beta(y)_{K+n} & \text{if } 1 \leq n \leq N, \\ v_n + \gamma_{K+N}^{(n-N)} & \text{if } N < n \leq N+m. \end{cases}$$

Then $w \in W^{(N)}$, and $h = w - v \in R$ is of the form $h = \sum_{n \in \mathbb{Z}} h_n u^n$ with

$$h_n = \begin{cases} \omega_\beta(y)_{K+n} & \text{if } -K+1 \leq n \leq 0, \\ \omega_\beta(y)_{K+n} - v_n & \text{if } 1 \leq n \leq N, \\ \gamma_{K+N}^{(n-N)} & \text{if } N < n \leq N+m, \\ 0 & \text{otherwise.} \end{cases}$$

By (6.12)–(6.13),

$$\begin{aligned} h(\beta^{-1}) = \sum_{n=-K+1}^{N} & \omega_{K+n} \beta^{-n} - \gamma_{K+N}^{(1)} \beta^{-N-1} - \cdots - \gamma_{K+N}^{(m)} \beta^{-N-m} \\ & - \sum_{n=1}^{N} v_n \beta^{-n} = 0, \end{aligned}$$

so that $h \in \tilde{f}R$ (cf. (3.4)). Proposition 3.3 shows that $\xi(v) = \xi(w)$ and completes the proof of (6.11).

Next we assert that $V_\beta = V_\beta^*$ (cf. (4.3)).

Indeed, if two elements $v, v' \in V_\beta$ with $\xi(v) = \xi(v')$ differ in only finitely many coordinates then $v - v' = h\tilde{f}$ for some $h = (h_n) \in R$. Choose an integer K with $h_n = 0$ and $v_n = v_n'$ for all $n \leq K$, and put

$$w_n = v_{n-K}, \qquad w_n' = v_{n-K}'$$

for every $n \geq 1$. Then $w, w' \in V_\beta^+$, w and w' differ in only finitely many coordinates, and

$$\rho_\beta(w) - \rho_\beta(w') = \sum_{n \geq 1} (w_n - w_n') \beta^{-n} = \sum_{n \in \mathbb{Z}} (v_n - v_n') \beta^{-n} = 0.$$

Since this is impossible according to the definition of beta-expansion we conclude that $V_\beta = V_\beta^*$, as claimed. From Proposition 4.1 we see that the restriction of ξ to V_β is bounded-to-one. $\square$

Let

$$\begin{aligned} Z_\beta(x) &= \{v \in V_\beta : \xi(v) = x\},\ x \in X_f = \mathbb{T}^m, \\ Z_\beta = Z_\beta(0) &= V_\beta \cap \tilde{f}\ell^\infty(\mathbb{Z},\mathbb{Z}) = \{v \in V_\beta : \xi(v) = 0\}. \end{aligned} \tag{6.14}$$

Lemma 6.2. *$Z_\beta \neq \{0\}$.*

Proof. Let $\omega^* = (\omega_n^*) \in V_\beta$ be the unique periodic point with

$$\omega_n^* = \omega_\beta^*(1)_n \tag{6.15}$$

for all sufficiently large $n \geq 1$ (cf. (6.4)–(6.5)). If $w \in \ell^\infty(\mathbb{Z},\mathbb{Z})$ is the point given by

$$w_n = \begin{cases} \omega_\beta^*(1)_n & \text{if } n > 0, \\ -1 & \text{if } n = 0, \\ 0 & \text{otherwise,} \end{cases}$$

then $\xi(w) = 0$. As $\xi(\sigma^n w) = 0$ for every $n \geq 0$ and $\sigma^{n_k} w \to \omega^*$ for an appropriately chosen sequence $n_k \to \infty$, $\xi(\omega^*) = 0$. $\square$

In [21] the authors prove that Question 1.2 has a positive answer if β is a quadratic Pisot number. Here we provide further support for the conjecture that the restriction of ξ to V_β is always almost one-to-one.

Theorem 6.3. *Let β be a Pisot number of degree $d \geq 2$, $f \in R$ an irreducible polynomial with $f(\beta) = 0$, and let $\alpha = \alpha_f$ be the expansive automorphism of $X_f = \mathbb{T}^m$ described in Proposition* 2.2. *We write $\xi\colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longmapsto \mathbb{T}^m$ for the group homomorphism* (1.1) *and define the two-sided beta-shift $V_\beta \subset \ell^\infty(\mathbb{Z},\mathbb{Z})$ by* (6.10).

Suppose that β is simple (i.e. $T_\beta^n 1 = 0$ for some $n \geq 1$), and that

$$Z_\beta = \{0\} \cup \{\sigma^n \omega^* : n \in \mathbb{Z}\}, \tag{6.16}$$

where $\omega^* \in V_\beta$ is the periodic point (6.15)*. Then the map $\xi\colon V_\beta \longmapsto \mathbb{T}^m$ is injective on the set of doubly transitive points.*

Remark 6.4. I am grateful to B. Solomyak for pointing out to me that (6.16) is equivalent to the condition — investigated in [8] — that every $x \in \mathbb{Z}[\beta^{-1}] \cap [0,1]$ has a finite beta-expansion: indeed, x must have an eventually periodic beta-expansion by [17], and the same argument as in Lemma 6.2 shows that the periodic extension of the tail of $\omega_\beta(x)$ lies in Z_β. Conversely, every element $z \in Z_\beta$ is the periodic extension of the tail of $\omega_\beta(x)$ for some $x \in \mathbb{Z}[\beta^{-1}] \cap [0,1]$. Sufficient conditions for every element in $\mathbb{Z}[\beta^{-1}] \cap [0,1]$ to have a finite beta-expansion were studied in [8] (cf. Example 6.5 (4) below).

The following examples illustrate the hypotheses of Theorem 6.3 and the results in [8].

Examples 6.5. (1) Let $f(u) = u^4 - 4u^3 + 3u^2 - 2u + 1$. Then f has roots

$$\beta = 3.23402\ldots, \qquad c_2 = 0.672378\ldots,$$
$$c_3, c_4 = 0.0467994\ldots \pm 0.676527\ldots\, i,$$

and β is Pisot. Furthermore,

$$\omega_\beta(1) = \omega^*_\beta(1) = (3, 0, 2, 1, 1, 1, \ldots),$$

so that β is not simple, and

$$\omega^* = (\ldots, 1, 1, 1, \ldots),$$
$$Z_\beta \supseteq \{j\omega^* : 0 \leq j \leq 2\}.$$

(2) Let f be a quadratic Pisot polynomial, i.e.

(a) $f(u) = u^2 - nu + 1$ with $n \geq 3$,
(b) $f(u) = u^2 - nu - 1$ with $n \geq 1$.

In case (a),

$$\omega_\beta(1) = \omega^*_\beta(1) = (n-1, n-2, n-2, \ldots),$$

β is not simple, and $Z_\beta = \{0, \omega^*\}$ with $\omega^* = (\ldots, n-2, \underline{n-2}, n-2, \ldots)$.

In case (b),

$$\omega^*_\beta(1) = (n-1, 0, n-1, 0, \ldots),$$

β is simple, and $Z_\beta = \{0, \omega^*, \sigma\omega^*\}$ with $\omega^* = (\ldots, 0, n-1, \underline{0}, n-1, 0, \ldots)$.

We can thus apply Theorem 6.3 in case (b), but not in case (a).

(3) Let $f(u) = u^4 - 2u^3 - u - 1$. Then f has roots

$$\beta = 2.277452390\ldots, \qquad c_2 = -0.5573174032\ldots,$$
$$c_3, c_4 = 0.1399325064\ldots \pm 0.8765142016\ldots\, i,$$

β is Pisot and

$$\omega^*_\beta(1) = (2, 0, 1, 0, 2, 0, 1, 0, \ldots).$$

However, as was shown in [8], $3\beta^{-2}$ has an infinite beta-expansion with eventual period (1200), and the periodic point

$$(\ldots, 1, 2, 0, 0, 1, 2, 0, 0, \ldots)$$

lies in Z_β. Hence the hypotheses of Theorem 6.3 are not satisfied.

(4) Let $f(u) = u^3 - 3u^2 - 2u - 1$ with roots

$$\beta = 3.62737\ldots$$
$$c_2, c_3 = -0.313683\ldots \pm 0.421053\ldots\, i.$$

Then β is Pisot,

$$\omega^*_\beta(1) = (3, 2, 0, 3, 2, 0, \ldots).$$

In this case

$$\omega^* = (\ldots, 0, 3, 2, \underline{0}, 3, 2, 0, \ldots),$$

where the zero coordinate is underlined, and

$$Z_\beta = \{0\} \cup \{\omega^*, \sigma\omega^*, \sigma^2\omega^*\}.$$

More generally, if a Pisot number β is a root of an irreducible polynomial $f(u) = u^m + f_{m-1}u^{m-1} + \cdots + f_1 u + f_0$ with $0 > f_0 \geq f_1 \geq \cdots \geq f_{m-1}$, then every element of $\mathbb{Z}[\beta^{-1}] \cap [0, 1]$ has a finite beta-expansion by [8], and Theorem 6.3 can be applied.

Proof of Theorem 6.3. Suppose that $x \in X_f = \mathbb{T}^m$ is homoclinic. Then every $v \in Z_\beta(x)$ must be of the following form: there exist an integer $N = N(v) \geq 0$ and points $\omega^{v,x,+}, \omega^{v,x,-} \in Z_\beta$ with

$$v_n = \omega_n^{v,x,+}, \qquad v_{-n} = \omega_{-n}^{v,x,-},$$

for all $n \geq N$. Furthermore, since $Z_\beta(x)$ is finite, the set of integers $\{N(v) : v \in Z_\beta(x)\}$ is bounded.

For $x = x^\Delta$, the fundamental homoclinic point of $\alpha = \alpha_f$, we have at least two pre-images in $Z_\beta(x^\Delta)$:

$$(\dots, 0, 0, \underline{1}, 0, 0, \dots) \text{ and } (\dots, 0, 0, \underline{0}, \omega^*_\beta(1)_1, \omega^*_\beta(1)_2, \dots).$$

In other words, there exist points $v, v' \in Z_\beta(x^\Delta)$ with $\omega^{v,x^\Delta,-} = \omega^{v,x^\Delta,+} = 0$ and $\omega^{v',x^\Delta,-} = 0$, $\omega^{v',x^\Delta,+} = \omega^*$.

Suppose that there exists a $v' \in Z_\beta(x^\Delta)$ with $\omega^{v',x^\Delta,-} = \sigma^k \omega^*$ for some $k \in \mathbb{Z}$. Since ω^* is periodic we write it as

$$\omega^* = (\dots, \omega_1^*, \dots, \underline{\omega_L}^*, \omega_1^*, \dots, \omega_L^*, \dots)$$

for some $L \geq 1$, where the zero coordinate is underlined. The point

$$w = (\dots, \omega_1^*, \dots, \omega_L^*, \omega_1^*, \dots, \underline{\omega_L + 1}^*, 0, 0, \dots)$$

in $\ell^\infty(\mathbb{Z}, \mathbb{Z})$ is the difference of two elements in Z_β and thus satisfies that $\xi(z) = 0$. Hence $\xi(v^\Delta - w) = x^\Delta$, where $v^\Delta \in V_\beta$ is given by (3.9).

We choose an $l < 0$ with $v'_{-n+l} = \omega_{-n}$ for every $n \leq 0$, set $v'' = \sigma^l v'$, and observe that

$$v = v'' + v^\Delta - w = (\dots, 0, 0, \underline{0}, v'_{l+1}, v'_{l+2}, \dots) \in V_\beta$$

and

$$\xi(v) = x^\Delta + \alpha^l x^\Delta.$$

The point $w = v^\Delta + \sigma^l v^\Delta \in V_\beta$ also has the property that $\xi(w) = x^\Delta + \alpha^l x^\Delta$. As $\sigma^{-1}v, \sigma^{-1}w \in Z_\beta(\alpha^{-1}x^\Delta + \alpha^{l-1}x^\Delta)$ satisfy that $(\sigma^{-1}v)_n = (\sigma^{-1}w)_n = 0$ for $n \leq 0$, they may be viewed as elements of V_β^+ which differ by an element of $\tilde{f}R$, and which therefore satisfy that $\rho_\beta(\sigma^{-1}v) = \rho_\beta(\sigma^{-1}w)$. However, the only pairs of distinct elements $y, y' \in V_\beta$ with $y \prec y'$ and $\rho_\beta(y) = \rho_\beta(y')$ are those for which there exists a $k \geq 1$ with

$$\begin{aligned} y_n &= y'_n \text{ for } n < k, \\ y'_k &= y_k + 1, \\ y'_{k+l} &= 0 \text{ for every } l \geq 1, \\ y_{k+l} &= \omega^*_\beta(1)_l \text{ for every } l \geq 1. \end{aligned}$$

Since $\sigma^{-1}v \prec \sigma^{-1}w$, $(\sigma^{-1}w)_1 = 0$, $(\sigma^{-1}v)_1 = 1$, and $(\sigma^{-1}v)_{1-l} = 1$ we obtain a contradiction which shows that $\omega^{v',x^\Delta,-} = 0$ for every $v' \in Z_\beta(x^\Delta)$. If $S \subset \mathbb{Z}$ is a sufficiently sparse bi-infinite set then the point $x = \sum_{k \in S} \alpha^k x^\Delta$ satisfies that $|\xi^{-1}(\{x\}) \cap V_\beta| = 1$, and an application of Lemma 4.4 completes the proof. $\square$

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