INTRODUCTION TO THE HABILITATIONSSCHRIFT: ALGEBRAIC METHODS IN HIGHER-DIMENSIONAL DYNAMICS

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1. Motivation

The theory of dynamical systems has developed in the last century from the beginning to a rich and broad field of mathematics.

The main idea of dynamical systems is the idea of a time-evolving system. Although time-evolving systems have been studied from the very beginning of mathematical physics, the mathematical idea is to study them not in particular examples, but in a general theory.

This has been motivated, for instance, by the non-predictability of examples like the 3-body problem: Assume we have three bodies in space, say a solar system with just the sun, the earth and the moon, and we want to find a formula telling us what will happen in time under the force of gravity. Even though this question leads to a well known differential equation, a general solution is not known. However, this is not the end of the problems in that example. Even if we had the possibility to simulate the behaviour of the system with the mass, position and velocity given (say with a computer of infinite accuracy), small errors in the initial data would make a huge difference after a surprisingly small time. This effect is known as chaos and has lead to many different mathematical notions of this behaviour (hyperbolicity, expansiveness).

Since this concrete example cannot be simulated in its long term behaviour to answer questions like 'Will the moon fall onto the earth?', the question 'What can be said about a system in general?' gains interest.

To start the mathematics we need to specify the meaning of the term *dynamical system*. We fix a space X, together with some mathematical structure (like a topology, group-structure or a probability measure). Since we are mainly interested in the long term behaviour of the system, we work with discrete time (instead of continuous time as in dynamical systems arising from differential equations). This means that we work with a map $T : X \to X$ which describes the position Tx of the point $x \in X$ after one unit of time. Depending on the structure of the set X, we will have to require some properties of T. Furthermore it is often convenient and not much of a restriction to assume that T is invertible.

Since a theory lives with (and dies with a lack of) examples, we now present some simple but illustrative examples of dynamical systems (X, T). We start with topological dynamical systems where X will always be a compact metric space and T will be a continuous map. Then we will consider measurable dynamical systems, where X is a probability space and T is measure preserving. In the last section we give examples of $\mathbb{Z}^d$-actions. During those three sections we will use these examples to illustrate various definitions, and questions which are important for the papers contained in my Habilitationsschrift.

In case this manuscript sparks your interest in dynamics, I will be delighted to hear that. If you want to see any of the mentioned papers ([ELMW], [ER], [ES], [EW], [EW00], [ME], [ME99]), I can send you a copy by email or mail.

2. Topological Dynamical Systems

In this section X is a compact metric space and $T : X \to X$ a continuous map. In the first examples, we will use the space $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ whose elements are of the form $x = r + \mathbb{Z}$ for some $r \in \mathbb{R}$.

Example 1. Let $X = \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and choose some $\alpha \in \mathbb{R}$, we define $T_\alpha(x) = x + \alpha$. What can be said about the *orbit*

$$\mathcal{O}(x) = \{x, T_\alpha(x), T_\alpha^2(x), \dots\}$$

of a point $x \in \mathbb{T}$? There are two cases: If $\alpha = \frac{p}{q} \in \mathbb{Q}$ is rational, then the orbit

$$\mathcal{O}(x) = \left\{x, x + \frac{p}{q}, x + 2\frac{p}{q}, x + 3\frac{p}{q}, \dots, x + (q-1)\frac{p}{q}\right\}$$

of any point $x \in \mathbb{T}$ is finite.

If α is irrational, then the orbit is infinite. Otherwise there would exist $m > n \geq 0$ such that

$$\begin{aligned} T_\alpha^m x &= T_\alpha^n x, \\ x + m\alpha &= x + n\alpha \pmod 1, \\ (m-n)\alpha &= k \text{ for some } k \in \mathbb{Z} \text{ and} \\ \alpha &= \frac{k}{m-n} \in \mathbb{Q}. \end{aligned}$$

If we now consider T as a homeomorphism of the compact metric space $\mathbb{T}$, we can ask if the orbit of a point is dense in $\mathbb{T}$. In the rational case $\alpha \in \mathbb{Q}$ this is of course not possible. Assume therefore that $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is irrational. It is easy to see that $\mathcal{O}(x) = \mathcal{O}(0) + x$, so it is sufficient to consider the orbit of the point $x = 0$. Let $\epsilon > 0$. Since $\mathbb{T}$ is compact, there exists $m > n \geq 1$ such that

$$|T_\alpha^m 0 - T_\alpha^n 0| < \epsilon,$$

here

$$|x| = \min\{|u| : u + \mathbb{Z} = x\} \text{ for some } x \in \mathbb{T}. \tag{1}$$

Then

$$\delta = |T_\alpha^{m-n} 0| = |(m-n)\alpha + \mathbb{Z}| < \epsilon$$

and the set of points

$$\{0, T_\alpha^{m-n} 0, T_\alpha^{2(m-n)} 0, \dots, T_\alpha^{\lfloor \frac{1}{\delta} \rfloor (m-n)} 0\} \subseteq \mathcal{O}(x)$$

is ϵ-dense, therefore the orbit $\mathcal{O}(x)$ is dense in $\mathbb{T}$.

Even though this systems has quite complicated orbits, it lacks the chaos as it appeared in the 3-body system. To see this, let $x, y \in \mathbb{T}$. Then

$$|T_\alpha x - T_\alpha y| = |(x+\alpha) - (y+\alpha)| = |x-y|$$

which shows that T is an isometry. Therefore

$$|T_\alpha^n x - T_\alpha^n y| = |x-y|$$

and small errors in the starting point remain small forever.

Example 2. Let again $X = \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and let $p > 1$ be a natural number. We define the map

$$T_p(x) = px \pmod 1.$$

There are many points $x \in \mathbb{T}$ with a finite orbit, in fact $|\mathcal{O}(x)| < \infty$ if and only if $x \in \mathbb{Q}$. To see this, assume $x = \frac{k}{l} \in \mathbb{Q}$, then every image point $T_p^n x = \frac{p^n k}{l}$ is also rational. However, since modulo 1 only finitely many points have denominator l, the orbit of x must be finite. Assume now the orbit of x is finite. Then there exists $m > n \geq 1$ such that

$$\begin{aligned} T_p^m x &= T_p^n x, \\ p^m x &= p^n x \pmod 1, \\ (p^m - p^n)x &= k \in \mathbb{Z} \text{ for some } k \text{ and} \\ x &= \frac{k}{p^m - p^n}. \end{aligned}$$

This shows that the orbit of irrational points $x \in \mathbb{R} \setminus \mathbb{Q}$ must be infinite. However, infinite orbits do not behave uniformly. There are dense orbits, but also infinite non-dense orbits. To construct such points one can make use of the p-adic expansion of real numbers (digit expansion to base p). For simplicity let $p = 2$. List all possible finite sequences (call them *words*) of 0 and 1, say

$$(w_1, w_2, \ldots) = (0, 1, 00, 01, 10, 11, 000, \ldots).$$

We concatenate all the words into one sequence and let a_i be the i-th digit in the resulting sequence. Define

$$x = \sum_{i=1}^{\infty} a_i 2^{-i}$$

then we claim that the orbit $\mathcal{O}(x)$ is dense in $\mathbb{T}$. It is enough to show that every interval of the form $[\frac{i}{2^l}, \frac{i+1}{2^l}]$ with $0 \leq i < 2^l$ contains an element of the orbit. However, since the 2-adic expansion of $\frac{i}{2^l}$ is just a finite word w_j there exists an n such that the 2-adic expansion of $T_2^n x$ begins with w_j. This shows that

$$T_2^n x \in [\frac{i}{2^l}, \frac{i+1}{2^l}]$$

and that the orbit of x is dense in $\mathbb{T}$. To construct a point with infinite but not dense orbit, one could list all words where no two 1's are next to each other and define by this a point y. The orbit of y is infinite but not dense, since $T_2^n y \notin [\frac{3}{4}, 1]$ for all $n \geq 1$.

We now come back to the question whether this dynamical system shows the kind of chaos we encountered in the 3-body problem. The system $(\mathbb{T}, T_p)$ is *expansive*. This means that there exists an *expansive constant* $\delta > 0$ such that for any two points $x, y \in \mathbb{T}$ with $x \neq y$ there exists $n \geq 0$ such that $|T_p^n x - T_p^n y| > \delta$. This is a strong version of the previously discussed form of chaos: no matter how close x and y are, at some time in their future they will be far apart.

Since $\mathbb{T}$ is a group and T_p is a group homomorphism, we can assume without loss of generality that $y = 0$. Let $\delta = \frac{1}{2p}$ and $x \in \mathbb{T} \setminus \{0\}$, we have to show that there exists $n \geq 0$ with $|T^n x| > \delta$. If $|x| > \delta$, set $n = 0$ and we are done. Assume now $0 < |x| \leq \delta$, there exists an $n \geq 1$ with

$$\frac{1}{2p^{n+1}} < |x| \leq \frac{1}{2p^n}.$$

Using the Definition (1) of the norm $|\cdot|$ on $\mathbb{T}$ one can show that

$$\delta < |T_p^n x| \leq \frac{1}{2},$$

which concludes the proof of expansiveness of T_p.

The above example is a non-invertible dynamical system and our notion of expansiveness is a one-sided notion since our 'time of different behaviour' n must satisfy $n \geq 0$. The next example is an invertible group automorphism and there we will use a two-sided version of expansiveness, in other words n will only satisfy $n \in \mathbb{Z}$. In [ELMW] the notion of expansiveness for certain algebraic dynamical system is studied.

Example 3. Let $X = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and put

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 + x_2 \end{pmatrix}.$$

One can also describe T as the map on $\mathbb{T}^2$ defined by the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

The map $T : \mathbb{T}^2 \to \mathbb{T}^2$ is invertible, its inverse is described by the matrix

$$B = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

(it is important that $B = A^{-1}$ has integer entries).

As we will see later, it is important to allow integer values $n \in \mathbb{Z}$ in the definition of expansiveness. For this reason one speaks in this context about the $\mathbb{Z}$-*action* defined by the map T. A $\mathbb{Z}$-action is a homomorphism

$$\phi : \mathbb{Z} \to \mathrm{Aut}(\mathbb{T}^2),$$

where $\mathrm{Aut}(\mathbb{T}^2)$ is the group of automorphism of $\mathbb{T}^2$, the homomorphism is given by

$$\phi(n) = T^n.$$

The $\mathbb{Z}$-action ϕ is *expansive* if there exists an expansive constant $\delta > 0$ such that for any two $x, y \in \mathbb{T}^2$ with $x \neq y$ there exists $n \in \mathbb{Z}$ (not necessarily in $\mathbb{N}$) with

$$|T^n x - T^n y| > \delta$$

(here $|x| = \min\{\|r\| : r \in \mathbb{R}^2, r + \mathbb{Z}^2 = x\}$). Again we can assume that $y = 0$ and $x \neq 0$. Choose a small open neighbourhood V of $0 \in \mathbb{R}^2$ such that

$\pi : \mathbb{R}^2 \to \mathbb{T}^2$ restricted to V is an isometry. If $x \notin \pi(V)$, we choose $n = 0$. Otherwise take $u \in V$ with $\pi(u) = x$. The matrix A is diagonalizable and has eigenvalues $\lambda = \frac{1+\sqrt{5}}{2}$ (the golden mean) and $\mu = \frac{1-\sqrt{5}}{2}$. The corresponding eigenvectors v_λ and v_μ form a basis of $\mathbb{R}^2$. There are real numbers a_1, a_2 with $u = a_1 v_\lambda + a_2 v_\mu$, we get

$$A^n u = \lambda^n a_1 v_\lambda + \mu^n a_2 v_\mu.$$

If now $a_1 \neq 0$, we can find $n \geq 0$ such that $\lambda^n a_1$ is relatively big, but $A^n u$ is still an element of V. This means we can choose a real number $\delta > 0$ (only depending on the geometry of V and the eigenvalues λ, μ) such that for this n

$$|T^n x| > \delta.$$

However, if $a_1 = 0$ then $A^n u \to 0$ for $n \to +\infty$ and $T^n x$ will stay close to 0 for positive n. In this case we have to consider negative values of n. Since $x \neq 0$, we have $a_2 \neq 0$ and we can again find an integer $n \leq 0$ such that $A^n u = \mu^n a_2 v_\mu$ is big, but still an element of V.

Together we see that for every $x \in \mathbb{T}^2 \setminus \{0\}$ there exists an $n \in \mathbb{Z}$ with $|T^n x| > \delta$.

The line spanned by the vector v_μ is called the *stable manifold* since a point on it will tend to 0 in its future. On the other hand the line through v_λ is the *unstable manifold* since a point on the line will have been arbitrarily close to 0 its past, in fact there $T^n x \to 0$ for $n \to -\infty$.

Both lines have irrational slope and one can show that they therefore are dense in $\mathbb{T}^2$. Those lines intersect in $\mathbb{T}^2$ in many points. Let w be one of the intersections (see for instance Figure 3). Since w is an element of the stable manifold, we have $T^n w \to 0$ for $n \to +\infty$. However, w is also an element of the unstable manifold and together we see that

$$T^n w \to 0 \text{ for } |n| \to \infty. \tag{2}$$

A point w satisfying Property (2) is called *homoclinic*. Two points $x, y \in \mathbb{T}^2$ are *homoclinic to each other* if

$$|T^n x - T^n y| \to 0 \text{ for } |n| \to \infty.$$

It is easy to see that x and y are homoclinic to each other if and only if $y = x + w$ for a homoclinic point w.

As in Example 2 the rational points $x \in \pi(\mathbb{Q}^2)$ are exactly those with finite orbit. A homoclinic point $w \neq 0$ is an example of a point with infinite but non-dense orbit.

Although in this example (as in Example 2) topologically almost all points have dense orbit (the points with dense orbits form a dense G_δ-set), it is not obvious how to find one. For the construction of a point with dense orbit we will make use of a *symbolic cover* (or *Markov partition*) of the dynamical system $(\mathbb{T}^2, T)$. This is a generalization of the p-adic expansion from Example 2. To define this, we first have to look at symbolic dynamical systems.

Example 4. Let $\mathcal{A} = \{1, \ldots, r\}$. We call $\mathcal{A}$ the alphabet and let $X = \mathcal{A}^{\mathbb{Z}}$ with the product topology. The elements $x \in X$ are functions from $\mathbb{Z}$ to $\mathcal{A}$, in other words bi-infinite sequences

$$x = (\dots\dots, x_{-1}, x_0, x_1, \dots) \in \mathcal{A}^{\mathbb{Z}}.$$

The space X is a metric space, a metric is given by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{m} & \text{if } m = \min\{|n| + 1 : x_n \neq y_n\}. \end{cases} \tag{3}$$

The *shift map* $\sigma : X \to X$ is defined by

$$(\sigma(x))_n = x_{n+1}, \tag{4}$$

which means $\sigma(x)$ is the sequence one gets after shifting all entries by one to the left. Then σ is a homeomorphism of the space X. The dynamical system (X, σ) is called the *full shift* (with alphabet $\mathcal{A}$).

The advantage of a symbolic system like the full shift is the ease one has in constructing points with certain properties, in fact we used this already in Example 2.

Example 5. In order to get a broader variety of symbolic systems, we now define the class of edge shifts. Let G be a finite directed graph with r edges and label every edge with a unique element of the alphabet $\mathcal{A} = \{1, \dots, r\}$. In Figure 1 this has been done for $r = 3$. Every path on G can be labelled

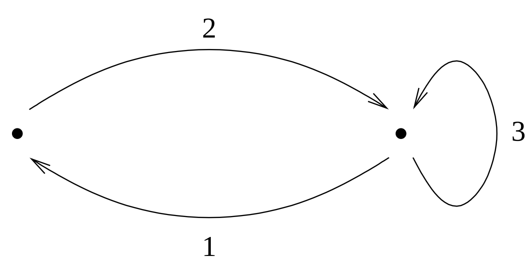


FIGURE 1. Graph with labelled edges

uniquely by a word in the alphabet $\mathcal{A}$. The space $X_G \subseteq \mathcal{A}^{\mathbb{Z}}$ is defined as the set of bi-infinite words arising from bi-infinite paths on G. In X_G we use the induced topology as a subset of the full shift. An element $x \in \mathcal{A}^{\mathbb{Z}}$ belongs to X_G if every symbol x_n is followed by an edge x_{n+1} which starts at the end point of x_n. In our example (see Figure 1) this means that 1 must be followed by 2 and 2 (resp. 3) can be followed either by 1 or by 3.

If every two vertices of the graph G are connected by at most one edge in one direction and by at most one in the other, then every bi-infinite path on G can also be described by the visited vertices. To make this clearer, consider the graph in Figure 2, which is essentially the same as in Figure 1, but where the labels are now at the vertices. Let $X_g \subseteq \{0,1\}^{\mathbb{Z}}$ be the set of labels of paths on G. The two spaces X_G (from Figure 1) and X_g are isomorphic, say $\phi : X_G \to X_g$ is the isomorphism as described above. To get dynamical systems we define on both spaces the shift map as in (4). The system (X_G, σ) is called the *edge shift* corresponding to the graph G. The system (X_g, σ) is the *golden mean shift* (we will justify the name later).

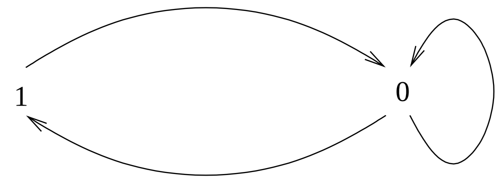


FIGURE 2. Graph with labelled vertices

The isomorphism ϕ satisfies the commutative diagram

$$\begin{array}{ccc} X_G & \xrightarrow{\sigma} & X_G \\ {\scriptstyle\phi}\downarrow & & {\scriptstyle\phi}\downarrow \\ X_g & \xrightarrow{\sigma} & X_g \end{array}$$

(which means that $\sigma \circ \phi = \phi \circ \sigma$) and is therefore called a *conjugacy* and the two systems (X_G, σ) and (X_g, σ) are *conjugate*.

As mentioned earlier the construction of points with certain properties is easy in symbolic spaces. To see this, we look again at the golden mean shift. The alphabet there consists of 0 and 1. The only restriction for points in X_g is that no two 1's can appear next to each other. Using the metric (3), we see that two points are close to each other if they agree in a large interval around the origin. To construct a point with dense orbit, list all possible finite words appearing in points in X_g:

$$(w_1, w_2, \ldots) = (0, 1, 00, 01, 10, 000, 001, 010, 100, 101, 0000, \ldots)$$

(the *language* of X_g). We construct a point $x = (x_i)_i \in X_g$ by writing only 0 for negative i and the positive half of x we set equal to the concatenation

$$w_1 0 w_2 0 w_3 0 \ldots$$

Then $x \in X_g$ and the orbit $\mathcal{O}(x) = \{\sigma^n x : n \geq 0\}$ is dense in X_g.

For a complete introduction to shift spaces (shifts of finite type, sofic shifts and their applications) see [LM].

Example 6 (Continuation of Example 3)**.** We want to construct a continuous surjective map $\phi : X_g \to \mathbb{T}^2$ such that the diagram

$$\begin{array}{ccc} X_g & \xrightarrow{\sigma} & X_g \\ {\scriptstyle\phi}\downarrow & & {\scriptstyle\phi}\downarrow \\ \mathbb{T}^2 & \xrightarrow{T} & \mathbb{T}^2 \end{array} \tag{5}$$

commutes (which means that $\phi \circ \sigma = T \circ \phi$). Such a map is called a *factor map* and the system $(\mathbb{T}^2, T)$ is a *factor* of (X_g, σ).

Let $w^\Delta \in \mathbb{T}^2$ be defined by the intersection of the line $\mathbb{R}v_\lambda$ with $\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathbb{R}v_\mu$ (see Figure 3).

Normalize v_λ and v_μ such that

$$v_\lambda = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + v_\mu.$$

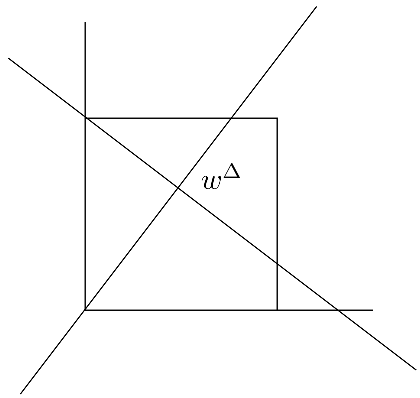


FIGURE 3. Definition of the point w^Δ

Now $w^\Delta = v_\lambda + \mathbb{Z}^2$, and similarly

$$\begin{aligned} T^n w^\Delta &= \lambda^n v_\lambda + \mathbb{Z}^2, \\ T^n w^\Delta &= \mu^n v_\mu + \mathbb{Z}^2. \end{aligned}$$

Let $x \in X_g$ and define

$$\phi(x) = \sum_{n\in\mathbb{Z}} x_n T^n w^\Delta = x_0 w^\Delta + \left(\sum_{n<0} x_n \lambda^n\right) v_\lambda + \left(\sum_{n>0} x_n \mu^n\right) v_\mu,$$

which is well defined since $|\lambda| > 1$ and $|\mu| < 1$.

The map $\phi : X_g \to \mathbb{T}^2$ also satisfies the commutative Diagram (5), and is continuous. It remains to show that ϕ is surjective.

Let $C = \{x \in X_g : x_0 = 0\}$ (such a set is called a *cylinder set*). We want to calculate $\phi(C)$. To do this we start with the question: Which real numbers $r \in \mathbb{R}$ can be expressed as

$$r = \sum_{n=1}^{\infty} x_n \lambda^{-n}$$

where x_n is a sequence of 0 and 1 such that no two 1's are next to each other? We claim that every $r \in [0,1]$ has such a representation.

If $r = 1$, we just need to calculate the geometric series

$$\sum_{m=0}^{\infty} \lambda^{-2m-1} = 1.$$

Let now $r_0 \in [0,1)$ and define inductively

$$\begin{aligned} s_{n+1} &= \lambda r_n \\ x_{n+1} &= \lfloor s_{n+1} \rfloor \\ r_{n+1} &= \{s_{n+1}\} = s_{n+1} - x_{n+1}, \end{aligned}$$

then

$$\lambda r_n = x_{n+1} + r_{n+1}. \tag{6}$$

It follows that $0 \le s_{n+1} < \lambda < 2$ and therefore $x_{n+1} \in \{0,1\}$ for every $n \ge 0$. If $x_{n+1} = 1$, then $s_{n+1} \in [1,\lambda)$ and $r_{n+1} \in [0, \lambda - 1)$. Since λ is the golden mean, we have $\lambda - 1 = \frac{1}{\lambda}$ and $s_{n+1} \in [0,1)$ and $x_{n+2} = 0$. Therefore the constructed sequence x_n satisfies the required property. Using (6) we get

$$\lambda^m r_0 = \lambda^{m-1}(x_1 + r_1) = \ldots = \lambda^{m-1}x_1 + \lambda^{m-2}x_2 + \cdots + x_m + r_m.$$

This shows that

$$\left| r_0 - \sum_{n=1}^{m} \lambda^{-n} x_n \right| = |\lambda^{-m} r_m| < \lambda^{-m}$$

and therefore

$$r_0 = \sum_{n=1}^{\infty} \lambda^{-n} x_n.$$

This provides a partial answer to our question, since every $r \in [0,1]$ can be expressed in the stated manner. The fact that only elements of $[0,1]$ appear in this way, can be shown similarly. The above procedure and statements are examples of *β-expansions*, where one replaces the basis p of the p-adic expansion by a real number $\beta > 1$.

For the calculation of the image $\phi(C)$ we also have to ask which real numbers $r \in \mathbb{R}$ can be expressed as $\sum_{n=1} x_n \mu^n$ (where $\mu = -\frac{1}{\lambda}$) with the same condition on the x_n as before. We claim that every $r_0 \in [-1, \frac{1}{\lambda}]$ has such a representation. Define

$$\begin{aligned} s_{n+1} &= -\lambda r_n \\ x_{n+1} &= \begin{cases} 1 & \text{if } s_{n+1} \in (\frac{1}{\lambda}, \lambda], \\ 0 & \text{otherwise,} \end{cases} \\ r_{n+1} &= s_{n+1} - x_{n+1} \in [-1, \frac{1}{\lambda}), \end{aligned}$$

then

$$-\lambda r_n = x_{n+1} + r_{n+1}. \tag{7}$$

If $x_{n+1} = 1$, then $s_{n+1} \in (\frac{1}{\lambda}, \lambda]$ and $r_{n+1} \in (\frac{1}{\lambda} - 1, \lambda - 1]$. Since $\frac{1}{\lambda} - 1 = -\frac{1}{\lambda^2}$ we get $s_{n+2} \in [-1, \frac{1}{\lambda})$ and $x_{n+2} = 0$. Now one can again use Equation (7) to prove that

$$r_0 = \sum_{n=1}^{\infty} x_n \mu^n.$$

Together with the previous statement for λ this shows that

$$\phi(C) \supseteq [0,1]v_\lambda + [-1, \frac{1}{\lambda}]v_\mu$$

and the right hand side is the rectangle R_C in Figure 4

Let $D = \{x \in X_g : x_0 = 1\}$, for every $x \in D$ we have necessarily $x_{-1} = x_1 = 0$. By a similar consideration as for the set C one shows that

$$\phi(D) \supseteq w^\Delta + [0, \frac{1}{\lambda}]v_\lambda + [-\frac{1}{\lambda^2}, \frac{1}{\lambda}]v_\mu,$$

where the right hand side is the rectangle R_D in Figure 4.

In Figure 4 one can see that R_C and R_D cover, modulo $\mathbb{Z}^2$, the space $\mathbb{T}^2$, therefore ϕ is onto.

The idea to construct a symbolic covering space using a homoclinic point w^Δ first appeared in [KV] and is pursued in [ES].

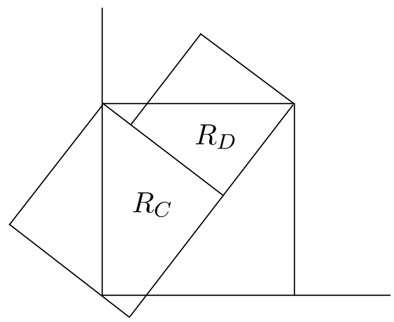


FIGURE 4. The two rectangles R_C and R_D cover $\mathbb{T}^2$.

Recalling the point x from Example 4 and applying ϕ, one gets a point $y = \phi(x) \in \mathbb{T}^2$ with a dense orbit.

Conjugacies and factor maps are important notions in the theory of dynamical systems. A main question is to decide when two systems are conjugate, or when one system is a factor of another. The map $\phi : (X_g, \sigma) \to (\mathbb{T}^2, T)$ from Example 5 can only be a factor map. However, this is because of topological reasons rather than dynamical ones. The space $\mathbb{T}^2$ is a connected topological space, X_g is not connected (the sets $C, D \subseteq X_g$ are open and closed).

Example 7. The shift spaces $X_2 = \{1,2\}^{\mathbb{Z}}$ and $X_3 = \{1,2,3\}^{\mathbb{Z}}$ are homeomorphic as topological spaces. To see this, we define a map $\phi : X_3 \to X_2$. Let $x \in X_3$ and let $\phi(x)$ be the resulting sequence after substituting in x every 2 by the word 21 and every 3 by 22. For instance, let

$$\begin{aligned} x &= \ldots 132.213 \ldots \in X_3 \\ \phi(x) &= \ldots 1\ 22\ 21.21\ 1\ 22 \ldots \in X_2 \end{aligned}$$

where the dot marks the origin ($x_0 =$ the symbol on the left of the dot). This map ϕ is a homeomorphism but not a conjugacy:

$$\begin{aligned} \sigma(x) &= \ldots 1322.13 \ldots \in X_3 \\ \phi(\sigma(x)) &= \ldots 1222121.122 \ldots \in X_2 \\ \sigma(\phi(x)) &= \ldots 122212.1122 \ldots \in X_2. \end{aligned}$$

To see that the dynamical systems (X_2, σ) and (X_3, σ) are not conjugate, it is sufficient to count the fixed points in both systems. A *fixed point* $x \in X$ for a dynamical system (X, T) is a point satisfying $Tx = x$. In the case of the dynamical system (X_2, σ) a point x is a fixed point if $\sigma x = x$ or $x_{n+1} = x_n$ for all $n \in \mathbb{Z}$. Therefore a fixed point $x \in X_2$ is just a constant sequence. Since there are two symbols, we have two fixed points. Similarly (X_3, σ) has three fixed points. A conjugacy $\phi : (X_3, \sigma) \to (X_2, \sigma)$ would have to map a fixed point to a fixed point and cannot be injective.

More generally we say $x \in X$ is a *periodic point* if $T^n x = x$ for some $n > 0$. A periodic point $x \in X$ has *period* n if $n > 0$ and $T^n x = x$. A periodic point has *least period* n if x has period n but not period m for $m < n$. In Example 6 we used the number of fixed points to see that (X_2, σ) and (X_3, σ) are 'really different' (not conjugate). A number (or mathematical structure) assigned to each topological dynamical system such that two conjugate systems have the same number assigned to them, is called a *(topological) invariant*. In Example 6 we used that the number of fixed points is an invariant. Similarly, the number of periodic points of period n is an invariant.

Another important invariant is the *topological entropy* $h(T)$. The exact definition is rather complicated (see Definition 8), therefore let us consider the entropy and its meaning in some examples. We discussed earlier the property of expansiveness. A system (X, T) is expansive if small distances will get big at some time. However, comparing the maps

$$\begin{aligned} T_2 \quad : \quad & \mathbb{T} \to \mathbb{T} \\ & x \mapsto 2x \text{ and} \\ T_3 \quad : \quad & \mathbb{T} \to \mathbb{T} \\ & x \mapsto 3x, \end{aligned}$$

we see that T_3 is faster in making small distances big. In those examples the topological entropy measures the rate how fast distances can get expanded using the map. One can show that

$$\begin{aligned} h(T_2) &= \log 2, \\ h(T_3) &= \log 3 \end{aligned}$$

In Example 1 where

$$\begin{aligned} T_\alpha \quad : \quad & \mathbb{T} \to \mathbb{T} \\ & x \mapsto x + \alpha \end{aligned}$$

distances stay the same and we have $h(T_\alpha) = 0$. In Example 3 where

$$\begin{aligned} T \quad : \quad & \mathbb{T}^2 \to \mathbb{T}^2 \\ & \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_2 \\ x_1 + x_2 \end{pmatrix} \end{aligned}$$

we already used the basis of the eigenvectors $\{v_\lambda, v_\mu\}$. Since $|\mu| < 1$, distances can only increase 'in the direction of v_λ' and for this reason, $h(T) = \log \lambda$.

We will now define the topological entropy and then use the definition to calculate the entropy of the full shift with alphabet $\mathcal{A} = \{1, \ldots, r\}$.

Definition 8. Let X be a compact metric space and $T : X \to X$ be a continuous map. Let d be a metric on X and $\epsilon > 0$. We say a set $E \subseteq X$ is (n, ϵ)-separated if for any two points $x, y \in E$ with $x \neq y$ there exists a $j \in [0, n-1]$ such that $d(T^j x, T^j y) > \epsilon$ (the two points get separated within the time from 0 to $n-1$). Let $s_n(\epsilon)$ be the largest cardinality of an (n, ϵ)-separated set, the topological entropy of T is defined as

$$h(T) = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log s_n(\epsilon).$$

Example 9. Let $X_r = \{1, \ldots, r\}^{\mathbb{Z}}$ be the full shift, let σ be the shift map defined in (4) and let d be the metric defined in (3).

To calculate the entropy $h(\sigma)$, set $\epsilon = \frac{1}{K+2}$. Then $d(x, y) > \epsilon$ if and only if

$$(x_{-K}, \ldots, x_K) \neq (y_{-K}, \ldots, y_K).$$

Similarly there exists an $j \in [0, n-1]$ such that $d(\sigma^j x, \sigma^j y) > \epsilon$ if and only if

$$(x_{-K}, \ldots, x_{K+n-1}) \neq (y_{-K}, \ldots, y_{K+n-1}).$$

Since there are r^{2K+n} words of length $2K+n$ using the alphabet $\{1, \ldots, r\}$, we get that

$$\begin{aligned} s_n(\epsilon) &= r^{2K+n} \\ \liminf_{n\to\infty} \frac{1}{n} \log s_n(\epsilon) &= \log r \text{ and} \\ h(\sigma) &= \log r. \end{aligned}$$

By a similar argument one can show that for the golden mean shift $\sigma : X_g \to X_g$ the entropy $h(\sigma) = \log \lambda$ is given by the logarithm of the golden mean (justifying its name). This shows that the two systems (X_g, σ) and $(\mathbb{T}^2, T)$ from Example 5 have the same entropy. In this sense the two systems are very close to each other although the factor map ϕ cannot be a conjugacy. In Section 3 of [ES] a symbolic cover of equal entropy is constructed more generally.

Although the calculation of the entropy can be quite challenging in examples, for many systems the entropy turns out to be a well known quantity (for instance the logarithm of an eigenvalue). In [ME99] an entropy formula from [LSW] is extended to more general systems and in [EW00] some algebraic techniques are used for the calculation of entropy and other invariants.

3. Measurable Dynamical Systems

In this section we give examples of measurable systems. A *measurable dynamical system* is given by a probability space $(X, \mathcal{B}, \mu)$ and a measure preserving map $T : X \to X$. This means we have a probability measure μ, which assigns to every measurable set $A \subseteq X$ (that means to every $A \in \mathcal{B}$) a measure $\mu(A) \in [0, 1]$. The map T has to satisfy

$$\mu(T^{-1}A) = \mu(A) \text{ for every } A \in \mathcal{B},$$

so that for a measurable set A the preimage $T^{-1}A$ should have the same measure.

In the last section we asked whether the orbit of a point is dense in the space. Now we can ask whether the points of the orbit distribute correctly in the space X (with respect to the measure μ): When is it true that the orbit of $x \in X$

$$\{x, Tx, \ldots, T^n x, \ldots\}$$

spends asymptotically the proportion $\mu(A)$ of its time in A?

This is surely not the case if A is invariant and satisfies $\mu(A) \in (0, 1)$, because for $x \in A$ every point $T^n x$ belongs to A.

We say T is *ergodic* if every invariant set $A \in \mathcal{B}$ satisfies $\mu(A) \in \{0, 1\}$. The ergodic theorem now states that if T is ergodic and f is an integrable function, then

$$\frac{1}{N} \sum_{n=0}^{N-1} f(T^i x) \to \int f \, d\mu$$

for $n \to \infty$ and almost all $x \in X$. Using the case where $f = 1_A$, we can (almost) answer the previous question. For almost any starting point $x \in X$ the orbit satisfies the property discussed above.

Example 10. Let $X = \mathbb{T}$ and $\mu = \lambda$ the Lebesgue measure on the Borel σ-algebra $\mathcal{B}$. We use the irrational translation map from Example 1

$$T_\alpha x = x + \alpha \text{ for a fixed } \alpha \in \mathbb{R} \setminus \mathbb{Q}.$$

We ask whether T_α is ergodic. One can show that here the answer is even better than for a general ergodic map, every orbit $\{x + n\alpha : n \in \mathbb{N}\}$ is uniformly distributed in $\mathbb{T}$ (see [WP]).

However, we can also ask if T_α is a mixing map. We say a measure-preserving map $T : X \to X$ is (strongly) *mixing* if for any two measurable sets $A, B \in \mathcal{B}$

$$\mu(A \cap T^{-n}B) \to \mu(A)\mu(B) \text{ for } n \to \infty.$$

In other words the two sets A and $T^{-n}B$ become more and more independent if n grows. This is version of chaos, since knowing where the point x is now (either in A or in $X \setminus A$) does not give you any information where it will be at time n (in B or not).

Again the irrational translation does not have this stronger version of chaos. Let $A = B = [0, \frac{1}{3})$. Since the orbit $\{n\alpha : n \in \mathbb{N}\}$ is dense, we can find large n with $n\alpha \in [\frac{1}{3}, \frac{2}{3})$. This shows that

$$\mu(A \cap T^{-n}B) = \mu\Big(\big[0, \frac{1}{3}\big) \cap \big[n\alpha, n\alpha + \frac{1}{3}\big)\Big) = 0,$$

and T_α cannot be mixing.

Example 11. To see a mixing system, we look at the multiplication by p

$$Tx = px \pmod 1$$

from Example 2 on the same measurable space $(\mathbb{T}, \mathcal{B}, \lambda)$. Then T is measure preserving. To see this, let $A \in \mathcal{B}$ and work in $[0, 1)$ instead of $\mathbb{T}$. The preimage

$$T^{-1}A = \frac{1}{p}A \cup \big(\frac{1}{p} \cup \frac{1}{p}A\big) \cup \cdots \cup \big(\frac{p-1}{p} + \frac{1}{p}A\big)$$

consists of p copies of the compressed set $\frac{1}{p}A$ and has the same measure as A.

The proof that T is mixing is not very complicated, but requires some approximation techniques from measure theory. To see the idea of the proof consider two intervals

$$A = \big[\frac{i}{p}, \frac{i+1}{p}\big) \text{ and } B = \big[\frac{j}{p}, \frac{j+1}{p}\big),$$

then for $n > 1$ the preimage is given by

$$\begin{aligned} T^{-n}B &= \bigcup_{k=0}^{p^{n-1}-1} \big[\frac{kp+j}{p^n}, \frac{kp+j+1}{p^n}\big) \\ &= \bigcup_{l=0}^{p-1} \bigcup_{m=0}^{p^{n-2}-1} \big[\frac{(lp^{n-2}+m)p+j}{p^n}, \frac{(lp^{n-2}+m)p+j+1}{p^n}\big). \end{aligned}$$

Taking the intersection with A we only get the part corresponding to $l = i$

$$A \cap T^{-n}B = \bigcup_{m=0}^{p^{n-2}-1} [\frac{i}{p} + \frac{mp+j}{p^n}, \frac{i}{p} + \frac{mp+j+1}{p^n})$$

whose measure equals

$$\mu(A \cap T^{-n}B) = \sum_{m=0}^{p^{n-2}-1} \frac{1}{p^n} = \frac{1}{p^2} = \mu(A)\mu(B).$$

This shows that A and $T^{-n}B$ (for $n > 1$) are already independent. More generally it is true that intervals $[a_1, a_2)$, $[b_1, b_2)$ with p-adic end points (that means with $a_1, a_2, b_1, b_2 \in \mathbb{Z}[\frac{1}{p}]$) become independent for big n. Using the fact that those intervals generate the σ-algebra $\mathcal{B}$, one can show that T_p is mixing.

In examples like the torus automorphism

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 + x_2 \end{pmatrix}.$$

from Example 3 there is a general characterization of ergodicity and the mixing property. The automorphism is ergodic (or mixing) if and only if no eigenvalue is a root of unity, so that the map T as above is ergodic and mixing. In [ER] we will study these properties for actions of non-abelian groups on compact abelian groups (see also Section 4).

4. $\mathbb{Z}^d$-actions and extensions

In this section we use instead of the one-dimensional time a higher dimensional time. This amounts to replacing the group $\mathbb{Z}$ and its action by a group G with more generators.

Mathematically it is a natural question after rephrasing the idea of dynamical systems in terms of $\mathbb{Z}$-actions to look at G-actions. However, there are also practical reasons to study them. The theory of edge shifts is used in computer science to adapt (by a recoding) data to the technical restrictions of a data storage media. Although many media are in fact two-dimensional (like a compact disc), they are used in a one-dimensional way (the data is arranged in many circles on the disc) and a one-dimensional recoding is used. However, some experimental media are now used as two-dimensional media and one reason to study $\mathbb{Z}^2$-actions is to develop a theory of coding in that context.

Let $G = \mathbb{Z}^d$. A $\mathbb{Z}^d$-action ϕ is defined by d commuting automorphism $T_1, \ldots, T_d$ of a space X

$$\phi(\mathbf{n}) = T_1^{n_1} \ldots T_d^{n_d} \text{ for } \mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d.$$

To develop a theory of $\mathbb{Z}^d$-actions, one needs a pool of examples which are interesting but allow analysis.

One class of examples are $\mathbb{Z}^d$-actions on compact abelian groups by automorphisms. Here X is a compact abelian group like $\mathbb{T}, \mathbb{T}^k, \mathbb{T}^{\mathbb{Z}^d}$ and closed subgroups of those, and the maps $T_1, \ldots, T_d$ are group automorphisms of X. An important tool in the analysis of such examples is harmonic analysis which assigns to each compact abelian group X a discrete abelian group $M = \hat{X}$ (the *dual group*). Since we have d automorphisms $T_1, \ldots, T_d$ on X, we can consider M as a module over the ring

$$R_d = \mathbb{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}].$$

If $f(u) = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) u^{\mathbf{n}} \in R_d$ (here $c_f(\mathbf{n}) \neq 0$ only for finitely many $\mathbf{n} \in \mathbb{Z}^d$) and $m \in M$, then

$$f(u)m = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) \hat{T_1}^{n_1} \cdots \hat{T_d}^{n_d} m$$

where $\hat{T_i} : M \to M$ is the dual map to $T_i : X \to X$. The converse of this construction assigns to every discrete R_d-module M a $\mathbb{Z}^d$-action by automorphisms on a compact abelian group $X = \hat{M}$. For complete discussion on the relationship between the dynamical system (X, ϕ) and the module M we refer to [KS].

Many dynamical questions can be translated by the above correspondence to algebraic questions on M. We present now some examples in the above framework.

Example 12. Let $d = 1$ and $M = \mathbb{Z}[\frac{1}{2}] = \{\frac{k}{2^l} : k \in \mathbb{Z}, l \in \mathbb{N}\}$. Then M is an R_1-module by defining

$$p(u_1)\frac{k}{2^l} = p(2)\frac{k}{2^l} \in \mathbb{Z}[\frac{1}{2}].$$

Another description for M is

$$M \cong R_1/(u_1 - 2) = \mathbb{Z}[u_1, u_1^{-1}]/(u_1 - 2).$$

Using this, one can show that the dual group is isomorphic to

$$X = \{x \in \mathbb{T}^{\mathbb{Z}} : 2x_n = x_{n+1} \text{ for every } n \in \mathbb{Z}\}$$

and the action is defined by the usual shift in X. Since

$$(\sigma x)_n = x_{n+1} = 2x_n$$

for every $x \in X$ and $n \in \mathbb{N}$ we see that the shift is the same as multiplication by 2

$$\sigma(x) = 2x$$

in the group.

Consider the projection map $\pi : X \to \mathbb{T}$ defined by $\pi(x) = x_0$, then we have the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X \\ \pi\downarrow & & \pi\downarrow \\ \mathbb{T} & \xrightarrow{T_2} & \mathbb{T}. \end{array}$$

The system (X, σ) is the invertible extension of the system $(\mathbb{T}, T_2)$ from Example 2.

To get higher dimensional examples, we now let $d = 2$.

Example 13. Let $M = \mathbb{Z}[\frac{1}{2}, \frac{1}{3}] = \{\frac{k}{6^l} : k \in \mathbb{Z}, l \in \mathbb{N}\}$ and define

$$f(u_1, u_2)\frac{k}{6^l} = f(2,3)\frac{k}{6^l} \in M.$$

Then M is an R_2-module which is isomorphic to

$$M \cong R_2/(u_1 - 2, u_2 - 3).$$

Using this description we can write the dual as

$$X = \{x \in \mathbb{T}^{\mathbb{Z}^2} : 2x_{\mathbf{n}} = x_{\mathbf{n}+(1,0)}, 3x_{\mathbf{n}} = x_{\mathbf{n}+(0,1)} \text{ for all } \mathbf{n} \in \mathbb{Z}^2\}.$$

The $\mathbb{Z}^2$-action is the $\mathbb{Z}^2$-*shift*, the map $\sigma_{\mathbf{m}} : X \to X$ is defined by

$$\big(\sigma_{\mathbf{m}}(x)\big)_{\mathbf{n}} = x_{\mathbf{n}+\mathbf{m}}. \tag{8}$$

Again $\sigma_{(1,0)}x = 2x$ and $\sigma_{(0,1)}x = 3x$ for $x \in X$, therefore X is a compact group where the multiplication by 2 and the multiplication by 3 are automorphisms.

In the last two examples the module M was always isomorphic to a quotient of the ring R_d, in other words M was a cyclic module. Although not all examples are of that form, systems defined by modules $M = R_d/\mathfrak{p}$ are the building blocks for more complicated examples and are therefore important for the theory. It is convenient to speak of the system defined by the prime ideal $\mathfrak{p}$.

Example 14. Let $\mathfrak{p} = (p) \subseteq R_d$ be the prime ideal generated by a prime number $p \in \mathbb{N}$. Then

$$M = R_d/\mathfrak{p} = \mathbb{F}_p[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$$

where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. The dual group is the *d-dimensional full shift* with alphabet $\mathbb{F}_p$

$$X = \mathbb{F}_p^{\mathbb{Z}^d},$$

the action is again the shift-action as in Equation (8).

Example 15. Let $d = 2$ and $\mathfrak{p} = (2, 1 + u_1 + u_2) \subseteq R_2$. Here the dual group is given by

$$X = \{x \in \mathbb{F}_p^{\mathbb{Z}^d} : x_{\mathbf{n}} + x_{\mathbf{n}+(1,0)} + x_{\mathbf{n}+(0,1)} = 0 \text{ for all } \mathbf{n} \in \mathbb{Z}^2\}$$

and the action is the $\mathbb{Z}^2$-shift. The dynamical system (X, σ) is called *Ledrappier's example* and was first introduced and studied because it is mixing but not 3-mixing.

A $\mathbb{Z}^d$-action T on X is 3-*mixing* if for any three measurable sets A, B, C

$$\lambda(A \cap T_{\mathbf{n}}^{-1}B \cap T_{\mathbf{m}}^{-1}C) \to \lambda(A)\lambda(B)\lambda(C)$$

if $\mathbf{n}, \mathbf{m}, \mathbf{n} - \mathbf{m} \to \infty$. This captures a strong form of chaos: a system is 3-mixing if information about a point in two different times does not give you any information about the point in another time.

In Ledrappier's example (as in all other algebraic examples) we use the Haar measure and since (by using freshman's dream)

$$x_{(0,0)} + x_{(2^n,0)} + x_{(0,2^n)} = 0$$

we see that knowing x at $(0,0)$ and at $(2^n, 0)$ tells you exactly x at $(0, 2^n)$. Therefore (X, σ) is not 3-mixing.

Similarly one can define the notion of r-mixing (or mixing of order r). In the recent preprint [EW] one can find a procedure how to construct similar examples which are r-mixing but not $r+1$-mixing for any $r \geq 2$.

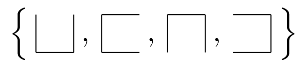
Knowing Ledrappier's example it is surprising that a mixing $\mathbb{Z}^d$-action on a compact abelian *connected* group is mixing of all orders. This has been shown in [SW] by using a strong result from the theory of diophantine approximations.

In [ELMW] we will study expansive $\mathbb{Z}^d$-action and determine whether subgroups of $\mathbb{Z}^d$ also act expansive. Related to that we will introduce and compare some notion of dynamical dimension. Example 12 and Example 13 are in many ways one-dimensional (although we have a $\mathbb{Z}^2$-action in Example 13), the individual elements of the action (the maps $\sigma_{\mathbf{n}}$) have finite entropy and some of them are expansive. If we set $d > 1$ in Example 14, then the individual elements have infinite entropy and no map $\sigma_{\mathbf{n}}$ is expansive. There one has to consider the whole $\mathbb{Z}^d$-action (instead of just one map), its 'dimension' is d.

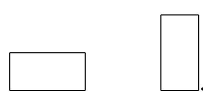
In [ER] actions of non-abelian groups G by automorphisms on compact abelian groups are studied. There many algebraic characterizations for dynamical properties have to be reformulated – or even fail to be true anymore. The characterization of ergodicity generalizes, the characterization for the mixing property does not generalize.

In [ES] and [ME] studies various aspects of symbolic $\mathbb{Z}^d$-actions, where $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ is a (closed and shift invariant) subset of the full shift and we use the restriction of the shift action to the space X.

Example 16. We define our alphabet as

$\mathcal{A} =$ 

and let X_{Dom} be the set of all elements $x \in \mathcal{A}^{\mathbb{Z}^2}$ where the open sides of the squares look to each other. In other words X_{Dom} is the set of all tilings of the plane by using the 2×1 and 1×2 dominoes



In [ME] a result about cocycles (see the Introduction of [ME]) from [KS98] is generalized to other tiling spaces. This is also related to the question: 'Can a given region be filled by using those tiles?'

Example 17. The 8×8 chess board can be tiled by using the domino tiles from Example 16.

Let R_1 be the region one gets after removing one corner square from the chess board. Then R_1 is not tile-able since every domino covers an area of 2 and the area of R_1 is not divisible by 2. Take now the region R_2 where the two bottom corner stones (the left most and the right most square in the lowest row) are removed. It is easy to see that one can tile the lowest row and the remaining rectangle separately, therefore showing that R_2 is tile-able with the domino tiles.

Let R_3 be the region one obtains by removing two opposite corner stones. We ask again: Is R_3 tile-able by using the domino tiles? Here the question is harder. The area of R_3 is 62, but R_3 is not tile-able. The idea for the proof here is that one has to count the black squares and the white squares (from the chess board pattern) of R_3 separately. Since we removed two squares with the same colour, we will have for instance 32 black and 30 white squares in R_3. As each domino will cover one black and one white square, the region R_3 cannot be tiled with the domino tiles.

This gives a necessary condition on the tile-ability which is easy to check. Are there other (in a sense algebraic) conditions like that one?

Every continuous cocycle of the dynamical system $(X_{\mathrm{Dom}}, \sigma)$ with values in a discrete group can be used to construct a necessary condition on the tile-ability of regions. The existence of a fundamental cocycles now means that there is a cocycle which produces the sharpest condition under all cocycles. However, in that example it turns out that the condition given by the fundamental cocycle is the black-white-argument above. This shows (in a sense) that we already had the best possible necessary condition for tile-ability.

In [ME] the existence of a fundamental cocycle for the domino shift is reproved and generalized to other tiling spaces.

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