CYCLICAL MONOTONICITY AND THE ERGODIC THEOREM

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Abstract. It is well known that *optimal* transport plans are *cyclically monotone* in a very general setting.

To obtain the reverse implication that cyclically monotone transport plans are optimal certain assumptions are required and the proof is non-trivial already if the costs are given by the squared euclidean distance on $\mathbb{R}^n$.

The aim of this note is to demonstrate that this assertion can be seen as a corollary to the ergodic theorem.

1. Introduction

We consider the *Monge-Kantorovich transport problem* for Borel probability measures μ, ν on Polish spaces X, Y. (cf. [Vil03, Vil09])

The set $\Pi(\mu, \nu)$ consists of all *transport plans*, that is, Borel probability measures on $X \times Y$ with X-marginal μ and Y-marginal ν. The *transport costs* associated to a *cost function* $c : X \times Y \to [0, \infty]$ and a transport plan π are given by

$$\langle c, \pi \rangle = \int_{X \times Y} c(x, y)\, d\pi(x, y).$$

The Monge-Kantorovich problem is then to determine the value

$$P_c := \inf\{\langle c, \pi\rangle : \pi \in \Pi(\mu, \nu)\} \tag{1}$$

and to identify an *optimal* transport plan $\hat{\pi} \in \Pi(\mu, \nu)$, i.e. a minimizer of (1).

Closely related to optimality is the notion of cyclical monotonicity. A set Borel set $\Gamma \subseteq X \times Y$ is *cyclically monotone* iff

$$c(x_1, y_2) - c(x_1, y_1) + \ldots + c(x_{n-1}, y_n) - c(x_{n-1}, y_{n-1}) + c(x_n, y_0) - c(x_n, y_n) \geq 0$$

for all $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) \in \Gamma$. A transport plan π is *cyclically monotone* if it assigns full measure to some cyclically monotone set Γ.

It was shown by Ambrosio and Pratelli [AP03] that every optimal transport plan is cyclically monotone for every general l.s.c. cost function $c : X \times Y \to [0, \infty]$.[1]

The situation concerning the reverse implication is more intricate. Ambrosio and Pratelli find that there exist cyclically monotone transport plans which are *not* optimal; notably the construction given in [AP03, Example 08.15] makes heavy use of the fact that c may attain the value ∞.

In [Vil03] it is asked whether cyclically monotone transport plans are always optimal in the case where c is the squared euclidean distance on $\mathbb{R}^n$. This problem was resolved in [Pra08] and [ST08] where it is proved that every cyclically monotone transport plan is optimal provided that c is continuous resp. merely l.s.c. but finitely valued.[2]

We propose a new approach to this assertion.

Theorem 1. *Let X, Y be Polish spaces equipped with probability measures μ resp. ν. Let $c : X \times Y \to [0, \infty)$ be a measurable cost function and let $\pi \in \Pi(\mu, \nu)$ be a finite cost transport plan which is cyclically monotone. Then π is optimal.*

[1] Indeed, (lower semi)-continuity is not important, cf. [BGMS09, BC09].

[2] We refer to [BGMS09] and [BC09] for stronger assertions.

2. Proof of Theorem 1

The novelty of the present approach is to connect the problem to the *ergodic theorem*[3] which we restate here for the convenience of the reader: Let (Z, κ) be a probability space and assume that $\sigma : Z \to Z$ is measure-preserving, i.e. that $\sigma(\kappa) = \kappa$. Then the ergodic theorem asserts that for every function $f \in L^1(\kappa)$ the limit

$$f^* = \lim_n \tfrac{1}{n} \textstyle\sum_{k=1}^n f \circ \sigma^k \tag{2}$$

exists almost surely and in $L^1(\kappa)$.[4]

Proof of Theorem 1. Let $\pi, \tilde{\pi} \in \Pi(\mu, \nu)$ be transport plans with finite costs. Assuming that π is concentrated on a cyclically monotone set Γ our aim is to show that π leads to lower costs than $\tilde{\pi}$.

To this end, we set $Z = (X \times Y)^{\mathbb{N}}$ and consider the *shift mapping*

$$\sigma : Z \to Z, \quad (x_i, y_i)_{i=1}^\infty \mapsto (x_{i+1}, y_{i+1})_{i=1}^\infty.$$

On Z we define the projections $P, Q : Z \to X \times Y$

$$P((x_i, y_i)_{i=1}^\infty) := (x_1, y_1), \text{resp. } Q((x_i, y_i)_{i=1}^\infty) := (x_1, y_2).$$

The transport plans π and $\tilde{\pi}$ give rise to a natural σ-invariant measure on Z:

Lemma 2.1. *There is a measure κ on Z such that $\sigma(\kappa) = \kappa, P(\kappa) = \pi, Q(\kappa) = \tilde{\pi}$.*

Proof. Let $(\pi_x)_{x \in X}$ be the disintegeration of π with respect to (X, μ) and let $(\tilde{\pi}^y)_{y \in Y}$ be the disintegration of $\tilde{\pi}$ with respect to (Y, ν). Then $R(x, B) := \pi_x(\{x\} \times B)$ and $S(y, A) := \pi^y(A \times \{y\})$ constitute transition kernels from X to Y resp. from Y to X. We identify Z with the product

$$X^{(1)} \times Y^{(1)} \times X^{(2)} \times Y^{(2)} \times \dots$$

where $X^{(n)}, Y^{(n)}$, $n \geq 1$ are copies of X and Y. Then we consider the discrete time Markov-process with initial distribution μ, and the probabilities of moving from $x \in X^{(n)}$ to $y \in Y^{(n)}$ resp. from $y \in Y^{(n)}$ to $x \in X^{(n+1)}$ given by R resp. S.

The resulting stationary distribution κ on the "path space" Z has the desired properties. (We refer the reader to [Kal02, Chapter 10] for more details on this procedure.) □

Letting $f = c \circ Q - c \circ P \in L^1(\kappa)$ we have $\int_{X \times Y} c \, d\tilde{\pi} - \int_{X \times Y} c \, d\pi = \int_Z F \, d\kappa$. The crucial step of the present proof is that, applying the ergodic theorem to the function f and integrating over (2) we have

$$\int_Z f \, d\kappa = \int_Z f^* \, d\kappa = \int_Z \lim_n \frac{1}{n} \sum_{k=1}^n f \circ \sigma^k \, d\kappa.$$

Rewriting this expression in terms of c we obtain

$$\int_{X \times Y} c \, d\tilde{\pi} - \int_{X \times Y} c \, d\pi = \int_Z \Big[\lim_n \frac{1}{n} \sum_{k=1}^n c(x_n, y_{n+1}) - c(x_n, y_n) \Big] \, d\kappa(x_i, y_i)_i \, . \tag{3}$$

To conclude the proof it is sufficient to show that the integrand on the right side of (3) is κ-almost surely non-negative. Note that $\kappa(\Gamma \times (X \times Y)^{\mathbb{N}}) = \pi(\Gamma) = 1$, hence, by σ-invariance of γ also $\gamma(\Gamma^N) = \bigcap_{n \geq 1} (\sigma^{-n}(\Gamma \times (X \times Y)^{\mathbb{N}})) = 1$. Thus it suffices to argue on sequences $(x_n, y_n)_{n=1}^\infty$ with $(x_n, y_n) \in \Gamma$, $n \geq 1$.

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[3]See for instance [Kal02, Theorem 9.6]

[4]The transformation σ is *ergodic* if all σ-invariant subsets of Z have measure 0 or 1. In this case $f^* \equiv \int f \, d\kappa$ but we will not need this.

It is instructive to note that the proof of Theorem 1 is immediate under the additional assumption that c is a bounded function: in this case cyclical monotonicity of Γ trivially implies

$$\liminf_n \frac{1}{n}\sum_{k=1}^n c(x_n, y_{n+1}) - c(x_n, y_n) \geq 0 \text{ whenever } (x_n, y_n)_{n=1}^\infty \in \Gamma^{\mathbb{N}},$$

thus π is optimal.

To prove Theorem 1 in the general case we need a little additional argument. Fix an auxiliary pair $(\bar{x}, \bar{y}) \in \Gamma$. Given $(x_i, y_i)_{i=1}^\infty \in \Gamma^{\mathbb{N}}$, cyclical monotonicity of Γ yields that

$$c(\bar{x}, y_1) - c(\bar{x}, \bar{y}) + \Big(\sum_{k=1}^n c(x_k, y_{k+1}) - c(x_k, y_k)\Big) + c(x_{n+1}, \bar{y}) - c(x_{n+1}, y_{n+1})$$

is non-negative for each $n \geq 0$. As c takes only values in $[0, \infty)$ this further implies

$$\liminf_n \Big[\frac{1}{n}\Big(\sum_{k=1}^n (c(x_k, y_{k+1}) - c(x_k, y_k)\Big) + \frac{c(x_{n+1}, \bar{y})}{n}\Big] \geq 0. \tag{4}$$

Setting $g((x_i, y_i)_{i=1}^\infty) := c(x_1, \bar{y})$ we have $c(x_{n+1}, \bar{y}) = g \circ \sigma^n((x_i, y_i)_{i=1}^\infty)$. As g is a finite function, g/n tends to 0 in measure (with respect to κ) and because σ is measure preserving, also $\lim_{n\to\infty} g\circ\sigma^n/n = 0$ in measure. Passing to a subsequence if necessary, we may also assume that this convergence holds κ-a.s. Together with (4) this proves that (3) is indeed non-negative and thus concludes the proof of Theorem 1. □

Using similar arguments as in [BGMS09, BC09] the above approach can be used to prove somewhat stronger versions of Theorem 1. As this does not involve interesting new ideas nor strengthens the results from [BGMS09, BC09] we do not go into this.

References

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