

AN OPTIMALITY PRINCIPLE WITH APPLICATIONS IN OPTIMAL TRANSPORT AND ITS OFFSPRING

MATHIAS BEIGLBÖCK AND CLAUDIUS GRIESSLER

ABSTRACT. A fundamental concept in optimal transport is c -cyclical monotonicity: it allows to link the optimality of transport plans to the geometry of their support sets. Recently, related concepts have been successfully applied in the multi-marginal version of the transport problem as well as in the martingale transport problem which arises from model-independent finance.

We establish a unifying concept of c -monotonicity / finitistic optimality which describes the geometric structure of optimizers to infinite-dimensional linear programming problems. This allows us to strengthen known results in martingale optimal transport and the infinitely marginal case of the optimal transport problem.

If the optimization problem can be formulated as a multi-marginal transport problem our contribution is parallel to a recent result of Zaev.

1. INTRODUCTION

1.1. Motivation from optimal transport. Consider the Monge-Kantorovich transport problem for probabilities μ, ν on Polish spaces X, Y , cf. [Vil03, Vil09]. The set $\Pi(\mu, \nu)$ of transport plans consists of all measures on $X \times Y$ with X -marginal μ and Y -marginal ν . Associated to a cost function $c : X \times Y \rightarrow \mathbb{R}_+$ and $\gamma \in \Pi(\mu, \nu)$ are the transport costs $\int c d\gamma$. The Monge-Kantorovich problem is then to determine the value

$$(OT) \quad \inf \left\{ \int c d\gamma : \gamma \in \Pi(\mu, \nu) \right\}$$

and to identify an *optimal* transport plan $\gamma^* \in \Pi(\mu, \nu)$, minimizing of (OT).

A fundamental concept in the theory of optimal transport is c -cyclical monotonicity which leads to a *geometric characterization* of optimal couplings. Its relevance was fully recognized by Gangbo and McCann [GM96].

We postpone precise definitions and just mention that heuristically, a transport plan is c -cyclically monotone if it cannot be improved by means of cyclical rerouting, i.e. by replacing the transfers

$$x_1 \rightarrow y_1, x_2 \rightarrow y_2, \dots, x_n \rightarrow y_n \quad \text{with} \quad x_1 \rightarrow y_2, x_2 \rightarrow y_3, \dots, x_n \rightarrow y_1.$$

Technically, the relation between optimality and c -cyclical monotonicity is rather intricate; it took a series of contributions ([AP03, Pra08, ST09, BGMS09] among others) to reach the following clear cut characterization:

Theorem 1.1. *Let $c : X \times Y \rightarrow [0, \infty)$ be Borel measurable and assume that $\gamma \in \Pi(\mu, \nu)$ is a transport plan with finite costs $\int c d\gamma \in \mathbb{R}_+$. Then γ is optimal if and only if γ is c -cyclically monotone.*

Date: April 29, 2014.

We acknowledge financial support through FWF-projects P21209 and P26736.

To understand if a transport behaves optimally on a finite number of points is an elementary and often feasible task. However, it is difficult to relate this to the transport between diffuse distributions since single points do not carry positive mass. Theorem 1.1 provides the required remedy to this obstacle: it connects the optimization problem for measures with optimality on a “pointwise” level.

1.2. Aims of this article. Several modifications and generalizations of the classical optimal transport problem have received interest in the literature, we list some which will also be considered in more detail below. A natural extension is the multi-marginal version of the transport problem where not just two but finitely many marginals are prescribed, see e.g. [Kel84, Car03, Pas11, Pas12, KP13]. Pass [Pas13a, Pas13b] considered the extensions of the transport problem to the case where a continuum of marginals is prescribed. Recently also martingale versions of the transport problem have received considerable attention (see [BHLP13, GHLT14, BJ13, HT13, DS13a, DS13b] among others) motivated by applications in model-independent finance.

Given the importance of c -cyclical monotonicity it is natural to search for a related concept applicable in the just mentioned versions of the transport problem. Kim and Pass [KP13] introduced a notion of c -monotonicity, necessary for optimality in the context of the multi-marginal transport problem ([KP13, Proposition 2.3]). This is used to develop a general condition on the cost function which is sufficient to imply existence of a Monge solution and uniqueness results in the multi-marginal optimal transport problem. In [BJ13], the authors introduced a concept of “finitistic optimality” which mimics c -cyclical monotonicity in the case of the 2-period martingale transport problem. A variational principle [BJ13, Lemma 2.1] then links finitistic optimality with optimality overall. This allows to determine optimal martingale transport plans in a number of instances.

The main goal of this article is to unify these notions and to make them applicable to the above mentioned variations of the transport problem. In (A) below we formulate an infinite-dimensional linear optimization problem which captures the just mentioned versions of the transport problem as special cases. We then introduce a version of finitistic optimality / c -monotonicity for this optimization problem and establish a “variational principle” (Theorem 2.4) which asserts that finitistic optimality is necessary for optimality overall.

In particular we obtain improved versions of the results from [BJ13] and [KP13, Proposition 2.3] as well as one half of the classical result stated in Theorem 1.1. To exemplify the applicability of Theorem 2.4 beyond optimization of finite products of spaces we prove a strengthened version of Pass’ Monge-type result for a continuum of marginals [Pas13a].

In independent work, Zaev [Zae14] obtains (among a number of further results) a theorem which is closely related to Theorem 2.4; [Zae14] works in the setup of a multi-marginal transport problem allowing for additional linear constraints. We will discuss the precise relation in Section 2.7 below.

2. FORMULATION OF THE PROBLEM AND THE OPTIMALITY CRITERION

2.1. The basic optimization problem. Throughout this article we assume that E is a Polish space and $c : E \rightarrow \mathbb{R}$ a Borel measurable cost function. Typical examples could be $E = M^2$, where M is a Riemannian

manifold, $E = (\mathbb{R}^d)^n$, or $E = C[0, T]$, the space of continuous functions $[0, T] \rightarrow \mathbb{R}$ with the topology of uniform convergence.

By \mathcal{F} we denote a set of Borel-measurable functions on E . We consider the probability measures γ on E for which $\int f d\gamma = 0$ for all $f \in \mathcal{F}$.¹ We denote the set of these measures by $\Pi_{\mathcal{F}}$. The optimization problem is then to minimize the total cost choosing from $\Pi_{\mathcal{F}}$, i.e.

$$(A) \quad \min_{\gamma \in \Pi_{\mathcal{F}}} \int c d\gamma.$$

We give a list of some more specific problems that can be posed this way. Given a product of spaces $\prod_{i \in I} X_i$ we will write p_{X_i} or in short p_i for the projection onto X_i .

2.2. Classical optimal transport and its multi-marginal version. To fit the classical Monge-Kantorovich problem (OT) into this setup, take $E = X \times Y$. To test whether a measure γ is a transport plan in $\Pi(\mu, \nu)$ it is sufficient to verify that

$$\int \varphi(x) d\pi(x, y) = \int \varphi(x) d\mu(x), \quad \int \psi(y) d\pi(x, y) = \int \psi(y) d\nu(y)$$

for all continuous bounded functions $\varphi : X \rightarrow \mathbb{R}, \psi : Y \rightarrow \mathbb{R}$. Hence, with

$$\mathcal{F}_1 = \{\varphi \circ p_X - \int \varphi d\mu, \psi \circ p_Y - \int \psi d\nu : \varphi \in C_b(X), \psi \in C_b(Y)\}$$

problem (A) is equivalent to the usual optimal transport problem (OT).

Of course, the same applies to the multi-marginal optimal transport problem where one considers

$$(1) \quad \inf\{\int c d\gamma : \gamma \in \Pi(\mu_1, \dots, \mu_n)\}$$

for probabilities μ_1, \dots, μ_n on Polish spaces X_1, \dots, X_n and $\Pi(\mu_1, \dots, \mu_n)$ consists of all probability measures γ on $E = X_1 \times \dots \times X_n$ satisfying $p_i(\gamma) = \mu_i, i = 1, \dots, n$. Here we take

$$(2) \quad \mathcal{F}_2 = \{\varphi \circ p_i - \int \varphi d\mu_i : \varphi \in C_b(X_i), 1 \leq i \leq n\}.$$

2.3. Optimal transport in the continuum marginal case. Recently Pass introduced an extension of the transport problem to the case of infinitely many marginals [Pas13a, Pas13b]. Specifically, in [Pas13a] the following problem was posed: for $I = [0, T]$, given a family $(\mu_t)_{t \in I}$ of probability measures on \mathbb{R} and a strictly concave function $h : \mathbb{R} \rightarrow \mathbb{R}$, determine

$$(B) \quad \inf_{\gamma \in \Pi_C(\mu_t)} \int h\left(\int_0^T f(t) dt\right) d\gamma(f),$$

where $\Pi_C(\mu_t)$ denotes the set of probability measures on $C[0, T]$ with marginals $(\mu_t)_{t \in I}$. Notably $(\mu_t)_{t \in I}$ can be assumed to be weakly continuous; otherwise there cannot be a measure on $C(I)$ with these marginals.

Under certain conditions Pass is able to show that this problem has a unique minimizer which he determines explicitly. He then lists several applications from parabolic equations to mathematical finance and quantum physics.

To view this in the framework of (A), simply set $E = C[0, T]$ and let

$$(3) \quad \mathcal{F}_3 = \{\varphi \circ p_t - \int \varphi d\mu_t : \varphi \in C_b(\mathbb{R}), t \in [0, T]\}.$$

¹ By asserting that $\int f d\gamma = 0$ we implicitly understand that this integral exists.

We use Theorem 2.4 to establish a strengthened version² of Pass' main result: following [Pas13a] we first describe a candidate optimizer: consider the quantile functions³ $q_t(x)$ of the probabilities $\mu_t, t \in [0, T]$. By weak continuity of $(\mu_t), t \mapsto q_t(x)$ is continuous for each $x \in (0, 1)$. Hence $q : x \mapsto (q_t(x))_{t \in [0, T]}$ defines a mapping from $(0, 1)$ to $C[0, T]$ and $\pi^* := q(\lambda|_{(0,1)}) \in \Pi_C(\mu_t)$. We then obtain

Theorem 2.1. *Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be concave and $(\mu_t)_{t \in I}$ a family of probability measures on \mathbb{R} , weakly continuous in t and such that*

$$\int_0^T \int |x| d\mu_t(x) dt < \infty, \quad \text{and} \quad \int |h| d\mu_t < \infty \quad \text{for all } t \in [0, T].$$

Then π^ is a minimizer of (B). If the infimum in (B) is finite and h is strictly concave, then π^* is the unique minimizer.*

2.4. Model-independent finance – Martingale Transport. For a general overview we refer to the survey of Hobson [Hob11]. Recent contributions on the general theory in discrete time include [ABPS13, HT13, BN13]. Here $E = \mathbb{R}_+^n$ or \mathbb{R}^n , the n -tuple (x_1, \dots, x_n) is interpreted as possible evolution of the stock price at future dates $t_1 < t_2 < \dots < t_n$. A possible price of a “path-dependent option” with payoff $c : E \rightarrow \mathbb{R}$ is then calculated as an integral

$$(4) \quad \int c d\gamma.$$

The basic problem in model-independent finance is to determine the minimal (or maximal) possible prices subject to appropriate constraints, i.e. to minimize (4) over a suitable class of probabilities γ .

According to the martingale pricing paradigm in mathematical finance possible measures of interest are *martingale measures*, i.e. probabilities γ such that the coordinate process on \mathbb{R}^n is a martingale (in its own filtration) with respect to γ . Thus γ is a *martingale measure* iff for each $l < n$, and each continuous bounded function $\varphi : \mathbb{R}^l \rightarrow \mathbb{R}$ one has

$$\int x_{l+1} \varphi(x_1, \dots, x_l) d\gamma = \int x_l \varphi(x_1, \dots, x_l) d\gamma;$$

this leads us to consider the family of functions

$$(5) \quad \mathcal{F}^{(m)} = \{ (p_{l+1} - p_l) (\varphi \circ p_{\{1, \dots, l\}}) : \varphi \in C_b(\mathbb{R}^l), l = 1, \dots, n-1 \}.$$

The martingale condition corresponds to asserting that $\int f d\gamma = 0$ for all $f \in \mathcal{F}^{(m)}$. In model-independent finance one typically assumes that additional information is given from market-data which again corresponds to asserting that $\int f d\gamma = 0$ for functions f in some family of functions \mathcal{H} .

The principle problem in model independent finance is then precisely the optimization problem (A) for $\mathcal{F}_4 = \mathcal{F}^{(m)} \cup \mathcal{H}$.

We list some particular choices for \mathcal{H} which have received particular interest: the instance $\mathcal{H} = \emptyset$ is not relevant for mathematical finance but more so in probability through its connection to martingale inequalities: we refer to [ABP⁺13, BS13, BN13, BN14] for recent developments in this direction.

²Among other conditions, Pass assumes that the quantile functions satisfy a property of uniform Riemann-integrability which may be difficult to verify.

³I.e. q_t is the generalized inverse of the cumulative distribution function of μ_t : $q_t(x) = \inf\{y : \mu_t((-\infty, y]) \geq x\}$

A notable result of Bouchard and Nutz [BN13] is that *every* martingale inequality in finite discrete time can be derived from a “dual”, elementary and deterministic inequality.

Provided that European call options on the underlying stock are liquidly traded, it is a reasonable mathematical idealization to assume that the marginal distribution of the stock price at a particular time instance is known from market data. In the mathematical finance literature the case where the marginal distribution at the terminal time t_n is given has received particular attention.⁴ In the present context this corresponds to asserting that $p_n(\gamma) = \mu$ for some probability μ , i.e. specifying

$$(6) \quad \mathcal{H} = \{\varphi \circ p_n - \int \varphi d\mu : \varphi \in C_b(R)\}.$$

More recently also the case where all intermediate marginals are assumed to be given has been considered under the name of martingale optimal transport; this corresponds to $\mathcal{H} = \mathcal{F}_2$ (where $X_1 = \dots = X_n = \mathbb{R}$).

2.5. A variational principle for martingale optimal transport. Having generalized the optimization problem, we need to adapt the optimality criterion. Our motivation stems from a result of [BJ13] which we discuss subsequently. We first recall the definition of *c-cyclical monotonicity*: a set $\Gamma \subseteq X \times Y$ is called *c-cyclical monotone* if for $(x_1, y_1), \dots, (x_l, y_l) \in \Gamma$, one always has, setting $y_{l+1} = y_1$,

$$\sum_{i=1}^l c(x_i, y_i) \leq \sum_{i=1}^l c(x_i, y_{i+1}).$$

A transport plan is called *c-cyclically monotone* if it is concentrated on a *c-cyclically monotone* set. Note that an equivalent way (cf. [Vil03, Exercise 2.21]) of stating cyclical monotonicity of Γ is: for each finite measure α concentrated on finitely many elements of Γ one has

$$\int c d\alpha \leq \int c d\alpha'$$

whenever α' has the same marginals as α .

In [BJ13] this notion was adapted for the martingale transport problem⁵ by adding a martingale component: for a measure α on \mathbb{R}^2 , a measure α' is called a *competitor* if

- (1) α and α' have the same marginals, and
- (2) $\int x_2 d\alpha_{x_1}(x_2) = \int x_2 d\alpha'_{x_1}(x_2)$ holds $p_1(\alpha)$ -almost surely (i.e., the difference $\alpha - \alpha'$ has the martingale property).

Using this, a set $\Gamma \subseteq \mathbb{R}^2$ is called *finitely optimal* if for each finite measure α concentrated on finitely many elements of Γ , one has $\int c d\alpha \leq \int c d\alpha'$, for each competitor α' of α .

The condition (2) in the definition of a competitor can be replaced by the following equivalent condition: for each bounded Borel-measurable f ,

⁴This case is naturally connected to the Skorokhod embedding problem, we refer to survey of Obłój [Obł04].

⁵The article [BJ13] is concerned with the case $E = \mathbb{R}^2$ where the minimization is taken over all transport plans which are martingale measures, i.e. the setup described in last part of Section 2.4, resp. $\mathcal{F} = \mathcal{F}^{(m)} \cup \mathcal{F}_2$ in the optimization problem (A).

we have

$$\int (x_2 - x_1) f(x_1) d\alpha(x_1, x_2) = \int (x_2 - x_1) f(x_1) d\alpha'(x_1, x_2).$$

However, this condition, together with the condition of equal marginals, can be written in short as

$$\int f d\alpha = \int f d\alpha'$$

for all $f \in \mathcal{F}^{(m)} \cup \mathcal{F}_2$, as above.

In [BJ13] it is shown that optimality implies finitistic optimality provided that c satisfies certain moment conditions and that the converse holds provided that c is continuous and bounded.

2.6. A general concept of finitistic optimality and main result. The above discussion leads us to the following definition:

Definition 2.2. For a measure α on the Polish space E and a set \mathcal{F} of measurable functions $E \rightarrow \mathbb{R}$, a competitor of α is a measure α' on E such that $\alpha(E) = \alpha'(E)$, and for all $f \in \mathcal{F}$ one has

$$(7) \quad \int f d\alpha = \int f d\alpha'.$$

If in addition

$$(8) \quad \int c d\alpha' < \int c d\alpha,$$

then α' is called a c -better competitor of α .

A set $\Gamma \subseteq E$ is called *finitely minimal / c -monotone* if no finite measure α concentrated on finitely many points in Γ has a c -better competitor. A measure γ on E is called *finitely minimal / c -monotone* if it is concentrated on a finitely minimal set.

Our goal is to establish that optimizers of the problem (A) are finitely minimal. To this end we require the following assumption on the family \mathcal{F} :

Assumption 2.3. (1) *There exists a function $g : E \rightarrow [0, \infty)$ such that each element of \mathcal{F} is bounded by some multiple of g . I.e. for each $f \in \mathcal{F}$ there is a constant $a_f \in \mathbb{R}_+$ such that $|f| \leq a_f g$.*

(2) *Either all functions in \mathcal{F} are continuous or \mathcal{F} is at most countable.*

Notably these properties are satisfied in all examples listed above.

Theorem 2.4. *Let E be a Polish space and $c : E \rightarrow \mathbb{R}$ a Borel measurable function. Let \mathcal{F} be a family of Borel-measurable functions on E satisfying Assumption 2.3 and assume that γ^* is a minimizer of the problem*

$$\min_{\gamma \in \Pi_{\mathcal{F}}} \int c d\gamma$$

and that $\int c d\gamma^ \in \mathbb{R}$. Then γ^* is finitely minimal / c -monotone.*

In applications it is natural to consider continuous or lower semi-continuous cost functions in which case the existence of an optimizer γ^* can often be established by compactness arguments. However this assumption does not simplify our arguments nor does it lead to a more specific result. We have therefore chosen to go with the general formulation above.

In the case of classical optimal transport one obtains a nicer result for the (most relevant) case where c is continuous: if γ is an optimal transport plan then $\text{supp } \gamma$ is c -cyclically monotone. It is natural to ask whether this

stronger assertion is also true in our setup. This is not the case: Juillet [Jui14] has found an example of a two-period martingale transport problem where the measures μ, ν are compactly supported and the cost function $c(x, y) = (y - x)^3$ is continuous but the support of the unique minimizer is not finitely optimal.

2.7. Connection with [Zae14]. The recent article [Zae14] is concerned with the multi-marginal transport problem described in Section 2.2 but allows for additional linear constraints. In our notation this corresponds to problem (A) on a set E which is a product $X_1 \times \dots \times X_n$ of Polish probability spaces and where \mathcal{F} is a superset of the set \mathcal{F}_2 defined in (6); several important extensions of the transport problem can be phrased in this form. Under continuity and (weak) integrability assumptions Zaev establishes the existence of an optimizer, a version of the classical Monge-Kantorovich duality as well as a necessary geometric condition for optimizers. The latter statement is equivalent to the assertion of Theorem 2.4 (applied to the setup of [Zae14]). Notably the proof given in [Zae14] is based on the duality result and different from the approach pursued in this article.

3. PROOF OF THEOREM 2.4

In the proof of Theorem 2.4 we will make use of the following result from [BGMS09], which is a consequence of a duality result by Kellerer [Kel84]:

Lemma 3.1 ([BGMS09, Proposition 2.1]). *Let (E, m) be a Polish probability space, and M an analytic⁶ subset of E^l , then one of the following holds true:*

- (i) *there exist m -null sets $M_1, \dots, M_l \subseteq E$ such that $M \subseteq \bigcup_{i=1}^l p_i^{-1}(M_i)$,*
or
- (ii) *there is a measure η on E^l such that $\eta(M) > 0$ and $p_i(\eta) \leq m$ for $i = 1, \dots, l$.*

Proof of Theorem 2.4. Without loss of generality we assume that $|c| \leq g$. We want to find a finitely minimal set Γ with $\gamma^*(\Gamma) = 1$. To obtain this, it is sufficient to show that for each $l \in \mathbb{N}$ there is a set Γ_l with $\gamma^*(\Gamma_l) = 1$ such that there is no c -better competitor α' with $\alpha'(E) \leq 1$, $\int g d\alpha' \leq l$ for any finite measure α concentrated on at most l points in Γ_l and satisfying $\alpha'(E) \leq 1, \int g d\alpha \leq l$. For then $\Gamma := \bigcap_{l \in \mathbb{N}} \Gamma_l$ is finitely minimal.

Hence, fix l and define a subset of E^l ,

$$M := \{(z_1, \dots, z_l) \in E^l :$$

$$\exists \text{ a measure } \alpha \text{ on } E, \alpha(E) \leq 1, \int g d\alpha \leq l, \text{supp } \alpha \subseteq \{z_1, \dots, z_l\},$$

$$\text{s.t. there is a } c\text{-better competitor } \alpha', \alpha'(E) \leq 1, \int g d\alpha' \leq l, |\text{supp } \alpha| \leq l\}.$$

Note that M is the projection of the set

$$\begin{aligned} \hat{M} = & \left\{ (z_1, \dots, z_l, \alpha_1, \dots, \alpha_l, z'_1, \dots, z'_l, \alpha'_1, \dots, \alpha'_l) \in E^l \times \mathbb{R}_+^l \times E^l \times \mathbb{R}_+^l : \right. \\ & \sum \alpha_i \leq 1, \sum \alpha_i g(z_i) \leq l, \sum \alpha'_i \leq 1, \sum \alpha'_i g(z'_i) \leq l, \sum \alpha_i = \sum \alpha'_i, \\ & \left. \sum \alpha_i f(z_i) = \sum \alpha'_i f(z'_i) \text{ for all } f \in \mathcal{F}, \sum \alpha_i c(z_i) > \sum \alpha'_i c(z'_i) \right\}. \end{aligned}$$

onto the first l coordinates. By our Assumption 2.3, the set \hat{M} is Borel, hence M is analytic.

⁶[BGMS09, Proposition 2.1] is stated only for Borel sets, however the same proof applies in the case where M is analytic.

We apply Lemma 3.1 to the space (E, γ^*) : if (i) holds, then define $N := \bigcup_{i=1}^l M_i$. Then $\Gamma_l := E \setminus N$ has full measure, $\gamma^*(\Gamma_l) = 1$. From the definitions of M and N it can be directly seen that Γ_l is as needed.

If (i) does not hold, (ii) has to. Hence, let us derive a contradiction from it.

Write p_i for the projection of an element of E^l onto its i -th component. We may assume that the measure η in (ii) is concentrated on M , and also fulfills $p_i(\eta) \leq \frac{1}{l}\gamma^*$ for $i = 1, \dots, l$.

We now apply the Jankow – von Neumann selection theorem to the set \hat{M} to define a Borel measurable mapping

$$z \mapsto (\alpha_1(z), \dots, \alpha_l(z), z'_1(z), \dots, z'_l(z), \alpha'_1(z), \dots, \alpha'_l(z))$$

such that

$$(z, \alpha_1(z), \dots, \alpha_l(z), z'_1(z), \dots, z'_l(z), \alpha'_1(z), \dots, \alpha'_l(z)) \in \hat{M}$$

for $z \in M$. Setting

$$\alpha_z := \sum_i \alpha_i(z) \delta_{z_i}, \alpha'_z := \sum_i \alpha'_i(z) \delta_{z'_i(z)}$$

we thus obtain kernels $z \mapsto \alpha_z, z \mapsto \alpha'_z$. We use these to define measures ω, ω' through

$$\omega(B) = \int \alpha_z(B) d\eta(z), \omega'(B) = \int \alpha'_z(B) d\eta(z).$$

By construction $\omega \leq \gamma^*$. Moreover ω' is a c -better competitor of ω : for each $f \in \mathcal{F}$ we have

$$\int f d\omega' = \iint f d\alpha'_z d\eta(z) = \iint f d\alpha_z d\eta(z) = \int f d\omega.$$

Note that the first and last equality are justified since $\int g d\alpha_z, \int g d\alpha'_z \leq l$ for all z . Similarly, since $|c| \leq g$, we obtain

$$\int c d\omega' = \iint c d\alpha'_z d\eta(z) < \iint c d\alpha_z d\eta(z) = \int c d\omega.$$

Summing up, we obtain a probability measure $\gamma' := \gamma^* - \omega + \omega'$ that fulfills $\int c d\gamma' < \int c d\gamma^*$ and $\gamma' \in \Pi_{\mathcal{F}}$. \square

4. THE CONTINUUM MARGINAL TRANSPORT PROBLEM REVISITED

This section is devoted to establishing Theorem 2.1. In order to simplify notations w.l.o.g. we work with $I = [0, T] = [0, 1]$ from now on.

Pass' result from [Pas13a] had a predecessor in an earlier paper by Carlier [Car03], who dealt with the following Monge-type problem: given a cost function g on \mathbb{R}^{n+1} which is continuous and strictly monotone of order 2, minimize

$$(C) \quad s \mapsto \int g(t, s(t)) d\mu_0,$$

where s runs through the Borel-functions $\mathbb{R} \rightarrow \mathbb{R}^n$ with $s_i(\mu_0) = \mu_i$ for $i = 1, \dots, n$, and μ_0, \dots, μ_n given. Under regularity assumptions and assuming that μ_0 does not charge points, Carlier used duality methods to demonstrate that there is a maximizer s which is unique μ_0 -almost surely, and which has nondecreasing components s_i . The finite-dimensional version of problem (B) does fall into the setting of (C), as for a strictly concave function $h : \mathbb{R} \rightarrow \mathbb{R}$,

the function $x \mapsto h(x_1 + x_2 + \dots + x_n)$ is continuous and strictly monotone of order 2. It is not hard to see how to obtain the analogous solution to the finite dimensional variant of (B) from Carlier's result; for a precise statement see Theorem 4.1 below. We introduce the notation π_n^* for the n -dimensional analogue of the measure π_n^* , that is given n probability measures μ_1, \dots, μ_n on \mathbb{R} , π_n^* is the measure uniformly distributed on the quantile functions q_i of μ_i . Then we can state:

Theorem 4.1. *Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be strictly concave and μ_1, \dots, μ_n be probability measures on \mathbb{R} such that*

$$\int |x| d\mu_i < \infty, \quad \int |h| d\mu_i < \infty, \quad \text{for } 1 \leq i \leq n.$$

Then π_n^ is the unique minimizer of*

$$(9) \quad \inf_{\gamma \in \Pi_n(\mu_1, \dots, \mu_n)} \int h(x_1 + \dots + x_n) d\gamma(x).$$

It is clear why the variational principle should come in useful for results as in Theorems 2.1 and 4.1. For in these situations, finite optimality of a set A (in \mathbb{R}^n or $C[0, 1]$, respectively) implies that \leq must be an order on A , i.e. if f and g are both in A , then either $f \leq g$ or $g \leq f$. Else, set $f' = \max\{f, g\}$ and $g' = \min\{f, g\}$, and let α be the measure $\frac{1}{2}\delta_f + \frac{1}{2}\delta_g$ and α' the measure $\frac{1}{2}\delta_{f'} + \frac{1}{2}\delta_{g'}$. It is clear that α' is a measure with the same marginals as α (on \mathbb{R}^n , or $C[0, 1]$, respectively). But due to the strict concavity of h , it is easy to see in both cases (C) and (B) that α' is a better competitor to α , contradicting the definition of local optimality. We call a set on which \leq is a total order a monotone set. The argument of optimality of π_n^* (π^* , respectively) is then completed by the following intuitive lemma. A proof of the first part can be found e.g. in [RR98, p 108] or [AGS08, 134]. The second part is a simple consequence since the distribution of a continuous process is determined by its finite dimensional marginal distributions.

Lemma 4.2. *Let γ be a probability measure on \mathbb{R}^n with marginals μ_1, \dots, μ_n . If there is a monotone Borel set M with $\mu(M) = 1$, then $\gamma = \pi_n^*$. Let γ be a probability measure on $C[0, 1]$ with marginals $(\mu_t)_{t \in I}$. If there is a monotone Borel set M with $\gamma(M) = 1$, then $\gamma = \pi^*$.*

Proof of Theorem 4.1. The set $\Pi(\mu_{t_1}, \dots, \mu_{t_n})$ is weakly compact. Due to the assumptions on first moments and h -moments of the marginal measures μ_i , the operator to be minimized is lower semi-continuous and bounded. Hence there is a finite minimizer. Strict concavity of h and an application of the variational principle yield that each finite minimizer must be concentrated on a finitely minimal, hence monotone set. By the preceding lemma, each minimizer must be equal to π_n^* . \square

Now we turn to proving Theorem 2.1: unfortunately, the nice and neat argument for Theorem 4.1 breaks down as $\Pi_C(\mu_t)$ need not be compact, as easy counterexamples show. Hence we have to find a way first to establish the existence of an optimizer at all. Here is how we want to proceed: we will solve a problem for a countable index set as an intermediate step, and use the result in the proof of Theorem 2.1 at the end of this section. Writing $Q = [0, 1] \cap \mathbb{Q}$, we define $\Pi_Q(\mu_q)$ as the set of probability measures on \mathbb{R}^Q with marginals $(\mu_q)_{q \in Q}$. Furthermore, we fix a sequence of finite partitions

(\mathcal{P}_n) of $[0, 1]$ with $\mathcal{P}_n \subseteq \mathcal{P}_{n+1} \subseteq Q$ and $\bigcup_n \mathcal{P}_n = Q$. We then replace the original problem (B) by

$$(B') \quad \inf_{\gamma \in \Pi_Q(\mu_q)} \int h \left(\liminf_{n \rightarrow \infty} \sum_{t_i \in \mathcal{P}_n} f_{t_i}(t_i - t_{i-1}) \right) d\gamma(f).$$

Writing π_Q^* for the Q -analogue of π^* and assuming monotonicity and boundedness of h for the time being, we claim:

Proposition 4.3. *Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be concave, decreasing, and non-positive. Provided that $\int |x| d\mu_q(x) < \infty$, $\int |h| d\mu_q < \infty$ for all $q \in Q$, the measure π_Q^* is a minimizer of Problem (B').*

The proof is preceded by Lemmas 4.4, 4.5, and 4.6. The assumptions here on h and the marginals are always as in Theorem 4.3.

Lemma 4.4. $\Pi_Q(\mu_q)$ is weakly compact.

Proof. By Prochorov's theorem: let $\varepsilon > 0$ be arbitrary. Then, with $Q = \{q_1, q_2, \dots\}$, for each q_k there exists a compact set $K_k \subseteq \mathbb{R}$ with $\mu_{q_k}(K_k) > 1 - \frac{\varepsilon}{2^k}$. The set $K = \prod_{k=1}^{\infty} K_k$ is a compact subset of \mathbb{R}^Q . For a measure $\gamma \in \Pi_Q(\mu_q)$ we have

$$\gamma(K) = \lim_{n \rightarrow \infty} \gamma(p_{q_1, q_2, \dots, q_n}^{-1}(K_1 \times K_2 \times \dots \times K_n)).$$

As for each n

$$\gamma(p_{q_1, q_2, \dots, q_n}^{-1}(K_1 \times K_2 \times \dots \times K_n)) > 1 - \sum_{k=1}^n \frac{\varepsilon}{2^k} \geq 1 - \varepsilon$$

we have $\gamma(K) \geq 1 - \varepsilon$, and Prochorov's theorem can be applied. \square

We introduce some notation:

$$\begin{aligned} s_n &: \mathbb{R}^Q \rightarrow \mathbb{R}, & f &\mapsto \sum_{t_i \in \mathcal{P}_n} f_{t_i}(t_i - t_{i-1}) \\ s_n^{(h)} &: \mathbb{R}^Q \rightarrow \mathbb{R}, & f &\mapsto \sum_{t_i \in \mathcal{P}_n} h(f_{t_i})(t_i - t_{i-1}), \\ \varphi_n &: \mathbb{R}^Q \rightarrow \mathbb{R} \cup \{-\infty\}, & f &\mapsto \inf_{k \geq n} s_k(f), \\ \varphi &: \mathbb{R}^Q \rightarrow \mathbb{R} \cup \{-\infty, \infty\}, & f &\mapsto \sup_n \varphi_n(f) = \liminf_n s_n(f). \end{aligned}$$

We continue with

Lemma 4.5. *For each n , the operators defined on $\Pi_Q(\mu_q)$,*

$$S_n : \gamma \mapsto \int h \circ s_n d\gamma$$

and

$$\Phi_n : \gamma \mapsto \int h \circ \varphi_n d\gamma$$

are lower-semi-continuous (w.r.t. weak convergence) and have minimizers. The values of the minima are finite.

Proof. The existence of minimizers will follow from lower-semi-continuity of the operators and compactness of $\Pi_Q(\mu_q)$. Hence, let $(\gamma_l)_{l \in \mathbb{N}}$ be a sequence in $\Pi_Q(\mu_q)$ converging weakly to some γ_0 .

We have

$$\varphi_n \leq s_n$$

and hence, by monotonicity and concavity of h that

$$h \circ \varphi_n \geq h \circ s_n \geq s_n^{(h)}.$$

For each $\gamma \in \Pi_Q(\mu_q)$ we have

$$\int s_n^{(h)} d\gamma = \sum_{t_i \in \mathcal{P}_n} (t_i - t_{i-1}) \int h(f_{t_i}) d\gamma(f) = \sum_{t_i \in \mathcal{P}_n} (t_i - t_{i-1}) \int h d\mu_{t_i}.$$

Hence in particular

$$\lim_{l \rightarrow \infty} \int s_n^{(h)} d\gamma_l = \int s_n^{(h)} d\gamma_0.$$

As $s_n^{(h)}$ is continuous, the prerequisites of Lemma 4.3. in [Vil09] are met for both S_n and Φ_n , and applying that result we get

$$\liminf_{l \rightarrow \infty} S_n(\gamma_l) \geq S_n(\gamma_0)$$

and

$$\liminf_{l \rightarrow \infty} \Phi_n(\gamma_l) \geq \Phi_n(\gamma_0).$$

Finally, the finiteness of the minimal values follows from h being bounded from above, the assumption on finite h -moments of the marginals, and $h \circ \varphi_n \geq h \circ s_n \geq s_n^{(h)}$. \square

Lemma 4.6. *For each $n \in \mathbb{N}$, the measure π_Q^* minimizes Φ_n on $\Pi_Q(\mu_q)$.*

Proof. We first show that in the case when h is strictly concave, the following stronger assertion is true: π_Q^* is the unique measure in $\Pi_Q(\mu_q)$ doing the following:

- (0) it minimizes Φ_n ,
- (1) among the minimizers of Φ_n it minimizes S_1 ,
- (2) among the measures fulfilling (0) and (1), it minimizes S_2 ,
- ...
- (k) among the measures fulfilling (0), (1), \dots , ($k-1$), it minimizes S_k
- ...

We show existence of a measure fulfilling all the conditions (0), (1), \dots : write K_0 for the set of minimizers of Φ_n . By the previous lemma, $K_0 \neq \emptyset$. Also, K_0 is compact: for it is a closed subset of the compact set $\Pi_Q(\mu_q)$, where closedness is due to the semi-continuity of Φ_n . Hence, among the minimizers of Φ , there is a minimizer of the lower-semi-continuous operator S_1 . Writing K_1 for the set of these minimizers, by the same argument as above, K_1 is nonempty and compact. Hence, the set K_2 of minimizers of S_2 on K_1 is nonempty and again compact. By induction we obtain a decreasing sequence of compact nonempty sets K_k . Hence the set $K = \bigcap_k K_k$ is nonempty and each of its elements fulfills properties (0), (1), \dots . Pick such an element and denote it by π_0 . We now apply the variational principle to show that π_0 must indeed be equal to π_Q^* : π_0 is concentrated on a set Γ that is locally optimal for each of the problems (k). Observe first that local optimality of Γ for problem (0) alone does not need to imply that Γ is monotone.⁷ However, local optimality of Γ for problem (1) - i.e. the optimization of

⁷What local optimality does imply is the following: if f, g are in Γ , and $\varphi_n(f) > \varphi_n(g)$, then one must have $\varphi_n((f-g)^+) = 0$. This is a weaker condition than \leq being an order

S_1 on the set K_0 - does imply that Γ must be monotone on \mathcal{P}_1 , that is, if $f, g \in \Gamma$, then either $f|_{\mathcal{P}_1} \leq g|_{\mathcal{P}_1}$ or $f|_{\mathcal{P}_1} \geq g|_{\mathcal{P}_1}$. For if there were f, g not ordered on \mathcal{P}_1 , then write $f' = \mathbf{1}_{\mathcal{P}_1} \max(f, g) + \mathbf{1}_{\mathcal{P}_1^c} f$ and $g' = \mathbf{1}_{\mathcal{P}_1} \min(f, g) + \mathbf{1}_{\mathcal{P}_1^c} g$. Set $\alpha = \frac{1}{2}\delta_f + \frac{1}{2}\delta_g$ and $\alpha' = \frac{1}{2}\delta_{f'} + \frac{1}{2}\delta_{g'}$, where δ_f denotes the Dirac-measure on f , etc. Then α' is apparently S_1 -better than α , but it is also a competitor of it: it clearly has the same marginals, and we have $\varphi_n(f') = \varphi_n(f)$ and $\varphi_n(g') = \varphi_n(g)$, as manipulating a function $f \in \mathbb{R}^Q$ on finitely many points does not change the value of φ_n . Hence, also $\Phi_n(\alpha') = \int h \circ \varphi_n d\alpha' = \int h \circ \varphi_n d\alpha = \Phi_n(\alpha)$. The existence of an S_1 -better competitor is a contradiction to local optimality, so Γ must indeed be monotone on \mathcal{P}_1 . Now for problem (2), we also find that Γ must be monotone on \mathcal{P}_2 : let $f, g \in \Gamma$, and assume, due to monotonicity of Γ on \mathcal{P}_1 , that $f|_{\mathcal{P}_1} \geq g|_{\mathcal{P}_1}$. If f and g were not ordered on \mathcal{P}_2 , then the same construction of f', g', α and α' as above (with \mathcal{P}_2 in place of \mathcal{P}_1) will give a contradiction to local optimality: note that $s_1(f') = s_1(f)$ and $s_1(g') = s_1(g)$, as $f' = f$ and $g' = g$ on \mathcal{P}_1 . Hence, $\Phi_n(\alpha') = \int h \circ \varphi_n d\alpha' = \int h \circ \varphi_n d\alpha = \Phi_n(\alpha)$, $S_1(\alpha') = \int h \circ s_1 d\alpha' = \int h \circ s_1 d\alpha = S_1(\alpha)$, and α' is really a competitor of α .

Iterating this argument one finds that Γ must indeed be monotone on each \mathcal{P}_k , and henceforth monotone. But then π_0 must be π_Q^* , because π_Q^* is the only measure in $\Pi_Q(\mu_q)$ concentrated on a monotone set. This last statement follows easily from Lemma 4.2.

Finally, we discuss the case where h is concave, but not necessarily strictly concave. Then, due to the finiteness of $\int |h| d\mu_q$ for all $q \in Q$, there is, for each $k \in \mathbb{N}$, a strictly concave function h_k such that $\int |h_k| d\mu_q < \infty$ for all $q \in \mathcal{P}_k$. Then by adapting the above argument, it is easy to see that π_Q^* is the only measure in $\Pi_Q(\mu_q)$ that

- (0) minimizes Φ_n
- (1') among the minimizers of Φ_n , it minimizes $\int h_1(s_1) d\gamma$,
- ...
- (k') among the measures fulfilling (0), (1'), ..., (k-1'), it minimizes $\int h_k(s_k) d\gamma$,
- ...

Proof of Proposition 4.3. Let γ be a measure in $\Pi_Q(\mu_q)$. Then for each n , according to the previous lemma

$$\int h \circ \varphi_n d\gamma \geq \int h \circ \varphi_n d\pi_Q^*.$$

As h is decreasing and non-positive, and φ_n increases to $\varphi = \liminf_n \varphi_n$, an application of monotone convergence finishes the proof. \square

Finally we can proof Theorem 2.1:

Proof of Theorem 2.1. First, note that due to the regularity assumption of $\int_0^1 \int |x| d\mu_t dt < \infty$, it is w.l.o.g to assume that h is non-positive. If we further assume for the time being that h is decreasing, we can apply Proposition 4.3 to see the optimality of π^* as follows: let p_Q be the projection $\mathbb{R}^I \rightarrow \mathbb{R}^Q$, and write p for its restriction on $C[0, 1]$. It is easy to see that p is a Borel isomorphism from $C[0, 1]$ onto \mathbb{R}_c^Q , the set of all elements of

on Γ , and explains why one works with the sequence of problems (k) rather than just with problem (0).

\mathbb{R}^Q that are restrictions of continuous functions on $[0, 1]$. For an arbitrary $\gamma \in \Pi_C(\mu_t)$, the measure $p(\gamma)$ is in $\Pi_Q(\mu_q)$ and clearly

$$\int h\left(\int_0^1 f dt\right) d\gamma = \int h\left(\liminf_{n \rightarrow \infty} \sum_{t_i \in \mathcal{P}_n} f_{t_i}(t_i - t_{i-1})\right) dp(\gamma).$$

But for the right-hand-side one also has, due to Theorem 4.3,

$$\int h\left(\liminf_n \sum_{t_i \in \mathcal{P}_n} f_{t_i}(t_i - t_{i-1})\right) dp(\gamma) \geq \int h\left(\liminf_n \sum_{t_i \in \mathcal{P}_n} f_{t_i}(t_i - t_{i-1})\right) d\pi_Q^*.$$

As the right-hand-side of this equals $\int h\left(\int_0^1 f dt\right) d\pi^*$ we have

$$\int h\left(\int_0^1 f dt\right) d\gamma \geq \int h\left(\int_0^1 f dt\right) d\pi^*.$$

If h is not decreasing, then assume first it is increasing. If in problem (B') we replace \liminf by \limsup one can show, with the proof of Theorem 4.3 and the above argument suitably adapted, that π^* must be again optimal. Finally, if h is neither increasing nor decreasing, then it can still be written as a sum $h_1 + h_2$, where h_1 is concave, increasing and non-positive, and h_2 is concave, decreasing and non-positive, and again π^* is an optimizer. (h_1 and h_2 will satisfy the regularity assumptions as long as h does.)

If the minimum is finite and h is strictly concave, each other minimizer must be concentrated on a finitely minimal, hence monotone set and thus be equal to π^* . \square

REFERENCES

- [ABP⁺13] B. Acciaio, M. Beiglöck, F. Penkner, W. Schachermayer, and J. Temme. A trajectorial interpretation of doob's martingale inequalities. *The Annals of Applied Probability*, 23(4):1494–1505, 2013.
- [ABPS13] B. Acciaio, M. Beiglöck, F. Penkner, and W. Schachermayer. A model-free version of the fundamental theorem of asset pricing and the super-replication theorem. *Mathematical Finance*, to appear, 2013.
- [AGS08] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, second edition, 2008.
- [AP03] L. Ambrosio and A. Pratelli. Existence and stability results in the L^1 theory of optimal transportation. In *Optimal transportation and applications (Martina Franca, 2001)*, volume 1813 of *Lecture Notes in Math.*, pages 123–160. Springer, Berlin, 2003.
- [BGMS09] M. Beiglöck, M. Goldstern, G. Maresch, and W. Schachermayer. Optimal and better transport plans. *J. Funct. Anal.*, 256(6):1907–1927, 2009.
- [BHLP13] M. Beiglöck, P. Henry-Labordère, and F. Penkner. Model-independent bounds for option prices: A mass transport approach. *Finance Stoch.*, 17(3):477–501, 2013.
- [BJ13] M. Beiglöck and N. Juillet. On a problem of optimal transport under marginal martingale constraints. *in revision*, 2013.
- [BN13] B. Bouchard and M. Nutz. Arbitrage and Duality in Nondominated Discrete-Time Models. *Ann. Appl. Probab.*, to appear, May 2013.
- [BN14] M. Beiglöck and M. Nutz. Martingale inequalities and deterministic counterparts. *arXiv preprint arXiv:1401.4698*, 2014.
- [BS13] M. Beiglöck and P. Siorpaes. Pathwise versions of the Burkholder-Davis-Gundy-inequality. *Bernoulli*, to appear, 2013.
- [Car03] G. Carlier. On a class of multidimensional optimal transportation problems. *Journal of Convex Analysis*, 10(2):517–529, 2003.
- [DS13a] Y. Dolinsky and M. H. Soner. Robust Hedging and Martingale Optimal Transport in Continuous Time. *Probab. Theory Relat. Fields*, to appear, 2013.

- [DS13b] Y. Dolinsky and M. H. Soner. Robust Hedging with Proportional Transaction Costs. *Finance Stoch. (to appear)*, February 2013.
- [GHLT14] A. Galichon, P. Henry-Labordère, and N. Touzi. A stochastic control approach to no-arbitrage bounds given marginals, with an application to lookback options. *The Annals of Applied Probability*, 24(1):312–336, 2014.
- [GM96] W. Gangbo and R. McCann. The geometry of optimal transportation. *Acta Math.*, 177(2):113–161, 1996.
- [Hob11] D. Hobson. The Skorokhod embedding problem and model-independent bounds for option prices. volume 2003 of *Lecture Notes in Math.*, pages 267–318. Springer, Berlin, 2011.
- [HT13] P. Henry-Labordere and N. Touzi. An Explicit Martingale Version of Brenier’s Theorem. *ArXiv e-prints*, February 2013.
- [Jui14] N. Juillet. Stability of the shadow projection and the left-curtain coupling. *preprint*, 2014.
- [Kel84] H.G. Kellerer. Duality theorems for marginal problems. *Z. Wahrsch. Verw. Gebiete*, 67(4):399–432, 1984.
- [KP13] Y.-H. Kim and B. Pass. A general condition for Monge solutions in the multi-marginal optimal transport problem. *SIAM J. Math. Anal.*, to appear, July 2013.
- [Obł04] J. Obłój. The Skorokhod embedding problem and its offspring. *Probab. Surv.*, 1:321–390, 2004.
- [Pas11] B. Pass. Uniqueness and Monge solutions in the multimarginal optimal transportation problem. *SIAM J. Math. Anal.*, 43(6):2758–2775, 2011.
- [Pas12] B. Pass. On the local structure of optimal measures in the multi-marginal optimal transportation problem. *Calc. Var. Partial Differential Equations*, 43(3-4):529–536, 2012.
- [Pas13a] B. Pass. On a class of optimal transportation problems with infinitely many marginals. *SIAM J. Math. Anal.*, 45:2557–2575, 2013.
- [Pas13b] B. Pass. Optimal transportation with infinitely many marginals. *J. Funct. Anal.*, 264(4):947–963, 2013.
- [Pra08] A. Pratelli. On the sufficiency of c -cyclical monotonicity for optimality of transport plans. *Math. Z.*, 258(3):677–690, 2008.
- [RR98] Svetlozar T. Rachev and Ludger Rüschendorf. *Mass transportation problems. Vol. I. Probability and its Applications* (New York). Springer-Verlag, New York, 1998.
- [ST09] W. Schachermayer and J. Teichmann. Characterization of optimal transport plans for the Monge-Kantorovich problem. *Proc. Amer. Math. Soc.*, 137(2):519–529, 2009.
- [Vil03] C. Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.
- [Vil09] C. Villani. *Optimal Transport. Old and New*, volume 338 of *Grundlehren der mathematischen Wissenschaften*. Springer, 2009.
- [Zae14] D. ZaeV. On the Monge-Kantorovich problem with additional linear constraints. *ArXiv e-prints*, April 2014.