**Math Finance (cont. time), SS19, Sheet 4**

1. Let $B$ be a Brownian motion wrt. a filtration $(\mathcal{F}_t)_{t\geq 0}$. Show that the process $(B_t^2 - t)_{t\geq 0}$ is a martingale.

2. Let $X = M + A$, where $M$ is a continuous martingale and $A$ is a continuous process which has finite variation, say $TV(A)_t$ is bounded for every $t$. Show that $\langle X \rangle = \langle M \rangle$.

3. Assume that $X, X_1, X_2$ are semi-martingales and that $H, H_1, H_2$ continuous adapted processes, $\alpha_1, \alpha_2 \in \mathbb{R}$. Show that the stochastic integral is linear, i.e.

$$\int_0^T \alpha_1 H_1 + \alpha_2 H_2 \, dX = \alpha_1 \int_0^T H_1 \, dX + \alpha_2 \int_0^T H_2 \, dX \tag{1}$$

$$\int_0^T H \, d(\alpha_1 X_1 + \alpha_2 X_2) = \alpha_1 \int_0^T H \, dX_1 + \alpha_2 \int_0^T H \, dX_2 \tag{2}$$

4. A process $H = (H_t)_{t\in[0,T]}$ is called simple if

$$H_t(\omega) = \sum_{i=1}^n H_i I_{(s_i, s_{i+1}]},$$

where $0 \leq s_1 \leq \ldots \leq s_n$ and each $H_i$ is $\mathcal{F}_{s_i}$-measurable and bounded. Let $X$ be a continuous adapted process.

For simple integrands we set

$$I_t := \int_0^u H_t \, dX_t := \sum_{i=1}^n H_i (X_{s_{i+1}\wedge u} - X_{s_i \wedge u}).$$

Show that that for all $t \geq 0$ and all sequences $(\mathcal{D}_n)_{n\in\mathbb{N}}$ of partitions of $[0, t]$, i.e.

$$\mathcal{D}_n := \{0 = t_0^n < t_1^n < \cdots < t_{n_k}^n = t\}$$

with $|\mathcal{D}_n| := \sup_{i\in\{1,\ldots,k_n\}} |t_i^n - t_{i-1}^n| \to 0$ as $n \to \infty$ it holds

$$I_t = \lim_{n\to\infty} \sum_{i=1}^{k_n} H_{t_i^n} (X_{t_i^n} - X_{t_{i-1}^n})$$

in probability.