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A VARIANT OF THE HALES-JEWETT THEOREM

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Abstract

It was shown by V. Bergelson that any set $B \subseteq \mathbb{N}$ with positive upper multiplicative density contains nicely intertwined arithmetic and geometric progressions: For each $k \in \mathbb{N}$ there exist $a, b, d \in \mathbb{N}$ such that $\{b(a+id)^j : i, j \in \{1, 2, \ldots, k\}\} \subseteq B$. In particular one cell of each finite partition of $\mathbb{N}$ contains such configurations. We prove a Hales-Jewett type extension of this partition theorem.

1. *Introduction*

Van der Waerden's Theorem ([**15**]) states that for any finite coloring of $\mathbb{N}$ one can find arbitrarily long monochromatic arithmetic progressions. In 1963 A. Hales and R. Jewett ([**10**]) gave a powerful abstract extension of van der Waerden's Theorem.

We introduce some notations to state their result. An *alphabet* Σ is a finite nonempty set. A *located word* α is a function from a finite set $dom\,\alpha \subseteq \mathbb{N}$ to Σ. The set of all located words will be denoted by $L(\Sigma)$. Note that for located words α, β satisfying $dom\,\alpha \cap dom\,\beta = \emptyset$, $\alpha \cup \beta$ is also located word. (Here it is convenient to view functions as sets of ordered pairs.) By $\mathcal{P}_f(\mathbb{N})$ we denote the set of all finite nonempty subsets of $\mathbb{N}$.

Theorem 1 (Hales-Jewett) *Let $L(\Sigma)$ be finitely coloured. There exist $\alpha \in L(\Sigma)$ and $\gamma \in \mathcal{P}_f(\mathbb{N})$ such that $dom\,\alpha \cap \gamma = \emptyset$ and $\{\alpha \cup \gamma \times \{s\} : s \in \Sigma\}$ is monochrome.*

The term $\alpha \cup \gamma \times \{s\}$ may be viewed as an analogue of the expression $a + c \cdot s$. In particular we use $\alpha \cup \gamma \times \{s\}$ for what should rigorously be $\alpha \cup (\gamma \times \{s\})$. Configurations of the form $\{\alpha \cup \gamma \times \{s\} : s \in \Sigma\}$ are often called *combinatorial lines*.

We will explain shortly how van der Waerden's Theorem can be derived from the Hales-Jewett Theorem. Let $\Sigma = \{0, 1, \ldots, k\}$ and assume that $\mathbb{N}$ is finitely coloured. Consider the map $f : L(\Sigma) \to \mathbb{N}, \alpha \mapsto 1 + \sum_{t \in dom\,\alpha} \alpha(t)$ and colour each $\alpha \in L(\Sigma)$ with the colour of $f(\alpha)$. Pick α and γ according to Theorem 1. Let $a = f(\alpha)$ and $d = |\gamma|$. Then for all $i \in \{0, 1, \ldots, k\}$, $a + id = f(\alpha \cup \gamma \times \{i\})$ and thus the arithmetic progression $\{a, a+d, \ldots, a+kd\}$ is monochrome.

Hales-Jewett type extensions of various other Ramsey theoretic results have been obtained. We mention two very deep theorems in this style: H. Furstenberg and Y. Katznelson ([**8**]) gave a density version of the Hales-Jewett Theorem, which generalizes Szemerédi's Theorem ([**14**]). V. Bergelson and A. Leibman ([**6**]) proved

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a polynomial Hales-Jewett theorem which extends the polynomial van der Waerden Theorem.

As shown in [**2**], Theorem 3.15, for every set $B \subseteq \mathbb{N}$ with positive upper multiplicative density (actually positive upper multiplictive Banach density is enough) and each $k \in \mathbb{N}$ there exist $a, b, d \in \mathbb{N}$ such that $\{b(a+id)^j : i, j \in \{0, 1, \dots, k\}\} \subseteq B$. A direct consequence is the following partition theorem:

Theorem 2 *For any finite coloring of $\mathbb{N}$ and $k \in \mathbb{N}$ there exist $a, b, d \in \mathbb{N}$ such that $\{b(a+id)^j : i, j \in \{1, 2, \dots, k\}\}$ is monochrome.*

(See [**1**]) for an algebraic proof of this result.)

The main theorem of this paper is an extension of the Hales-Jewett Theorem which is strong enough to yield Theorem 2 but also implies some other corollaries. (Call a family $\mathcal{F}$ of subsets of $\mathbb{N}$ *partition regular* if for any finite coloring of $\mathbb{N}$ there exists some $F \in \mathcal{F}$ which is monochrome.)

Theorem 3 *Let $\mathcal{F}$ be a partition regular family of finite subsets of $\mathbb{N}$ which contains no singletons and let Σ be a finite alphabet. For any finite colouring of $L(\Sigma)$ there exist $\alpha \in L(\Sigma), \gamma \in \mathcal{P}_f(\mathbb{N})$ and $F \in \mathcal{F}$ such that $dom\,\alpha, \gamma$ and F are pairwise disjoint and*

$$\{\alpha \cup (\gamma \cup \{t\}) \times \{s\} : s \in \Sigma, t \in F\}$$

is monochrome.

Similarly as the Hales-Jewett Theorem implies van der Waerden's Theorem, Theorem 3 can be applied to derive Theorem 2. Assume that $\mathbb{N}$ is finitely coloured. Fix $k \in \mathbb{N}$, let $\mathcal{F} = \{\{a, a+d, \dots, a+kd\} : a, d \in \mathbb{N}\}$ be the set of all $(k+1)$-term arithmetic progressions, put $\Sigma = \{0, 1, \dots, k\}$ and define $f : L(\Sigma) \to \mathbb{N}$ by $f(\alpha) = \prod_{t \in dom\,\alpha} t^{\alpha(t)}$. Colour each $\alpha \in L(\Sigma)$ with the colour of $f(\alpha)$ and choose α, γ and $F = \{a, a+d, \dots, a+kd\}$ according to Theorem 3. Then for all $i, j \in \{0, 1, \dots, k\}$,

$$f\big(\alpha \cup (\gamma \cup \{a+id\}) \times \{j\}\big) = \prod_{t \in dom\,\alpha} t^{\alpha(t)} \cdot \prod_{t \in \gamma} t^j \cdot (a+id)^j =$$

$$= \underbrace{\prod_{t \in dom\,\alpha} t^{\alpha(t)}}_{=\bar{b}} \cdot \Big(\underbrace{a \cdot \prod_{t \in \gamma} t}_{=\bar{a}} + i \cdot \underbrace{d \cdot \prod_{t \in \gamma} t}_{=\bar{d}}\Big)^j = \bar{b}(\bar{a} + i\bar{d})^j$$

has the same colour.

Note that we may replace $\mathcal{F}$ by an arbitrary partition regular family of finite subsets of $\mathbb{N}$ which contains no singletons. In this way we see that there exist $a, r \in \mathbb{N}$ and $F \in \mathcal{F}$ such that $\{b(rt)^j : t \in F, j \in \{0, 1, \dots, k\}\}$ is monochrome. (This stronger version also follows from the algebraic proof of Theorem 2, see [**1**], Corollary 4.3.)

Another application of Theorem 3 is that it allows to lower the assumption on the set which is coloured in Theorem 2.

Corollary 1 *Let $k, r \in \mathbb{N}$. There exists $K \in \mathbb{N}$ such that for all $A, D \in \mathbb{N}$ the following holds: Whenever*

$$S_K(A, D) = \{(A + i_1 D)(A + i_2 D) \dots (A + i_m D) : m, i_1, i_2, \dots, i_m \in \{0, 1, \dots, K\}\}$$

is partitioned into sets $B_1, B_2, \ldots, B_r$ one can find $b, a, d \in \mathbb{N}$ and $s \in \{1, 2, \ldots, r\}$ such that $\{b(a+id)^j : i, j \in \{1, 2, \ldots, k\}\} \subseteq B_s$.

Proof. Let $\Sigma = \{0, 1, \ldots, k\}$ and $\mathcal{F} = \{\{a, a+d, \ldots, a+kd\} : a, d \in \mathbb{N}\}$. Using a standard compactness argument (cf. [**11**], section 5.5) we see that for fixed $r \in \mathbb{N}$ there exists some $N \in \mathbb{N}$ such that for any colouring of $L(\Sigma)$ into r colours one can choose α, γ and $F = \{a, a+d, \ldots, a+kd\}$ according to Theorem 3 and additionally require $dom\,\alpha, \gamma, F \subseteq \{1, 2, \ldots, N\}$. Let $f : L(\Sigma) \to \mathbb{N}, f(\alpha) = \prod_{t \in dom\,\alpha}(A + tD)^{\alpha(t)}$ and put $K = kN$. Thus $f(\alpha) \in S_K(A, D)$ for all $\alpha \in L(\Sigma)$, $dom\,\alpha \subseteq \{1, 2, \ldots, N\}$. For each $s \in \{1, 2, \ldots, r\}$ let C_s be the set of all $\alpha \in L(\Sigma)$, $dom\,\alpha \subseteq \{1, 2, \ldots, N\}$ such that $f(\alpha) \in B_s$. Choose $s \in \{1, 2, \ldots, r\}$ and $\alpha, \gamma, F = \{a, a+d, \ldots, a+kd\}$ such that $dom\,\alpha, \gamma, F$ are pairwise disjoint subsets of $\{1, 2, \ldots, N\}$ and $\alpha \cup (\gamma \cup \{a+id\}) \times \{j\} \subseteq C_s$ for all choices of $i, j \in \{0, 1, \ldots, k\}$. Applying f it follows that $\{\overline{b}(\overline{a} + i\overline{d})^j : i, j \in \{0, 1, \ldots, k\}\} \subseteq B_s$, where $\overline{b} = \prod_{t \in dom\,\alpha}(A + tD)^{\alpha(t)}, \overline{a} = (A + aD)\prod_{t \in \gamma}(A + tD)$, and $\overline{d} = dD \prod_{t \in \gamma}(A + tD)$. □

Many structures considered in Ramsey theory have the nice property to be 'unbreakable' in the sense that if a sufficiently large structure of a certain type is partitioned into a specified in advance number of cells, at least one cell will again contain a large strucure of the same type. For example it follows from van der Waerden's Theorem that for all $r, k \in \mathbb{N}$ there exists some $K \in \mathbb{N}$, such that whenever a K-term arithmetic progression is partitioned into r cells at least one cell contains a k-term arithmetic progression. Unfortunately the intertwined additive-multiplicative structures in Theorem 2 do not posses this property as shown in [**1**, Theorem 4.9] that for all $K \in \mathbb{N}$ there exist $A, D \in \mathbb{N}$ and a partition $B_1 \cup B_2 = \{(A + iD)^j : i, j \in \{0, 1, \ldots, K\}\}$ such that no B_i contains a configuration of the form $\{b(a+id)^j : i \in \{0, 1, 2\}, j \in \{0, 1\}\}$. Corollary 1 gives a vague hint that the situation could be different for sets

$$S_k(a, b, d) = \{b(a + i_1 d)(a + i_2 d) \ldots (a + i_m d) : m, i_1, i_2, \ldots, i_m \in \{0, 1, \ldots, k\}\},$$

where $a, b, d \in \mathbb{N}$.

QUESTION 1. Fix $k, r \in \mathbb{N}$. Does there exist some $K \in \mathbb{N}$ such that for all $A, B, D \in \mathbb{N}$ and any partition $\bigcup_{s=1}^{r} B_s = S_K(A, B, D)$ one can find $a, b, d, k \in \mathbb{N}$ and $s \in \{1, 2, \ldots, r\}$ such that $S_k(a, b, d) \subseteq B_s$?

Note added in revision: Imre Leader (private communication) has found an example answering Question 1 in the negative.

2. *Preliminaries*

We give a short outline of the ideas behind our proofs. Lemma 1 states that any large enough, that is, piecewise syndetic set contains structures as in Theorem 3. The argument behind this is abstract but simple in nature. Then one shows that for any colouring of $\Sigma(L)$ there is a piecewise syndetic set A of combinatorial lines having the same colour. Thus Lemma 1 can be applied to A to yield Theorem 3.

It was shown by Furstenberg and Glasner ([**7**]) that any piecewise syndetic set of integers contains a piecewise syndetic set of arithmetic progressions. Their proof is based on the theory of compact semigroups applied to the Stone-Čech compactification $\beta\mathbb{Z}$ (in the form of the enveloping semigroup of an appropriate dynamical system) and so far no purely elementary proof of this strong version of van der Waerden's Theorem is known. In [**4**] similar statements are proved for various notions of largeness in an abstract semigroup S. This extension is possible since - to some extend - the algebraic theory of βS resembles that of $\beta\mathbb{Z}$. In our case we have to work with the partial semigroup of located words $L(\Sigma)$ respectively with its Stone-Čech compactification. We reassure the reader that under the right technical precautions this setting is closely related to the one encountered when facing $\beta\mathbb{Z}$ (which of course is an esoteric enough object). Below we review the most important facts needed for the proof of Theorem 3 in Section 3.

A *partial semigroup* $(S,\cdot)$ (as introduced in [**3**]) is a set S together with a binary operation $\cdot$ that maps a subset of $S \times S$ into S and satisfies the associative law $x\cdot(y\cdot z) = (x\cdot y)\cdot z$ in the sense that if either side is defined, so is the other and they are equal. (The notion of a partial semigroup was introduced in [**3**].)

Given a partial semigroup $(S,\cdot)$ and $x \in S$ let $\phi(x) = \{y \in S : x\cdot y \text{ is defined}\}$. $(S,\cdot)$ is called *adequate* if $\bigcap_{x\in F}\phi(x) \neq \emptyset$ for all finite nonempty $F \subseteq S$.

We shall deal mostly with the adequate semigroup $(L(\Sigma),\uplus)$ of located words over an alphabet Σ where we let $\alpha\uplus\beta = \alpha\cup\beta$ for $\alpha,\beta \in L(\Sigma)$ if $dom\,\alpha \cap dom\,\beta = \emptyset$ and leave $\alpha\uplus\beta$ undefined otherwise.

Given a discrete space S we take the Stone-Čech compactification βS of S to be the set of all ultrafilters on S, the points of S being identified with the principal ultrafilters. Given a set $A \subseteq S$ let $\overline{A} = \{p \in \beta S : A \in p\}$. The family $\{\overline{A} : A \subseteq S\}$ forms a clopen basis of βS. A semigroup structure on S can be extended to βS. The resulting algebraic properties of βS turn out to be extremly useful in Ramsey theory, see [**11**] for an extensive treatment of this topic and related material.

In the case of partial semigroups a slightly different approach is used. For an adequate partial semigroup $(S,\cdot)$ let $\delta S = \bigcap_{x\in S}\overline{\phi(x)} \subseteq \beta S$. For $x \in S$ and $A \subseteq S$ we let $x^{-1}A = \{y \in \phi(x) : x\cdot y \in A\}$. (Note that this corresponds to the usual definition if $(S,\cdot)$ is a group.) For $p,q \in \delta S$ put

$$p\cdot q = \{A \subseteq S : \{x \in S : x^{-1}A \in q\} \in p\}.$$

Then $(\delta S,\cdot)$ is a *compact right topological semigroup*, i.e. δS is compact with the topology inherited from βS and for each $q \in \delta S$ the map $p \mapsto p\cdot q$ is continuous ([**3**], Proposition 2.6).

We will need some facts on the algebraic properties of compact right topological semigroups: Every compact right topological semigroup $(T,\cdot)$ posses an *idempotent*, i.e. there is some $e \in T$ such that $e\cdot e = e$. The set of all idempotents of T is partially ordered if we let $e \leq f \iff e\cdot f = f\cdot e = e$. Idempotents which are minimal with respect to this ordering are called *minimal idempotents*. For any idempotent $f \in T$ there exists some minimal idempotent $e \in T$ such that $e \leq f$. A different characterisation of the minimal idempotents in T can be given via the ideals of T. (A nonempty set $I \subseteq T$ is called an ideal if $I\cdot T \cup T\cdot I \subseteq I$.) Every compact right topological semigroup T has a smallest ideal $K(T)$. An idempotent $e \in T$ is a minimal idempotent if and only if $e \in K(T)$. (See [**11**] for a comprehensive introduction to the theory of compact right topological semigroups.)

3. *Proof of the main Theorem*

Let $(S, \cdot)$ be a partial semigroup, and let $\mathcal{F}$ be a family of subsets of S. $\mathcal{F}$ is called invariant if for all $s \in S$ and $F \in \mathcal{F}$ the following conditions hold:

(1) If $s \cdot F = \{s \cdot f : f \in F\}$ is defined, then $s \cdot F \in \mathcal{F}$.
(2) If $F \cdot s = \{f \cdot s : f \in F\}$ is defined, then $F \cdot s \in \mathcal{F}$.

$\mathcal{F}$ is *adequately partition regular* if for any finite set $G \subseteq S$ and any finite partition of $\bigcap_{x \in G} \phi(x)$ there exists some $F \in \mathcal{F}$ which is entirely contained in one cell of the partition.

If $(S, \cdot)$ is a partial semigroup, $A \subseteq S$ is called *piecewise syndetic* if there exists some $p \in K(\delta S)$ such that $A \in p$. Piecewise syndetic subsets of semigroups admit a simple combinatorial characterisation (see [**11**, Theorem 4.40]). For instance $A \subseteq (\mathbb{Z}, +)$ is piecewise syndetic if and only if $\bigcup_{n=1}^{r} A - n$ contains arbitrarily long intervals. However the combinatorial and the algebraic definition of piecewise syndetic lead to different concepts in the case of partial semigroups. (See [**12**, **13**] for a detailed analysis of this phenomenon.) For our intended application in the proof of Theorem 3 the algebraic version is appropriate.

It is a fairly easy combinatorial exercise to show that van der Waerden's Theorem corresponds to the fact that piecewise syndetic sets in $\mathbb{Z}$ contain arbitrarily long arithmetic progressions. The following lemma translates this idea to our setting.

Lemma 1 *Let S be an adequate partial semigroup, let $\mathcal{F}$ be an adequately partition regular invariant family of finite subsets of S and assume that $A \subseteq S$ is piecewise syndetic. Then there exists $F \in \mathcal{F}$ such that $F \subseteq A$.*

Proof. Let $P = \{p \in \delta S : \text{For all } A \in p \text{ exists } F \in \mathcal{F} \text{ such that } F \subseteq A\}$. It is sufficient to show that P is an ideal of δS, since this implies $P \supseteq K(\delta S)$.

Let G be a finite subset of S. Since for any finite partition of $\bigcap_{x \in G} \phi(x)$ there exists some $F \in \mathcal{F}$ which is entirely contained in one cell of the partition we have that $P_G = \{p \in \overline{\bigcap_{x \in G} \phi(x)} : \text{For all } A \in p \text{ exists } F \in \mathcal{F} \text{ such that } F \subseteq A\} \neq \emptyset$ by [**11**], Theorem 3.11.

Each P_G is closed, so by compactness of δS, $P = \bigcap_{G \subseteq S, |G| < \infty} P_G \neq \emptyset$.

To see that P is a left ideal, let $p \in P$ and $q \in \delta S$. Assume that $A \in q \cdot p$, i.e. $\{s : s^{-1}A \in p\} \in q$. Thus we may pick $s \in S$ such that $s^{-1}A \in p$. Since $p \in P$ there exists $F \in \mathcal{F}$ such that $F \subseteq s^{-1}A$. This is equivalent to $sF \subseteq A$ and $sF \in \mathcal{F}$ by the invariance of $\mathcal{F}$. Since A was arbitrary we see that $q \cdot p \in P$.

To show that P is a right ideal, pick $p \in P, q \in \delta S$ and $A \in p \cdot q$. Thus $\{s \in S : s^{-1}A \in q\} \in p$, so by the choice of p we may pick $F \in \mathcal{F}$ such that $F \subseteq \{s \in S : s^{-1}A \in q\}$. Since F is finite $\bigcap_{s \in F} s^{-1}A \in q$, so pick $t \in \bigcap_{s \in F} s^{-1}A$. Then $F \cdot t \subseteq A$ and, once again by the invariance of $\mathcal{F}$, $F \cdot t \in \mathcal{F}$. Thus $p \cdot q \in P$. $\square$

Let v be a 'variable' not in Σ. $L(\Sigma \cup \{v\})$ is the set of all located words over the alphabet $\Sigma \cup \{v\}$. By $L(\Sigma; v) = L(\Sigma \cup v) \setminus L(\Sigma)$ we denote the set of all located words over the alphabet $\Sigma \cup \{v\}$ in which v occurs. The elements of $L(\Sigma; \{v\})$ are often called *variable words* while one refers to elements of $L(\Sigma)$ as *constant words*. $L(\Sigma; v)$ is an ideal of $L(\Sigma \cup \{v\})$ and consequently $\delta L(\Sigma; v)$ is an ideal of $\delta L(\Sigma \cup \{v\})$.

For $s \in \Sigma$ and $\alpha \in L(\Sigma \cup \{v\})$, let $\theta_s(\alpha)$ be the result of replacing each occurence of v in α by s. More formally, $dom\, \theta_s(\alpha) = dom\, \alpha$ and for $t \in dom\, \theta_s(\alpha)$

$$\theta_s(\alpha)(t) = \begin{cases} s & \text{if } \alpha(t) = v \\ \alpha(t) & \text{if } \alpha(t) \in \Sigma. \end{cases}$$

The mapping $\theta_s : L(\Sigma \cup \{v\}) \to L(\Sigma)$ gives rise to the continuous extension $\tilde{\theta}_s : \beta L(\Sigma \cup \{v\}) \to \beta L(\Sigma)$. Whenever $\alpha \uplus \beta$ is defined for $\alpha, \beta \in L(\Sigma)$, so is $\theta_s(\alpha) \uplus \theta_s(\beta)$ and it equals $\theta_s(\alpha \uplus \beta)$. So $\theta_s : L(\Sigma \cup \{v\}) \to L(\Sigma)$ is a *homomorphism of partial semigroups.* Moreover θ_s is surjective. By [**3**], Proposition 2, this implies that the restriction of $\tilde{\theta}_s$ to $\delta L(\Sigma \cup \{v\})$ is a continuous homomorphism $\delta L(\Sigma \cup \{v\}) \to \delta L(\Sigma)$. Since $\theta_s \restriction L(\Sigma)$ is the identity the same holds true for $\tilde{\theta}_s \restriction \delta L(\Sigma)$.

We are now able to give the proof of our main Theorem:

Proof of Theorem 3. Pick a minimal idempotent $e \in \delta L(\Sigma)$. Let $M = \{p \in \delta L(\Sigma \cup \{v\}) : \tilde{\theta}_s(p) = e$ for all $s \in \Sigma\}$. Since, for each $s \in \Sigma$, $\tilde{\theta}_s$ is the identity on $\delta L(\Sigma)$ we have $\tilde{\theta}_s(e) = e$. Thus M is nonempty. M is the intersection of homomorphic preimages of the closed semigroup $\{e\}$, thus it is a closed semigroup as well. Fix a minimal idempotent $q \in M$. We want to show that q is also minimal in $\delta L(\Sigma \cup \{v\})$. Pick a minimal idempotent $f \in \delta L(\Sigma \cup \{v\})$ such that $f \le q$, i.e. $f \uplus q = q \uplus f = f$. Let $s \in \Sigma$. We have $\tilde{\theta}_s(f) = \tilde{\theta}_s(f \uplus q) = \tilde{\theta}_s(f) \uplus \tilde{\theta}_s(q) = \tilde{\theta}_s(f) \uplus e$ and, analoguosly, $\tilde{\theta}_s(f) = e \uplus \tilde{\theta}_s(f)$, thus $\tilde{\theta}_s(f) \le e$. Since e was chosen to be minimal in $L(\Sigma)$ it follows that $\tilde{\theta}_s(f) = e$. $s \in \Sigma$ was arbitrary, thus we have $f \in M$. Since q is minimal in M it follows that $q = f$, so q is in fact a minimal idempotent in $\delta L(\Sigma \cup \{v\})$.

By the ultrafilter property of e, choose a monochrome set $B \subseteq L(\Sigma)$ such that $B \in e$. Thus $\overline{B}$ is a neighbourhood of e, so pick, by continuity of $\tilde{\theta}_s, s \in \Sigma$, a neighbourhood $\overline{A}$ of q such that $\tilde{\theta}_s[\overline{A}] \subseteq \overline{B}$ for all $s \in \Sigma$. Then $A \in q \in \delta L(\Sigma \cup \{v\})$, so A is piecewise syndetic in $L(\Sigma \cup \{v\})$ and $\theta_s[A] \subseteq B$ for all $s \in \Sigma$.

$\mathcal{F}$ is a partition regular family which contains no singletons. This implies that for any $m \in \mathbb{N}$ and any finite colouring of $\{n \in \mathbb{N} : n > m\}$ there exists a monochrome set $F \in \mathcal{F}$. (Extend the colouring of $\{n \in \mathbb{N} : n > m\}$ to a colouring of $\mathbb{N}$ by giving all elements of $\{1, 2, \ldots, m\}$ new and mutually different colours. Any $F \in \mathcal{F}$ which is monochrome with respect to this colouring is contained in $\{n \in \mathbb{N} : n > m\}$ since F has more then one element.)

Consequently, $\mathcal{F}' = \{\{\{(t, v)\} : t \in F\} : F \in \mathcal{F}\}$ is an adequately partition regular family of subsets of $L(\Sigma \cup \{v\})$ and thus

$$\mathcal{F}'' = \{\{\beta \uplus \{(t, v)\} : t \in F\} : \beta \in L(\Sigma; v), F \in \mathcal{F}, dom\, \beta \cap F = \emptyset\}$$

is an invariant adequately partition regular family. So we may apply Lemma 1 and pick $G \in \mathcal{F}''$ such that $G \subseteq A$. Each variable word $\beta \in L(\Sigma; v)$ can be written in the form $\alpha \cup \gamma \times \{v\}$ for uniquely determined $\alpha \in L(\Sigma), \gamma \in \mathcal{P}_f(\mathbb{N}), dom\, \alpha \cap \gamma = \emptyset$. Hence G is of the form $\{\alpha \uplus (\gamma \cup \{t\}) \times \{v\} : t \in F\}$, where $\alpha \in L(\Sigma), \gamma \in \mathcal{P}_f(\mathbb{N})$ and $F \in \mathcal{F}$. It follows that for each $s \in \Sigma$ and each $t \in F$,

$$\theta_s(\alpha \uplus (\gamma \cup \{t\}) \times \{v\}) = \alpha \cup (\gamma \cup \{t\}) \times \{s\} \in B.$$

$\square$

It would be preferable to show Theorem 3 in a purely elementary way. While it is not difficult to substitute Lemma 1 by a combinatorial argument, the author does not know how to do this in the case of the other part of the proof.

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