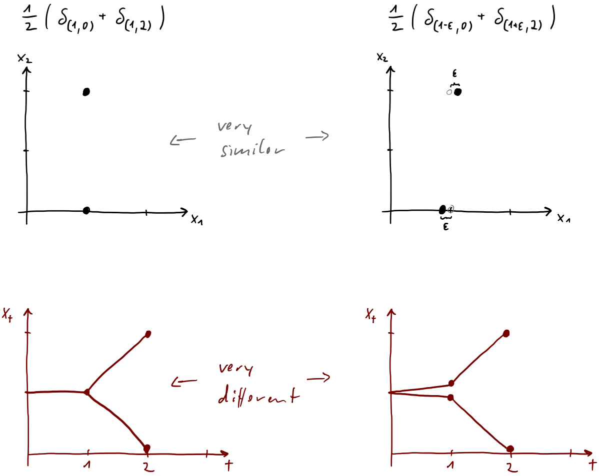

1/2 (δ(1,0) + δ(1,2))
1/2 (δ(1-ε,0) + δ(1+ε,2))
x2
x1
ε
very similar
very different
Xt
1
2
t

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here $\pi \in \text{Cpl}(\mathbb{P}, \mathbb{Q})$ is causal if

$$\forall t \le N : (Y_1, \dots, Y_t), (X_{t+1}, \dots, X_N) \text{ cond. ind. given } (X_1, \dots, X_t)$$

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where $\Delta x_i = x_i - x_{i-1}, x_0 = 0$.

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hedging – discrete time *or* X, Y cont. semi-martingales

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hedging – discrete time *or* X, Y cont. semi-martingales

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definition: $c(X)$...derivative, $m + (H \cdot X)$ is ε-*almost super-hedge* if

$$\mathbb{E}_{\mathbb{P}}(c(X) - m - (H \cdot X)_T)_+ \leq \varepsilon$$

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$$\mathbb{E}_\mathbb{Q}(c(Y) - m - (H^Y \cdot Y))_+ \quad \leq \quad \mathbb{E}_\mathbb{P}(c(X) - m - (H^X \cdot X))_+ + 2(k+L)\, \mathcal{W}_{\text{ad}}(\mathbb{P},\mathbb{Q})$$

hedging – discrete time *or* X, Y cont. semi-martingales

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Note: $X_t = \sigma_1 B_t^1$, $Y_t = \sigma_2 B_t^2$, c...call option $\Longrightarrow$ sharp up to $1/2\pi$

hedging – discrete time *or* X, Y cont. semi-martingales

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Note: $X_t = \sigma_1 B^1_t$, $Y_t = \sigma_2 B^2_t$, c...call option $\Longrightarrow$ sharp up to $1/2\pi$

$$H\ \tilde{L}\text{-Lip} \Rightarrow \quad \mathbb{E}_\mathbb{Q}(c(Y) - m - (H^X \cdot Y))_+ \quad \leq \quad \mathbb{E}_\mathbb{P}(c(X) - m - (H^X \cdot X))_+ + \Big(\tilde{L}\ \mathcal{W}_{\text{ad}}(\mathbb{P},\delta_0) + 2(k+L)\Big)\ \mathcal{W}_{\text{ad}}(\mathbb{P},\mathbb{Q})$$