Universität Wien WS 2014
Fakultät für Mathematik
Mathias Beiglböck
Christa Cuchiero

Mathematical Finance 1
Exercise sheet 1

Let us recall the definition of an arbitrage opportunity in a one-period market with two assets, a risk-free bond $B \equiv 1$ and a stock S whose price at time 0 is $S_0 > 0$ and at time 1 a random variable defined on some probabiltiy space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definiton: A strategy $(H^0, H^1) \in \mathbb{R}^2$ is called an *arbitrage opportunity* if $V_0 := H^0 B_0 + H^1 S_0 = 0$, but $V_1 := H^0 B_1 + H^1 S_1 \geq 0$, $\mathbb{P}$-a.s and $\mathbb{P}[V_1 > 0] > 0$. *No-arbitrage* means that there exists no such opportunity.

1. Consider the following one-period model for a stock price S:

$$S_1 = S_0 Z, \quad S_0 > 0,$$

where Z is a random variable taking the values $\frac{7}{6}$ and $\frac{2}{3}$ with probability p and $1-p$ respectively, i.e.,

$$\mathbb{P}\left[Z = \frac{7}{6}\right] = p \quad \text{and} \quad \mathbb{P}\left[Z = \frac{2}{3}\right] = 1 - p, \quad p \in (0,1).$$

Compute p such that $\mathbb{E}[S_1] = S_0$.

2. Consider similar to the example of the lecture a financial market with one period and two assets: a bond B such that $B_0 = B_1 = 1$ and a stock S with $S_0 > 0$ and S_1 a random variable which takes two values, either $S_0 u$ or $S_0 d$, both with postive probability, where u, d are constants with $u > d > 0$.

i) Prove that there exists an arbitrage opportunity if $d < 1 < u$ does not hold by finding the corresponding strategies.

In the following let $S_0 = 1$, $d = 3/4$ and $u = 5/4$.

Please turn over!

ii) Consider a European call option C with strike price $K = 1$. Find a replicating portfolio and apply the no-arbitrage principle to determine the price C_0 of the option.

iii) Find a measure $\mathbb{Q}$ such that $\mathbb{E}_{\mathbb{Q}}[S_1] = S_0$ and show that $\mathbb{E}_{\mathbb{Q}}[C_1] = C_0$.

3. Consider a one-period model of a financial market consisting of one stock and one bond $B \equiv 1$. Assume that the stock price at time 0 is $S_0 = 1$, and at time 1 can have any of the three values d, m and u, each with strictly positive probability, where we assume that $0 < d < m < u$. Under which conditions is this model free of arbitrage?