Stochastic Analysis, WS18/19, Sheet 2

1. If (M_n) is a simple symmetric random walk then $(M_n^2 - n)$ is a martingale. Use this to calculate $\mathbb{E}T_{-a,b}$ for $a, b \in \mathbb{N}, T_{-a,b} = \inf\{n : M_n \in \{-a, b\}\}$.

2. Assume that (M_n) is a simple symmetric random. For which adapted (H_n) do we have $\sup_n \|X_n\|_2 < \infty$ for the martingale $X_n = (H \cdot M)_n, n \in \mathbb{N}$. Can you detect a Hilbert-space isometry?

3. Assume that $\{X_n\}_n$ is a UI family and that $\lim_n X_n$ exists almost surely. Show that the limit also exists in L^1.

4. Show that $1), 2), 3)$ and $1'), 2'), 3')$ are equivalent, where

   1) $B_0 = 0$.
   2) B has independent increments.
   3) For all $s \leq t$, $B_t - B_s \sim N(0, t-s)$.

   and

   1') $B_0 = 0$.
   2') $(B_{t_1}, \ldots, B_{t_n})$ is centered Gaussian for all $t_1, \ldots, t_n \geq 0$.
   3') For all $s \leq t$, $Cov(B_s, B_t) = s \wedge t$.