Stochastic Analysis

Sheet 1

Problem 1. *Let* $(X_1, X_2, \dots)$ *be i.i.d. random variables with the uniform distribution on the interval* $(0, 1)$*. Let* $(Y_1, Y_2, \dots)$ *be i.i.d. random variables such that each* Y_i *has the standard Gaussian distribution* $\mathcal{N}(0, 1)$*. Construct explicitly a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and such sequences of random variables on this space. You may also present a construction in the canonical setup.*

Problem 2. *Let* $(X_n)_{n \geq 1}$ *be Gaussian random variables that converge in probability to a random variable* X*. Show that* X *is Gaussian, that the mean of* X *is the limit of the means of* X_n*, and that the variance of* X *is the limit of the variances of* X_n*. Does this imply that* $X_n \to X$ *also in* L^2*?*

Problem 3. *The lost problem*

Sheet 2

Problem 4. *Let* $(X_t)_{t\in T}$ *be a Gaussian process on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*. Denote by*

$$H = \overline{\operatorname{span}}\{\, X_t : t \in T \,\}$$

the closure in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ *of the linear span of the random variables* X_t*.*

Show that H *is a Gaussian space, that is, every finite collection of elements of* H *is jointly Gaussian.*

Problem 5. *Let* $(B_t)_{t\geq 0}$ *be a* pre-Brownian motion*, that is, a centered Gaussian process such that*

$$\mathbb{E}[B_s B_t] = \min\{s, t\}, \qquad s, t \geq 0.$$

Show that B *is a pre-Brownian motion if and only if one of the following equivalent conditions holds:*

(a) $B_0 = 0$*, and for all* $0 \leq s < t$*, the increment* $B_t - B_s$ *is independent of* $\sigma(B_r : r \leq s)$ *and is distributed as* $\mathcal{N}(0, t-s)$*.*

(b) $B_0 = 0$*, the increments* $(B_t - B_s)_{0\leq s<t}$ *are independent, and each increment* $B_t - B_s$ *has distribution* $\mathcal{N}(0, t-s)$*.*

(It is fine if you just show a) or b).)

Problem 6. *Let* $(B_t)_{t\geq 0}$ *be a pre-Brownian motion. Show that each of the following processes is also a pre-Brownian motion:*

(i) $(-B_t)_{t\geq 0}$*.*

(ii) For $\lambda > 0$*, the scaled process* $B_t^{(\lambda)} := \lambda^{-1} B_{\lambda^2 t}$*.*

(iii) For $s \geq 0$*, the shifted process* $B_t^{(s)} := B_{t+s} - B_s$*.*

Problem 7. *Let* $(B_t)_{t\geq 0}$ *be a pre-Brownian motion with continuous sample paths, hence a Brownian motion. Construct another stochastic process* $(\widetilde{B}_t)_{t\geq 0}$ *which is a pre-Brownian motion but whose paths are not continuous.*

Sheet 3

Problem 8. *Use a version of Kolmogorov's continuity theorem to show that the sample paths of Brownian motion are almost surely α-Hölder continuous for every $\alpha \in (0, \frac{1}{2})$.*

Problem 9. *Let $(B_t)_{t\in[0,1]}$ be a Brownian motion, and define the process $(R_t)_{t\in[0,1]}$ by*

$$R_t := B_t - tB_1.$$

(i) Compute the finite-dimensional distributions of the process $(R_t)_{t\in[0,1]}$. (If you observe that this is a Gaussian process, it is sufficient that you determine the covariance structure.)
(ii) Show that the process $(R_t)_{t\in[0,1]}$ has the same finite-dimensional distributions as the time-reversed process $(R_{1-t})_{t\in[0,1]}$.

Problem 10. *There are two further standard ways to define the Brownian bridge on $[0,1]$.*
(i) As a centered Gaussian process $(\beta_t)_{t\in[0,1]}$ with covariance

$$\mathbb{E}[\beta_s\beta_t] = \min\{s,t\} - st, \qquad s,t \in [0,1].$$

(ii) As the regular conditional law of Brownian motion $(B_t)_{t\in[0,1]}$ given $B_1 = 0$ (with $B_0 = 0$ understood), that is, a process $(\widetilde{\beta}_t)_{t\in[0,1]}$ whose finite-dimensional distributions satisfy

$$\mathcal{L}\big((\widetilde{\beta}_{t_1},\dots,\widetilde{\beta}_{t_k})\big) = \mathcal{L}\big((B_{t_1},\dots,B_{t_k}) \,\big|\, B_1 = 0\big), \qquad 0 < t_1 < \cdots < t_k < 1.$$

Conclude that the three objects above all have the same finite-dimensional distributions, in particular

$$\mathcal{L}\big((R_{t_1},\dots,R_{t_k})\big) = \mathcal{L}\big((B_{t_1},\dots,B_{t_k}) \,\big|\, B_1 = 0\big)$$

for all $0 < t_1 < \cdots < t_k < 1$. You may justify the conditioning either by an explicit Gaussian computation or via the limit

$$\mathbb{E}\big[f(B_{t_1},\dots,B_{t_k}) \,\big|\, B_1 = 0\big] = \lim_{\varepsilon\downarrow 0} \mathbb{E}\big[f(B_{t_1},\dots,B_{t_k}) \,\big|\, |B_1| \le \varepsilon\big]$$

for bounded continuous f.

Problem 11. *Let $\Omega := \mathbb{R}^{[0,1]}$ be the set of all functions $\omega : [0,1] \to \mathbb{R}$. Equip Ω with the product topology (equivalently, the topology of pointwise convergence), and let $\mathcal{B}$ be its Borel σ-algebra. Denote by*

$$\mathcal{C} := \{\, \omega \in \Omega : \omega \text{ is continuous on } [0,1] \,\}$$

the subset of continuous functions. Show that $\mathcal{C} \notin \mathcal{B}$. In other words, the set of continuous functions is not measurable with respect to the Borel σ-algebra generated by the product topology on $\mathbb{R}^{[0,1]}$.

Sheet 4

Problem 12. *Let* $(B_t)_{t>0}$ *be a standard Brownian motion, and define the process*

$$\widehat{B}_t := t\,B_{1/t}, \qquad t > 0,$$

and $\widehat{B}_0 := 0$.

Verify that $(\widehat{B}_t)_{t\geq 0}$ *is again a Brownian motion. In particular, show that it is a centered Gaussian process with covariance*

$$\mathbb{E}[\widehat{B}_s \widehat{B}_t] = \min\{s,t\}, \qquad s,t \geq 0.$$

Pay attention to the behaviour at $t = 0$ *and justify the continuity of the sample paths at* 0.

Problem 13. *Let* $(B_t)_{t\in[0,1]}$ *be a Brownian motion. For each* $n \in \mathbb{N}$, *consider the equidistant partition of* $[0,1]$

$$\pi_n := \left\{0, \tfrac{1}{n}, \tfrac{2}{n}, \dots, 1\right\},$$

and define the quadratic variation along this partition by

$$Q_n := \sum_{k=1}^{n} (B_{k/n} - B_{(k-1)/n})^2.$$

Show that $\mathbb{E}[Q_n] = 1$ *and compute* $\mathrm{Var}(Q_n)$. *Conclude that* $Q_n \to 1$ *in* L^2, *and hence also in probability.*

Problem 14. *Let* $(B_t)_{t\in[0,1]}$ *be a Brownian motion, and let* Q_n *be as in the previous problem. Show that* $Q_{2^m} \to 1$ *almost surely as* $n \to \infty$.

(Hint: For $m \in \mathbb{N}$, *consider the filtration* $\mathcal{F}_m := \sigma(B_{k/2^m} : 0 \leq k \leq 2^m)$, *and define*

$$M_m := Q_{2^m} - 1.$$

Show that $(M_m, \mathcal{F}_m)_{m\geq 0}$ *is a martingale. Compute its second moments and apply the martingale convergence theorem to conclude almost sure convergence along the dyadic partitions.)*

Problem 15. *Let* $(B_t)_{t\geq 0}$ *be a Brownian motion. Define*

$$X_t := \frac{1}{t}\int_0^t B_s\,ds, \qquad t > 0.$$

Use Blumenthal's zero–one law to determine

$$\mathbb{P}\left(\lim_{t\downarrow 0} X_t = 0\right).$$

Sheet 5

Problem 16. *Let* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *be a filtered probability space, and let* τ *be a stopping time taking values in a countable set* $S \subset [0, \infty]$*. Recall that*

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}.$$

Show that for every $A \in \mathcal{F}_\tau$ *there exists a family* $(A_t)_{t\in S}$ *with* $A_t \in \mathcal{F}_t$ *for all* $t \in S$ *such that*

$$A = \bigcup_{t\in S} (A_t \cap \{\tau = t\}).$$

Problem 17. *Let* $(\mathcal{F}_t)_{t\geq 0}$ *be a right-continuous filtration. For a random time* $\tau : \Omega \to [0, \infty]$*, show that the following statements are equivalent.*

(a) τ *is a stopping time.*

(b) For every $t \geq 0$*, the event* $\{\tau < t\}$ *belongs to* $\mathcal{F}_t$*.*

Problem 18. *Let* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *be a filtered probability space with right-continuous filtration, and let* $(\tau_n)_{n\in\mathbb{N}}$ *be a sequence of stopping times such that* $\tau_n \downarrow \tau$ *almost surely for some stopping time* τ*. Show that*

$$\mathcal{F}_\tau = \bigcap_{n=1}^{\infty} \mathcal{F}_{\tau_n}.$$

Problem 19. *Let* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *be a filtered probability space with right-continuous filtration. Show that a random time* $\tau : \Omega \to [0, \infty]$ *is a stopping time if and only if there exists a sequence of stopping times* $(\tau_n)_{n\in\mathbb{N}}$*, each taking values in a countable subset of* $[0, \infty]$*, such that* $\tau_n \downarrow \tau$ *almost surely. A convenient choice is*

$$\tau_n(\omega) := \inf\{k2^{-n} : k \in \mathbb{N},\ k2^{-n} \geq \tau(\omega)\}.$$

Sheet 6

Remark 1. *Let $D \subset [0,1]$ and $f : D \to \mathbb{R}$ be a function. For real numbers $c < d$, the number of upcrossings of the interval (c,d) by f is defined as*

$$U_{c,d}(f) := \sup\Big\{\, k \in \mathbb{N} : \exists\, t_1 < s_1 < \cdots < t_k < s_k \text{ in } D \text{ with } f(t_i) \le c,\ f(s_i) \ge d \,\Big\}.$$

Problem 20. *Let D be a countable dense subset of the interval $[0,1]$, and let $f : D \to \mathbb{R}$ be a bounded function. Show the following implication. Assume there exists a function $F : [0,1] \to \mathbb{R}$ which extends f on D, has left limits everywhere, and is right-continuous. Prove that for every interval (c,d) with rational endpoints $c < d$, the function f has only finitely many upcrossings of the interval (c,d).*

Problem 21. *With the same assumptions on D and f as above, suppose that for every interval (c,d) with rational endpoints $c < d$, the function f has only finitely many upcrossings of the interval (c,d). Show that there then exists a function $F : [0,1] \to \mathbb{R}$ extending f on D, which has left limits at every point $t \in [0,1]$ and right limits at every point $t \in [0,1]$.*

Problem 22. *Given a finite sequence $x_0, \ldots, x_N \in \mathbb{R}$ show that the number of upcrossings of the interval $[0,1]$ can be bounded in the form $\sum_{k<N} h_k(x_0, \ldots, x_k)(x_{k+1} - x_k) + |x_N|$.*

(Hint: you may want to think of a buy low, sell high strategy.)

Problem 23. *Let $(X_k)_{k=0}^N$ be a submartingale in discrete time with respect to some filtration. Denote by $\overline{X}_k := \max\{X_0, \ldots, X_k\}$ the running supremum. Prove the following form of Doob's maximal inequality: for every $\lambda > 0$,*

$$\lambda\, \mathbb{P}\Big(\overline{X}_N > \lambda\Big) \le \mathbb{E}\big[(X_N)^+\big].$$