OPTIMAL TRANSPORT AND SKOROKHOD EMBEDDING

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Abstract. It is well known that several solutions to the Skorokhod problem optimize certain "cost"- or "payoff"-functionals. We use the theory of Monge-Kantorovich transport to study the corresponding optimization problem. We formulate a dual problem and establish duality based on the duality theory of optimal transport. Notably the primal as well as the dual problem have a natural interpretation in terms of model-independent no arbitrage theory.

In optimal transport the notion of c-monotonicity is used to characterize the geometry of optimal transport plans. We derive a similar optimality principle that provides a *geometric characterization* of optimal stopping times. We then use this principle to derive the Root- and Rost solutions to the Skorokhod embedding problem.

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1. Introduction

Throughout this paper we denote by λ a measure on the real line which has barycenter 0 and finite second moment. Let B be a Brownian motion on some stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. It is well known that then there exists a stopping time τ which solves[1] the Skorokhod embedding-problem

$$B_\tau \sim \lambda, \quad \mathbb{E}[\tau] = \int x^2 \, \lambda(dx)\,. \tag{1.1}$$

There exist a variety of different constructions of a stopping time τ which solves the *embedding problem* (1.1), we refer to the survey of Obloj [Obł04].

Starting with Hobson's seminal paper [Hob98] the Skorokhod embedding problem has received significant attention in the mathematical finance community due to its relevance in the theory of model-independent finance, see [Hob11] for an overview. Here we do not elaborate on this connection; we just mention that a large class of problems corresponds to an optimization problem which relates to (1.1) and which we now formalize.

1.1. The Primal Problem.

We consider the set of *stopped paths*

$$S = \{(f, s) : f : [0, s] \to \mathbb{R} \text{ is continuous}, f(0) = 0\}. \tag{1.2}$$

Throughout the paper we consider a functional

$$\gamma : S \to \mathbb{R}.$$

The primal problem which we study consists in

$$P_\gamma(\lambda) = \sup\Big\{\mathbb{E}\big[\gamma((B_t)_{t\leq\tau})\big] : \ \tau \text{ solves } (1.1)\Big\}. \tag{1.3}$$

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[1] In particular we allow here that the stopping time τ depends on external randomization. The condition $\mathbb{E}[\tau] = \int x^2 \, \lambda(dx)$ is imposed to exclude trivial (degenerate) solution of the embedding problem.

We say that the problem is *well-posed* iff $\mathbb{E}\Big[\gamma((B_t)_{t\le\tau})\Big]$ exists with values in $[-\infty,\infty)$ for all τ which satisfy (1.1) and is finite for one such τ.

Typical examples of the functional γ which are relevant in model-independent finance are given by the running maximum $\gamma((f,s)) := \bar{f} := \max_{t\le s} f(t)$ or the convex/concave functions of time, e.g., $\gamma((f,s)) = h(s)$, where $h:\mathbb{R}_+\to\mathbb{R}$ is convex/concave.

The set of all *randomized stopping times* (see (4.2) below) solving the Skorokhod Problem (1.1) is *compact* in a natural sense and as a consequence we will establish:

Proposition 1.1. *Let $\gamma : S \to \mathbb{R}$ be upper semi-continuous. Then* (1.3) *admits a maximizer $\hat\tau$ whenever the optimization problem is well-posed.*

Here we can talk about the continuity properties of γ since S carries a naturally Polish topology: Let $(f,s),(g,t)\in S$ and assume wlog $s\le t$. We then say that (f,s) and (g,t) are ε-close if

$$\max\Big(t-s, \sup_{0\le u\le s}|f(u)-g(u)|, \sup_{s\le u\le t}|g(u)-g(s)|\Big) < \varepsilon. \tag{1.4}$$

1.2. **The Dual Problem.** Related to the above primal problem is a dual problem which has a financial interpretation in terms of robust super-hedging. In the formulation given below we take the probability space to be the path space $C(\mathbb{R}_+)$ of continuous functions starting in 0 equipped with the Wiener measure $\mathbb{W}$ and we take $(B_t)_{t\ge0}$ to be the canonical process, i.e. $B_t(\omega)=\omega(t)$.

Theorem 1.2. *Let $\gamma : S \to \mathbb{R}$ be upper semi-continuous, bounded from above*[2]*, predictable and assume that* (1.3) *is well posed. Put*

$$D_\gamma(\lambda) = \inf\left\{\int \psi(y)\,d\lambda(y) : \exists(H),\ \begin{array}{l}(H\cdot B)_t + \psi(B_t) \ge \gamma((B_s)_{s\le t},t)\\ \textit{for all } t\in\mathbb{R}_+,\ \mathbb{W}\textit{-a.s.}\end{array}\right\},$$

where H runs through all predictable processes with $\int_0^t \mathbb{E}H_s^2\,ds \le at+b$ for some $a,b>0$.

Then there is no duality gap, i.e.

$$P_\gamma(\lambda) = D_\gamma(\lambda). \tag{1.5}$$

1.3. **Variational Principle.** A basic and fundamental notion in the theory of optimal transport is *c-cyclical monotonicity* which we recall in (3.6) below. The remarkable feature of this optimality criterion is that the optimality of the measure π is linked to the *geometry* of the support set $\mathrm{supp}(\pi)$. Often this is key to understanding the transport problem.

We establish a corresponding result which applies to the theory of Skorokhod embedding. Let B be a Brownian motion (on some stochastic basis $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge0},\mathbb{P})$) and τ a stopping time. Let Γ be a Borel subset of the set S defined in (1.2). We say that τ is concentrated on Γ if $\mathbb{P}$-a.s.

$$((B_t)_{t\le\tau},\tau)\in\Gamma. \tag{1.6}$$

Suppose now that $\hat\tau$ is an optimizer of the primal problem $P_\gamma(\lambda)$ for some function γ. Intuition from optimal transport suggest that in this case there exists a set $\Gamma\subseteq S$ which supports $\hat\tau$ and reflects the optimality of $\hat\tau$:

Suppose that $\hat\tau$ stops a path once it reaches (g,t) and some other path is still living in (f,s). Assume also that

$$\begin{array}{c} f(s)=g(t) \text{ and}\\ \text{“stop in } (f,s)\text{, don't stop in } (g,t)\text{” leads to a better } \gamma\text{-payoff.}\end{array} \tag{1.7}$$

[2]It is possible to relax this condition, see Remark 5.2 below.

Then we will say that $((f,s),(g,t))$ is a *bad pair*[3] w.r.t. γ and write BP for the set of all $((f,s),(g,t))$ satisfying (1.7). If $\hat\tau$ is optimal we should not encounter bad pairs for which (f,s) is still living with respect to $\hat\tau$ while the process is dying in (g,t). After all, in this case it would be better to switch the role of (f,s) and (g,t) under the stopping time $\hat\tau$.

The following result formalizes this heuristic idea.

Theorem 1.3 (Variational Principle)**.** *Assume that $\gamma : S \to \mathbb{R}$ is upper semi-continuous, the optimization problem* (1.3) *is well-posed and that $\hat\tau$ is an optimizer of $P_\gamma(\lambda)$. There is a* stopping set $\Gamma \subseteq S$ *such that $\hat\tau$ is supported by Γ and there are no bad pairs with respect to Γ, i.e. if $\big((f,s),(g,t)\big) \in \mathsf{BP}$, then at least one of the following applies:*

(1) (f,s) is not before the right end of Γ, i.e. (f,s) has no proper extension $(f',s') \in \Gamma$.
(2) $(g,t) \notin \Gamma$.

*We call a set Γ verifying (1) and (2) γ-*monotone.

Writing

$$\Gamma^< := \{(f,s) : \exists (g,t) \in \Gamma, f = g_{\llcorner[0,s]}, s < t\} \tag{1.8}$$

for the set of all (f,s) which are before (the right end of) Γ, the properties (1), (2) can also be expressed as

$$\mathsf{BP} \ \cap \ \Gamma^< \times \Gamma = \emptyset. \tag{1.9}$$

Notice that, in general, the sets $\Gamma^<$ and Γ are *not* disjoint. In fact, for a stopping time τ there exists a set Γ such that τ is supported by Γ and $\Gamma^< \cap \Gamma = \emptyset$ iff the stopping time τ depends only on the evolution of B but not on external randomization, i.e. if τ is a stopping time for the filtration generated by B.

In Section 2 we will use Theorem 1.3 to give short derivations of the Root- and the Rost solution of the Skorokhod problem.

1.4. **Connections with the Literature.** The idea to relate the theory of optimal transport with model-independent finance first appeared in the papers [GHT12, BHP12].

While the article [BHP12] is concerned with a discrete time setup, Galichon, Henry-Labordere, and Touzi [GHT12] study the Skorokhod embedding problem as an optimal stopping problem. By connecting the Skorokhod embedding problem to a free boundary problem they derive the Azema-Yor solution to the Skorokhod-embedding.

Through the Dambis-Dubins-Schwarz theorem, the optimization problems (1.3) and (5.4) are related to the pricing of financial derivatives whose payoff is invariant under time-changes; this idea goes back to [Hob98]. In mathematical finance terms, Theorem 1.2 is a robust super-replication theorem comparable to the recent result of Dolinsky and Soner [DS12]. Dolinsky and Soner processed through a discretization of the problem. Opposed to our result this allows to treat also functionals γ which are not necessarily invariant w.r.t. time-changes. On the other hand, in [DS12] it is necessary to assume stricter conditions on the continuity of the functional γ, hence excluding functionals involving the quadratic variation. For related duality results in a quasi-sure context we refer to [PRT13].

The idea to consider an analogue of c-cyclical monotonicity in the martingale context comes from [BJ12] where the corresponding notion is introduced in a discrete time framework and applied to obtain a 1-dimensional martingale analogue of Brenier's theorem. A different and more explicit approach to this Brenier-type result is given by Henry-Labordere and Touzi in [HT13].

[3] The formal definition will be given in (2.1) (and (6.2) resp.) below.

1.5. **Organization of the Article.** In Section 2 we establish the Root- and the Rost-embedding based on Theorem 1.3. In Section 3 we recall some principal definitions and results from optimal transport. In Section 4 we consider *randomized* stopping times on the Wiener space and establish some basic properties. In Section 5 we develop the dual side of the problem and prove Theorem 1.2. In Sections 6 and 7 we will establish Theorem 1.3 by combining the duality theory of optimal transport with the Choquet-capacity theorem.

2. Particular embeddings

In this section we explain how Theorem 1.3 can be used to derive particular solutions to the Skorokhod embedding problem. We first define the notion of "bad pairs".

Definition 2.1. *Write* $(g \oplus h, t+u)$ *for the path obtained from concatenating* (g,t) *and* $(h,u) \in S$. *Then the set of* bad pairs *for* $\gamma : S \to \mathbb{R}$ *is given by*

$$\mathsf{BP} = \Big\{((f,s),(g,t)) : f(s) = g(t), \ \textit{for all } (h,u) \in S, u > 0$$
$$\gamma((f \oplus h, s+u)) + \gamma((g,t)) \quad < \quad \gamma((f,s)) + \gamma((g \oplus h, t+u))\Big\}.$$

2.1. **The Root embedding.** A set $\mathcal{R} \subseteq \mathbb{R} \times \mathbb{R}_+$ is a *barrier* if $(x,s) \in \mathcal{R}$ and $s < t$ implies that $(x,t) \in \mathcal{R}$. Root [Roo69] established that there exists a barrier $\mathcal{R}$ such that the Skorokhod problem is solved by the stopping time

$$\tau_{\mathrm{Root}} = \inf\{t \geq 0 : (B_t, t) \in \mathcal{R}\}. \tag{2.1}$$

To see this let $\gamma(f,t) = h(t)$, where $h : \mathbb{R}_+ \to \mathbb{R}$ is a strictly concave function such that

$$\sup\{\mathbb{E}[h(\tau)] : \tau \text{ solves the Skorokhod-problem}\} \tag{2.2}$$

is well posed and pick a maximizer $\hat\tau$ of (2.2) by Proposition 1.1. Then we have:

Theorem 2.2. *There exists a barrier* $\mathcal{R}$ *such that* $\hat\tau = \inf\{t \geq 0 : B_t \in \mathcal{R}\}$. *In particular the Skorokhod problem has a solution of barrier-type* (2.1).

Proof. Pick, by Theorem 1.3, a γ-*monotone* set $\Gamma \subseteq S$ such that $\mathbb{P}(\hat\tau \in \Gamma) = 1$. Note that due to the concavity of h the set of bad pairs is given by

$$\mathsf{BP} = \{((f,s),(g,t)) \in S : f(s) = g(t), t < s\}.$$

As Γ is γ-monotone, $\Gamma^< \times \Gamma \cap \mathsf{BP} = \emptyset$. Define a left and a right barrier by

$$\mathcal{R}_L := \{(x,s) : \exists (g,t) \in \Gamma, g(t) = x, t \leq s\}, \tag{2.3}$$
$$\mathcal{R}_R := \{(x,s) : \exists (g,t) \in \Gamma, g(t) = x, t < s\}. \tag{2.4}$$

and denote the respective hitting times by τ_L and τ_R. We claim that $\tau_L \leq \hat\tau \leq \tau_R$ a.s.

Note that $\tau_L \leq \hat\tau$ holds by definition of τ_L. To show the other inequality pick ω satisfying $((B_t(\omega))_{t \leq \hat\tau(\omega)}, \hat\tau(\omega)) \in \Gamma$ and assume for contradiction that $\tau_R(\omega) < \hat\tau(\omega)$. Then there exists $s < \hat\tau(\omega)$ such that $B_s(\omega) \in \mathcal{R}_R$. By definition of the right barrier, this means that there is some $(g,t) \in \Gamma$ such that $t < s$ and $g(t) = B_s(\omega)$. But then $(f,s) := ((B_u(\omega))_{u \leq s}, s) \in \Gamma^<$, hence $\big((f,s),(g,t)\big) \in \mathsf{BP} \cap \Gamma^< \times \Gamma$ which is the desired contradiction.

It remains to show that $B_{\tau_L} \sim B_{\tau_R}$. This is evident from the properties of one-dimensional Brownian motion but can also be seen by a "softer" argument: Consider

$$\mathcal{R}_L^\varepsilon := \{(x,s) : \exists (g,t) \in \Gamma, g(t) = x, t + \varepsilon \leq s\}$$

and the corresponding hitting time τ_ε. Then the law of B_{τ_ε} tends to the law B_{τ_L} in the total variation norm. To see this, write B^ε for Brownian motion started at time $-\varepsilon$ and note that $B^\varepsilon_{\tau_L} \sim B_{\tau_\varepsilon}$. □

A consequence of this proof is that (on a given stochastic basis) there exists exactly one solution[4] of the Skorokhod embedding problem which maximizes (2.2): Assume that maximizers τ_1 and τ_2 are given. Then we can use an independent coin-flip to define a new maximizer $\bar{\tau}$ which is with probability $1/2$ equal to τ_1 and with probability $1/2$ equal to τ_2. By Theorem 2.2, $\bar{\tau}$ is of barrier-type and hence $\tau_1 = \tau_2$.

We also note that the above proof of Theorem 2.3 is based on a heuristic derivation of the optimality properties of the Root-embedding given by Hobson in [Hob11]. Indeed Hobson's approach was the starting point of the present paper.

2.2. **The Rost embedding.** A set $\mathcal{R} \subseteq \mathbb{R} \times \mathbb{R}_+$ is an *inverse barrier* if $(x, s) \in \mathcal{R}$ and $s > t$ implies that $(x, t) \in \mathcal{R}$. It has been shown by Rost that under the condition[5] $\lambda(\{0\}) = 0$ there exists an inverse barrier such that the corresponding hitting time (in the sense of (2.1)) solves the Skorokhod problem. We derive this using an argument almost identical to the one above:

Let $\gamma(f, t) = h(t)$, where $h : \mathbb{R}_+ \to \mathbb{R}$ is a strictly *convex* function such that the problem to maximize $\mathbb{E}[h(\tau)]$ over all solutions to the Skorokhod-problem (1.1) is well posed. Pick, by Proposition 1.1, a maximizer $\hat{\tau}$. Then we have:

Theorem 2.3. *There exists an inverse barrier $\mathcal{R}$ such that $\hat{\tau} = \inf\{t \geq 0 : (B_t, t) \in \mathcal{R}\}$. In particular the Skorokhod problem can be solved by a hitting time of an inverse barrier.*

Proof. Pick, by Theorem 1.3, a γ-*monotone* set $\Gamma \subseteq S$ such that $\mathbb{P}(\hat{\tau} \in \Gamma) = 1$. Note that due to the convexity of h the set of bad pairs is given by

$$\mathsf{BP} = \{((f, s), (g, t)) \in S : f(s) = g(t), s < t\}.$$

As Γ is γ-monotone, $\Gamma^< \times \Gamma \cap \mathsf{BP} = \emptyset$. Define a left and a right inverse barrier by

$$\mathcal{R}_L := \{(x, s) : \exists (g, t) \in \Gamma, g(t) = x, s < t\}, \tag{2.5}$$

$$\mathcal{R}_R := \{(x, s) : \exists (g, t) \in \Gamma, g(t) = x, s \leq t\}. \tag{2.6}$$

and denote the respective hitting times by τ_L and τ_R. We claim that $\tau_R \leq \hat{\tau} \leq \tau_L$ a.s.

Note that $\tau_R \leq \hat{\tau}$ holds by definition of τ_R. To show the other inequality pick ω satisfying $((B_t(\omega))_{t \leq \hat{\tau}(\omega)}, \hat{\tau}(\omega)) \in \Gamma$ and assume for contradiction that $\tau_L(\omega) < \hat{\tau}(\omega)$. Then there exists $s < \hat{\tau}(\omega)$ such that $B_s(\omega) \in \mathcal{R}_L$. By definition of the left barrier, this means that there is some $(g, t) \in \Gamma$ such that $s < t$ and $g(t) = B_s(\omega)$. But then $(f, s) := (B_u(\omega))_{u \leq s}, s) \in \Gamma^<$, hence $\big((f, s), (g, t)\big) \in \mathsf{BP} \cap \Gamma^< \times \Gamma$ which is the desired contradiction.

Similar to the previous proof we have $B_{\tau_L} \sim B_{\tau_R}$. □

As in the case of the Root-embedding we obtain that the maximizer of $\mathbb{E}[h(\tau)]$ is unique.

2.3. **Remarks.** It is well known (see for instance [Obł04, Hob11]) that the Root- and Rost-embedding can be shown to maximize $\mathbb{E}[h(\tau)]$ for convex resp. concave h. In the above approach we have turned this upside down: the optimization problem is used as an *auxiliary tool* to derive the Root- and Rost-solution of the Skorokhod problem.

The arguments used here do not use the properties of one-dimensional Brownian motion. We believe that the above approach generalizes to a multi-dimensional setup and (sufficiently regular) continuous Markov-processes. Also it does not matter for the argument whether the starting distribution is a Dirac in 0 as in our setup or rather a more general distribution.

[4]This was first established in [Ros76], together with the optimality property of Root-solution.

[5]It is not hard to see that without this condition some additional randomization is required.

We also mention two recent accounts on the Root-embedding given in [CW12] and [Od13]. These approaches are based on PDE techniques and in particular allow for much more explicit description of the barrier than the proof given in Theorem 2.3.

3. The classical Transport Problem

To establish Theorem 1.2 and Theorem 1.3 we link the Skorokhod embedding problem to the *Monge-Kantorovich optimal transport*. This allows us to use the duality theorem of optimal transport and techniques related to c-cyclical monotonicity in the context of Brownian motion.

In abstract terms the transport problem (cf. [Vil03, Vil09]) can be stated as follows: For probabilities μ, ν on Polish spaces X, Y the set $\mathsf{Cpl}(\mu, \nu)$ of *transport plans* consists of all *couplings* between μ and ν. These are all measures on $X \times Y$ with X-marginal μ and Y-marginal ν. Associated to a *cost function* $c : X \times Y \to [0, \infty]$ and $\pi \in \mathsf{Cpl}(\mu, \nu)$ are the *transport costs* $\int_{X\times Y} c(x, y)\, d\pi(x, y)$. The Monge-Kantorovich problem is then to determine the value

$$\inf\Big\{ \int c\, d\pi : \pi \in \mathsf{Cpl}(\mu, \nu)\Big\} \tag{3.1}$$

and to identify an *optimal* transport plan $\hat\pi \in \mathsf{Cpl}(\mu, \nu)$, i.e. a minimizer of (3.1). Going back to Kantorovich, this is related to the following dual problem. Consider the set $\Phi(\mu, \nu)$ of pairs (φ, ψ) of integrable functions $\varphi : X \to [-\infty, \infty)$ and $\psi : Y \to [-\infty, \infty)$ which satisfy $\varphi(x) + \psi(y) \le c(x, y)$ for all $(x, y) \in X \times Y$. The dual part of the Monge-Kantorovich problem then consists in maximizing

$$J(\varphi, \psi) = \int_X \varphi\, d\mu + \int_Y \psi\, d\nu \tag{3.2}$$

for $(\varphi, \psi) \in \Phi(\mu, \nu)$. In the literature duality has been established under various conditions, see for instance [Vil09, p 98f] for a short overview.

Theorem 3.1 (Monge-Kantorovich Duality)**.** *Let $(X, \mu), (Y, \nu)$ be Polish probability spaces and $c : X \times Y \to [0, \infty]$ be lower semi-continuous*[6] *(*[Kel84, Theorem 2.2]*). Then*

$$\inf\Big\{ \int c\, d\pi : \pi \in \Pi(\mu, \nu)\Big\} = \sup\Big\{J(\varphi, \psi) : (\varphi, \psi) \in \Phi(\mu, \nu)\Big\}. \tag{3.3}$$

Moreover the duality relation pertains if the optimization in the dual problem is restricted to bounded functions φ, ψ.

We will need the following straightforward corollary:

Corollary 3.2. *Let $\tilde c : X \times Y \times [0, t_0] \to \mathbb{R}$ be upper semi-continuous and bounded from above. Then*

$$\sup\Big\{ \int \tilde c\, d\pi : \pi \in \mathcal{P}(X \times Y \times [0, t_0]), \mathsf{proj}_X(\pi) = \mu, \mathsf{proj}_Y(\pi) = \nu\} \tag{3.4}$$

$$= \inf\Big\{J(\varphi, \psi) : (\varphi, \psi) \in L^\infty(\mu) \times L^\infty(\nu), \varphi(x) + \psi(y) \ge \tilde c(x, y, t)\Big\}. \tag{3.5}$$

A basic and important goal is to characterize minimizers through a tractable property of their support sets: a Borel set $\Gamma \subseteq X \times Y$ is *c-cyclically monotone* iff

$$c(x_1, y_2)-c(x_1, y_1) + \ldots + c(x_{n-1}, y_n)-c(x_{n-1}, y_{n-1}) + c(x_n, y_1)-c(x_n, y_n) \ge 0 \tag{3.6}$$

whenever $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) \in \Gamma$. A transport plan π is *c-cyclically monotone* if it assigns full measure to some cyclically monotone set Γ.

Concerning the origins of c-cyclical monotonicity in convex analysis and the study of the relation to optimality we mention [Roc66, KS92, Rüs96, GM96]. Intuitively speaking,

[6] If c takes only values in $[0, \infty)$, then it suffices to assume plain measurability ([BS09]).

c-cyclically monotone transport plans resist improvement by means of cyclical rerouting and optimal transport plans are expected to have this property. Indeed we have:

Theorem 3.3. *Let $c : X \times Y \to \mathbb{R}_+$ be a lower semi-continuous cost function. Then a transport plan is optimal if and only if it is c-cyclically monotone.*

Even in the case where c is the squared Euclidean distance this a non trivial result, posed as an open question by Villani in [Vil03, Problem 2.25]. Following contributions of Ambrosio and Pratelli [AP03], this problem was resolved by Pratelli [Pra08] and Schachermayer-Teichmann [ST09] who established the clear-cut characterization stated in Theorem 3.3.[7]

Notably Theorem 1.3 above is only a first step towards a full *characterization* of optimality as provided in Theorem 3.3. To obtain a necessary and sufficient condition for optimality in the Skorokhod embedding case (1.3) an extension from *pairs* to a finite number of stopped paths will be required.

4. Preliminaries on stopping times and filtrations

4.1. **Spaces and Filtrations.** In this section we mainly discuss the formal aspects of filtrations, measure theory, etc. Confident readers might want to skip this section.

We consider the space $\Omega = C(\mathbb{R}_+)$ of continuous paths with the topology of uniform convergence on compact sets. The elements of Ω will be denoted by ω. As explained above we consider the set S of all continuous functions defined on some initial segment $[0, s]$ of $\mathbb{R}_+$; we will denote the elements of S by (f, s) and (g, t). The set S admits a natural partial ordering; we say that (g, t) extends (f, s) if $t \geq s$ and the restriction $g_{\llcorner[0,s]} = f$. In this case we write $(f, s) \prec (g, t)$.

For two sets A, B the projection from $A \times B$ to A (resp. B) will be denoted by proj_A (resp. proj_B). For a map $T : X \to Y$ and a measure μ on X the push forward of μ by T will be denoted by $T(\mu)$. The set of all probability (resp. sub-probability) measures on a Polish space Z will be denoted by $\mathcal{P}(Z)$ (resp. $\mathcal{P}^{\leq 1}(Z)$). The set of all finite nonnegative measures on a set Z will be denoted by $\mathcal{M}(Z)$. The complement of a set A will be denoted by $\complement A$.

For our arguments it will be important to be precise about the relationship between the sets $C(\mathbb{R}_+) \times \mathbb{R}_+$ and S. We therefore discuss the underlying filtrations in some detail.

We consider three different filtrations on the Wiener space $C(\mathbb{R}_+)$, the canonical or natural filtration $\mathcal{F}^0 = (\mathcal{F}^0_t)_{t\in\mathbb{R}_+}$, the right-continuous filtration $\mathcal{F}^+ = (\mathcal{F}^+_t)_{t\in\mathbb{R}_+}$, and the augmented filtration $\mathcal{F}^a = (\mathcal{F}^a_t)_{t\in\mathbb{R}_+}$ obtained from $(\mathcal{F}^0_t)_{t\in\mathbb{R}_+}$ by including all $\mathbb{W}$-null sets in $\mathcal{F}^0_0$. As the Brownian motion is a continuous Feller process, $\mathcal{F}^a$ is automatically right-continuous, all $\mathcal{F}^a$-stopping times are predictable and all right-continuous $\mathcal{F}^a$-martingales are continuous. In particular, the $\mathcal{F}^a$-optional and the $\mathcal{F}^a$-predictable σ-algebra coincide (see e.g. [RY99, Corollary IV 5.7]). This will allow us to use the following result.

Theorem 4.1 ([DM78, Theorem 78])**.** *For every $\mathcal{F}^a$-predictable process $(X_t)_{t\in\mathbb{R}_+}$ there is an $\mathcal{F}^0$-predictable process $(X'_t)_{t\in\mathbb{R}_+}$ which is indistinguishable from $(X_t)_{t\in\mathbb{R}_+}$. If τ is an $\mathcal{F}^a$-stopping time, there exists an $\mathcal{F}^0$-stopping time τ' such that $\tau = \tau'$ a.s.*

Of course, every $\mathcal{F}^a$-martingale has a continuous version. Not so commonly used but entirely straightforward is the following: if M is an $\mathcal{F}^0$-martingale then there is a version M' of M which is an $\mathcal{F}^0$-martingale and almost all paths of M' are continuous.

[7] We refer to [BGMS09] and [BC10] for more general results, in particular it turns out that lower semi-continuity of the cost function is not required.

The message of the proposition below is that a process $(X_t)_{t\in\mathbb{R}_+}$ is $\mathcal{F}^0$-predictable iff $X_t(\omega)$ can be calculated from the restriction $\omega_{\llcorner[0,t]}$. We introduce the mapping

$$r : C(\mathbb{R}_+)\times\mathbb{R}_+ \to S, \quad r(\omega, t) = (\omega_{\llcorner[0,t]}, t). \tag{4.1}$$

Note that the topology on S introduced in (1.4) coincides with the terminal topology induced by the mapping r; in particular r is continuous.

Proposition 4.2. *(1) A set $D \subseteq C(\mathbb{R}_+)\times\mathbb{R}_+$ is $\mathcal{F}^0$-predictable iff there is a Borel set $A \subseteq S$ with $D = r^{-1}(A)$.*
(2) A process $X = (X_t)_{t\in\mathbb{R}_+}$ is $\mathcal{F}^0$-predictable iff there is a Borel measurable $H : S \to \mathbb{R}$ such that $X = H \circ r$.

Remark 4.3. *In the following we will say that $H : S \to \mathbb{R}$ is* continuous / right-continuous */ etc. if the corresponding property holds for the process $H \circ r$. Similarly we say that $H_1, H_2 : S \to \mathbb{R}$ are* indistinguishable *if this holds for the processes $H_1 \circ r, H_2 \circ r$ w.r.t. Wiener measure. We will also often used the notation $H(\omega_{\llcorner[0,t]}) = H(\omega_{\llcorner[0,t]}, t)$.*

To establish Proposition 4.2 we use a result from [DM78]. Adjoin to $\mathbb{R}$ a coffin state ð. We let D be the set of all cadlag paths with *lifetime*, i.e. all cadlag functions $f : \mathbb{R}_+ \to \mathbb{R}\cup\{ð\}$ for which $\{t : f(t) = \infty\}$ is a closed, possibly empty, half line $[a, \infty)$. Define a killing operation $k_t : D \to D$ by

$$(k_t\omega)_s = \begin{cases} \omega_s & \text{if } s < t \\ ð & \text{if } s \geq t. \end{cases}$$

Proposition 4.2 then follows from the following result.

Theorem 4.4 ([DM78, Theorem 97])**.** *Let $Z = (Z_t)_{t\in\mathbb{R}_+}$ be an adapted process with paths in D. Then Z is $\mathcal{F}^0$-predictable iff $Z_t = Z_t \circ k_t$ for all $t \in \mathbb{R}_+$.*

We will primarily work with the *natural filtration* $\mathcal{F}^0$ on $\Omega = C(\mathbb{R}_+)$.

4.2. **Preliminaries on stopping times.** Working on the path space $C(\mathbb{R}_+)$, a stopping time τ is a mapping which assigns to each path ω the time $\tau(\omega)$ at which the path is stopped. Assuming that a stopping time depends on some external randomization we may think that a path ω is not stopped at a particular point $\tau(\omega)$ but rather that there exists a subprobability μ_ω on $\mathbb{R}$ such that the path ω is stopped randomly according to the law μ_ω. Let us make this idea precise.

We consider the space

$$\mathsf{M} := \{\mu \in \mathcal{P}^{\leq 1}(C(\mathbb{R}_+)\times\mathbb{R}_+) : \mu(d\omega, dt) = \mu_\omega(dt)\mathbb{W}(d\omega), \mu_\omega \in \mathcal{P}^{\leq 1}(\mathbb{R}) \text{ for } \mathbb{W} \text{ a.e. } \omega\},$$

where $(\mu_\omega)_{\omega\in\Omega}$ is a disintegration of μ w.r.t. the first coordinate $\omega \in \Omega$.

We equip M with the weak topology induced by the continuous bounded functions on $C(\mathbb{R}_+)\times\mathbb{R}_+$.

In particular, we will be interested in the subset RST of all elements which are "adapted". Formally we define the set RST of all *randomized stopping times*[8] to consist of all $\mu \in \mathsf{M}$ satisfying one of the equivalent properties in the following theorem.

Theorem 4.5. *Let $\mu \in \mathsf{M}$. Then the following are equivalent:*

[8] The relation of usual (non-randomized) stopping times and randomized stopping times is analogous to the relation of (Monge) transport-maps to (Kantorovich) transport-plans in theory of optimal transport.

(1) *There is a Borel function* $H : S \to [0,1]$ *such that* H *is right-continuous, decreasing and*

$$\mu_\omega([0,s]) := 1 - H(\omega_{\llcorner[0,s]}) \tag{4.2}$$

defines a disintegration of μ *w.r.t. to* $\mathbb{W}$.

(2) *For every disintegration* $(\mu_\omega)_{\omega\in\Omega}$ *of* μ, *for all* $t \in \mathbb{R}_+$ *and every Borel set* $A \subseteq [0,t]$ *the random variable*

$$X_t(\omega) = \mu_\omega(A)$$

is $\mathcal{F}^a_t$*-measurable.*

(3) *There is a disintegration* $(\mu_\omega)_{\omega\in\Omega}$ *of* μ *such that for all* $t \in \mathbb{R}_+$ *and all* $f \in C_b(\mathbb{R}_+)$ *such that the support of* f *lies in* $[0,t]$ *the random variable*

$$X_t(\omega) = \mu_\omega(f)$$

is $\mathcal{F}^0_t$*-measurable.*

(4) *There exists a Brownian motion* B' *on some stochastic basis* $\Omega' = (\Omega', \mathcal{F}, (\mathcal{F}'_t), \mathbb{P}')$ *and a stopping time* τ' *on* Ω' *such that the function*

$$\Phi : \Omega' \to C(\mathbb{R}_+) \times \mathbb{R}_+, \quad \Phi(\omega', t') = ((B'_s)_{s\in\mathbb{R}_+}(\omega'), \tau'(\omega))$$

satisfies $\mu = \Phi(\mathbb{P}')$.

Proof. The argument is fairly straightforward. We establish that (2) implies (1). Consider a disintegration $(\mu_\omega)_{\omega\in\Omega}$ of μ. Define a process $\bar{H}$ by

$$\bar{H}_t(\omega) := 1 - \mu_\omega([0,t]).$$

Then $\bar{H}$ is right continuous and $\mathcal{F}^a$-adapted, hence $\mathcal{F}^a$-progressive. By Theorem 4.1 and Proposition 4.2 there exist a Borel function H on S such that $\bar{H}$ is indistinguishable from $H \circ r$. This function H is as required.

□

Remark 4.6. (1) *The function* H *in* (4.2) *is unique up to indistinguishability (cf. Remark 4.3). We will designate this function* H^μ *in the following. This function has a natural interpretation.* $H^\mu(f,s)$ *is the probability that a particle is still alive at time* s *given that it has followed the path* f. *We call* H^μ *the* lively-hood function *associated to* μ.

(2) *We will say* μ *is a non-randomized stopping time or a deterministic stopping time iff there is a disintegration* $(\mu_\omega)_{\omega\in\Omega}$ *of* μ *such that* μ_ω *is a Dirac-measure (of mass 1) for every* ω. *Clearly this means that* $\mu_\omega = \delta_{\tau(\omega)}$ *a.s. for some usual stopping time* τ. μ *is a deterministic stopping time iff there is a version of* H^μ *which only attains the values* 0 *and* 1.

Definition 4.7. *A randomized stopping time is* finite *iff* $\mu(C(\mathbb{R}_+) \times \mathbb{R}_+) = 1$. *The set of all finite randomized stopping times will be denoted by* RST^1.

Note, that on the set RST equipped with the topology inherited from M, it is sufficient to consider continuous, bounded, adapted processes as test functions.

Proposition 4.8. *The set* RST *is closed.*

Proof. Assume that a sequence $(\mu^n)_{n\in\mathbb{N}}$ in RST tends to μ. Fix a continuous bounded random variable $Y : C(\mathbb{R}_+) \to \mathbb{R}_+$ and a continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$ which has support in $[0,t]$. Then, using the above notation,

$$\mathbb{E}_{\mathbb{W}}[YX_t^n] = \int_{C(\mathbb{R}_+)} Y(\omega)\mu_\omega^n(f)\,\mathbb{W}(d\omega) = \int_{C(\mathbb{R}_+)} Y(\omega)\int_{\mathbb{R}_+} f(t)\,\mu_\omega^n(dt)\,\mathbb{W}(d\omega) \tag{4.3}$$

$$= \int_{C(\mathbb{R}_+)\times\mathbb{R}_+} Y(\omega)f(t)\,\mu^n(d\omega,dt) \to \int_{C(\mathbb{R}_+)\times\mathbb{R}_+} Y(\omega)f(t)\,\mu(d\omega,dt) = \mathbb{E}_{\mathbb{W}}[YX_t]. \tag{4.4}$$

That is, X_t^n converges to X_t weakly in $L^2(\Omega,\mathbb{W})$. All X_t^n are $\mathcal{F}_t^0$-measurable, hence also X_t can be taken to be $\mathcal{F}_t^0$-measurable.

□

4.3. **Randomized Stopping of Martingales.** Given $\mu \in \mathsf{M}$ and $s \in \mathbb{R}_+$ we define the measure $\mu\wedge s \in \mathsf{M}$ to be the random time which is the minimum of μ and s; formally this means that for $\omega\in\Omega$ and $A\subseteq\mathbb{R}_+$

$$(\mu\wedge s)_\omega(A) := \mu_\omega(A\cap[0,s)) + \delta_s(A)(1-\mu_\omega([0,s))).$$

Assume that $(U_s)_{s\in\mathbb{R}_+}$ is a process on Ω. Then the stopped process (U_s^μ) is given by

$$U_s^\mu(\omega) := \int U_t(\omega)\,(\mu\wedge s)_\omega(dt),$$

where $((\mu\wedge s)_\omega)_{\omega\in C(\mathbb{R}_+)}$ denotes a disintegration of $\mu\wedge s$.

Recall (Definition 4.7) that the random time μ is *finite* if the measure μ has mass one. If $(U_t)_{t\geq 0}$ is uniformly integrable, then we may consider

$$U_\mu := U_\infty^\mu = \lim_{s\to\infty} U_s^\mu.$$

Of course the optional stopping theorem applies:

Proposition 4.9. *Let $\mu \in \mathsf{RST}$ and let $(M_t)_{t\in\mathbb{R}_+}$ be a martingale. Then $(M_t^\mu)_{t\in\mathbb{R}_+}$ is a martingale and we have*

$$M_0 = \mathbb{E}_{\mathbb{W}}[M_t^\mu].$$

If $(M_t)_{t\in\mathbb{R}_+}$ is uniformly integrable then

$$M_0 = \mathbb{E}_{\mathbb{W}}[M_\mu].$$

Subsequently we will use that this property actually *characterizes* whether a given random time is a stopping time:

Proposition 4.10. *(1) Let $\mu \in \mathsf{M}$. Then $\mu \in \mathsf{RST}$ iff for every martingale $M = (M_t)_{t\in\mathbb{R}_+}$ and all $t\in\mathbb{R}_+$*

$$M_0 = \mathbb{E}_{\mathbb{W}}[M_t^\mu]. \tag{4.5}$$

(2) If μ is a finite time, then $\mu\in\mathsf{RST}$ iff

$$M_0 = \mathbb{E}_{\mathbb{W}}[M_\mu] \tag{4.6}$$

for every uniformly integrable martingale M.

Subsequently we will often use the following notation: Let $f : C(\mathbb{R}_+) \to \mathbb{R}$ be a continuous bounded function. Then we write f^M for the $\mathcal{F}^0$-martingale defined through

$$f_t^M := \mathbb{E}[f|\mathcal{F}_t^0] \tag{4.7}$$

whose paths are almost surely continuous.

Then, by Proposition 4.10, $\mu \in \mathsf{M}$ is a randomized stopping time if and only if for all continuous bounded functions $f : C(\mathbb{R}_+) \to \mathbb{R}$

$$\int_{C(\mathbb{R}_+)\times\mathbb{R}_+} f(\omega)\, d\mu(\omega,t) = \int_{C(\mathbb{R}_+)\times\mathbb{R}_+} f_t^M(\omega)\, d\mu(\omega,t).$$

Proof of Proposition 4.10. We show (2), the proof of (1) is the same. The first implication follows from optional stopping, so assume that (4.6) holds for all uniformly integrable martingales. If μ is not a randomized stopping time then for some t there exist an $\mathcal{F}_t^0 \otimes \mathcal{B}(\mathbb{R}_+)$-measurable set $A \subseteq C(\mathbb{R}_+) \times \mathbb{R}_+$ and a Borel set $D \in \mathcal{F}_\infty$ such that

$$\mu(A \cap (\theta_t^{-1}(D) \times \mathbb{R})) \neq \mu(A)\mathbb{W}(D),$$

where $\theta_t : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ is defined by

$$\theta_t(\omega) = (\omega_s - \omega_t)_{s\geq t}.$$

It follows that there also exist a bounded continuous function $H : C(\mathbb{R}_+) \to \mathbb{R}$ such that

$$\int \mathbb{1}_A(\omega,t)\cdot H\circ\theta_t(\omega)\,\mu(d\omega,dt)) \neq \mu(A)\mathbb{E}_{\mathbb{W}}[H].$$

Subtracting $\mathbb{E}_{\mathbb{W}}[H]$ from H, we may here assume that $\mathbb{E}_{\mathbb{W}}[H] = 0$. We then define a bounded martingale M by

$$M_\infty(\omega) := \mathbb{1}_A(\omega)\cdot H\circ\theta_t(\omega), \quad M_t := \mathbb{E}[M_\infty|\mathcal{F}_t^a].$$

Then $M_0 = 0$ while

$$\mathbb{E}[M_\mu] = \int \mathbb{1}_A \cdot H\circ\theta_t\,\mu(d\omega) \neq 0.$$

This is the desired contradiction. □

4.4. **Relation with Skorokhod-Embedding.** As is customary in the theory of Skorokhod embedding we consider stopping times which are *minimal*, that is, we are interested in finite randomized stopping times μ such that $(B_t^\mu)_{t\in\mathbb{R}_+}$ is uniformly integrable. Given a centered probability measure λ on $\mathbb{R}$ we denote by

$$\mathsf{RST}(\lambda)$$

the set of minimal stopping times μ such that $B_\mu \sim \lambda$. From now on we make the assumption that λ has finite second moment[9]

$$V := \int x^2\,\lambda(dx) < \infty. \tag{4.8}$$

Denote by T the projection

$$T : C(\mathbb{R}_+)\times\mathbb{R}_+ \to \mathbb{R}_+.$$

For $\mu \in \mathsf{RST}(\lambda)$ we then have (by uniform integrability of (B_t^μ) and Jensen's inequality)

$$\mathbb{E}_\mu[T] = \lim_{t\to\infty}\mathbb{E}_{\mu\wedge t}[T] = \lim_{t\to\infty}\mathbb{E}_{\mathbb{W}}[\langle B\rangle_{\mu\wedge t}] = \lim_{t\to\infty}\mathbb{E}_{\mathbb{W}}[B^2_{\mu\wedge t}] = \lim_{t\to\infty}\mathbb{E}_{\mathbb{W}}[\mathbb{E}[B_\mu|\mathcal{F}_t]^2] \leq \mathbb{E}[B_\mu^2] = V.$$

On the other hand if $B_\mu \sim \lambda$ then $\mathbb{E}_\mu[T] < \infty$ implies that $(B_t^\mu)_{t\leq\infty}$ is uniformly integrable: $(B_t)^2 - t =: M_t$ defines a martingale and

$$0 = \mathbb{E}_{\mathbb{W}}[M_0] = \mathbb{E}_{\mathbb{W}}[M_t^\mu] = \mathbb{E}_{\mathbb{W}}[(B_t^\mu)^2] - \mathbb{E}_\mu[T\wedge t].$$

Hence $\mathbb{E}_{\mathbb{W}}[(B_t^\mu)^2] \leq \mathbb{E}_\mu T$ such that $(B_t^\mu)_{t\in\mathbb{R}_+}$ is an L^2-martingale and a posteriori $\mathbb{E}_\mu[T] = \mathbb{E}_{\mathbb{W}}[B_\mu^2] = V$.

[9] Modulo some technicalities one could replace this condition by assuming that λ has only finite first moment.

Summing up we obtain the following fact (which is of course well known in theory of Skorokhod-embedding):

Lemma 4.11. *Using the above notations and assumptions, the following are equivalent for $\mu \in \mathsf{RST}$ with $B_\mu = \lambda$:*

(1) μ *is minimal.*
(2) $\mathbb{E}_\mu[T] < \infty$.
(3) $\mathbb{E}_\mu[T] = V$.

The main reason why we consider *randomized* stopping times is that they have the following property:

Theorem 4.12. *The set* $\mathsf{RST}(\lambda)$ *is compact.*

Proof. By Prohorov's theorem we have to show that $\mathsf{RST}(\lambda)$ is tight and that $\mathsf{RST}(\lambda)$ is closed.

Tightness.

Fix $\varepsilon > 0$ and take R so large that $V/R \leq \varepsilon/2$. Then, for any $\mu \in \mathsf{RST}(\lambda)$ we have $\mu(T > R) \leq \varepsilon/2$. As $C(\mathbb{R}_+)$ is Polish there is a compact set $\tilde{K} \subseteq C(\mathbb{R}_+)$ s.t. $\mathbb{W}(\complement \tilde{K}) \leq \varepsilon/2$. Put $K := \tilde{K} \times [0, R]$. Then K is compact and we have for any $\mu \in \mathsf{RST}(\lambda)$

$$\mu(\complement K) \leq \mathbb{W}(\complement \tilde{K}) + \mu(T > R) \leq \varepsilon.$$

Hence, $\mathsf{RST}(\lambda)$ is tight.

Closedness. Take a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathsf{RST}(\lambda)$ converging to some μ. Putting $h : C(\mathbb{R}_+) \times \mathbb{R}_+, (\omega, t) \mapsto \omega(t)$ we have to show that $h(\mu) = \lambda$ and that $\mathbb{E}_\mu[T] < \infty$. Note that h is a continuous map. Take any $g \in C_b(\mathbb{R})$. Then $g \circ h \in C_b(C(\mathbb{R}_+) \times \mathbb{R}_+)$. Thus, we have that

$$\int g \, d\lambda = \lim_n \int_{C(\mathbb{R}_+)\times\mathbb{R}_+} g \circ h \, d\mu_n = \int_{C(\mathbb{R}_+)\times\mathbb{R}_+} g \circ h \, d\mu = \int g \, dh(\mu).$$

Hence, we have $h(\mu) = \lambda$. Moreover, the set $\{(\omega, t) : t \leq L\}$ is closed. Hence, by the Portmanteau Theorem we have that for any $L \geq 0$

$$\limsup \mu_n(t \leq L) \leq \mu(t \leq L).$$

This readily implies that $\mathbb{E}_\mu[T] \leq \limsup \mathbb{E}_{\mu_n}[T] = V < \infty$. □

4.5. **Joinings / Tagged Stopping Times.** We now add another dimension: assume that (Y, ν) is some Polish probability space. The set of all *tagged random times* or *joinings* $\mathsf{JOIN}(\mathbb{W}, \nu) = \mathsf{JOIN}(\nu)$ is given by

$$\Big\{\pi \in \mathcal{P}^{\leq 1}(C(\mathbb{R}_+) \times \mathbb{R}_+ \times Y), \mathsf{proj}_{C(\mathbb{R}_+)\times\mathbb{R}_+}(\pi_{\llcorner C(\mathbb{R}_+)\times\mathbb{R}_+\times B}) \in \mathsf{RST}, B \in \mathcal{B}(\mathbb{R}), \mathsf{proj}_Y(\pi) \leq \nu\Big\}.$$

We shall also write $\mathsf{JOIN}^1(\mathbb{W}, \nu)/\mathsf{JOIN}^1(\nu)$ for the subset of measures which have mass 1.

Remark 4.13. *Write* pred *for the σ-algebra of $\mathcal{F}^0$-predictable sets in $C(\mathbb{R}_+) \times \mathbb{R}_+$.*

We call a set $A \subseteq C(\mathbb{R}_+) \times \mathbb{R}_+ \times Y$ predictable if it is an element of $\mathsf{pred} \otimes \mathcal{B}(Y)$. *We will say that a function defined on $C(\mathbb{R}_+) \times \mathbb{R}_+ \times Y$ is predictable if it is measurable w.r.t.* $\mathsf{pred} \otimes \mathcal{B}(Y)$.

5. The Optimization Problem and Duality

5.1. **The Primal Problem.**

Recall that we consider a function $\gamma : S \to \mathbb{R}_+$ which is *continuous* –(or at least upper semi-continuous).

A randomized stopping time μ gives rise to the probability measure $\mu_S := r(\mu)$. Given an $\mathcal{F}^a$-predictable function $\tilde{\gamma}$ on $C(\mathbb{R}_+) \times \mathbb{R}_+$ we can find a Borel function γ on S such that $\gamma \circ r$ is indistinguishable from $\tilde{\gamma}$ and then

$$\int_{C(\mathbb{R}_+)\times\mathbb{R}_+} \tilde{\gamma}(\omega, t)\, \mu(d(\omega, t)) = \int_S \gamma(f, s)\, \mu_S(d(f, s)). \tag{5.1}$$

As long as there is no danger of confusion we will not distinguish strictly between $\tilde{\gamma}$ and γ as well as μ and μ_S, respectively.

We assume that there exists at least one $\mu \in \mathsf{RST}(\lambda)$ which satisfies that

$$\int \gamma(\omega, t)\, d\mu(\omega, t) > -\infty. \tag{5.2}$$

and that the integral in (5.2) is less than ∞ for all $\mu \in \mathsf{RST}(\lambda)$. The maximization problem introduced in the introduction can then also be written as

$$P_\gamma(\mathbb{W}, \lambda) = \sup\left\{\int \gamma(\omega, t)\, d\mu(\omega, t), \mu \in \mathsf{RST}(\lambda)\right\}. \tag{5.3}$$

It is straightforward to see that the functional (5.1) is upper semi-continuous provided that γ is (upper semi-) continuous. (This is spelled out in detail for instance in [Vil09, Chapter 4] in the context of classical optimal transport.) In particular (5.3) then admits an optimizer according to the compactness properties derived above.

5.2. **The dual problem.**

Theorem 5.1. *Let $\gamma : C(\mathbb{R}_+) \times \mathbb{R}_+ \to \mathbb{R}$ be upper semi-continuous, bounded from above and predictable. Put*

$$D_\gamma(\mathbb{W}, \lambda) = \inf\left\{\int \psi(y)\, d\lambda(y) : \psi \in L^1(\lambda), \exists\varphi, \begin{array}{l} \varphi \text{ is a continuous martingale}, \varphi_0 = 0 \\ \varphi_t(\omega) + \psi(\omega(t)) \geq \gamma(\omega, t) \text{ for all } t,\ \mathbb{W}\text{-a.s.} \end{array}\right\}$$

where φ runs through all continuous $\mathcal{F}^a$-martingales with $\mathbb{E}\varphi_t^2 \leq at + b$ for some $a, b > 0$. Then we have the duality relation

$$P_\gamma(\mathbb{W}, \lambda) = D_\gamma(\mathbb{W}, \lambda). \tag{5.4}$$

By the martingale-representation theorem, Theorem 5.1 and Theorem 1.2 are equivalent.

As usual the inequality $P_\gamma(\mathbb{W}, \lambda) \leq D_\gamma(\mathbb{W}, \lambda)$ is straight-forward to verify.

Remark 5.2. *The assumption that γ is bounded from above can be relaxed: Assume that φ, ψ is an admissible dual pair such that $\psi : \mathbb{R} \to \mathbb{R}, \varphi : C(\mathbb{R}_+) \times \mathbb{R}_+ \to \mathbb{R}_+$ are lower semi-continuous. Then the duality relation for the function γ follows from Theorem 5.1 if we consider $\tilde{\gamma}(\omega, t) := \gamma(\omega, t) - (\varphi_t(\omega) + \psi(\omega(t)))$.*

A more natural assumption on the function γ would be that $D_\gamma(\mathbb{W}, \lambda) < \infty$ but presently we are not able to establish Theorem 5.1 in this case.

The key idea for the proof of Theorem 5.1 is to translate the embedding problem for λ into a transportation problem between the Wiener measure $\mathbb{W}$ and λ using the cost function

$$c(\omega,t,y) = \begin{cases} \gamma(\omega,t) & \text{if } \omega(t)=y \\ -\infty & \text{else,} \end{cases}$$

for $(\omega,t,y) \in C(\mathbb{R}_+)\times\mathbb{R}_+\times\mathbb{R}$. The result of choosing this special cost function is that $P_\gamma(\mathbb{W},\lambda) = P_c^{C(\mathbb{R}_+)\times\mathbb{R}_+\times\mathbb{R}}(\mathbb{W},\lambda)$, where

$$P_c^{C(\mathbb{R}_+)\times\mathbb{R}_+\times\mathbb{R}}(\mathbb{W},\lambda) = \sup\left\{\int c(\omega,t,y)\,\pi(d\omega,dt,dy), \pi\in \mathsf{JOIN}^1(\mathbb{W},\lambda), \mathbb{E}_\pi[T]\le V\right\}, \tag{5.5}$$

where T is the projection on $\mathbb{R}_+$, $V=\int x^2\,\lambda(dx)$ and we used $Y=\mathbb{R}$ in the definition of $\mathsf{JOIN}^1(\mathbb{W},\lambda)$ (see Section 4.5).

To see this, define $p(\omega,t,y)=(\omega,t)$. Then, if $\pi\in\mathsf{JOIN}^1(\mathbb{W},\lambda)$ is concentrated on $\{(\omega,t,y):\omega(t)=y\}$ we have $\mu=p(\pi)\in\mathsf{RST}(\lambda)$ and $\int c\,d\pi=\int\gamma\,d\mu$.

On the other hand, let $h(\omega,t)=\omega(t)$. If $\mu\in\mathsf{RST}(\lambda)$ then $\pi=(id,h)(\mu)\in\mathsf{JOIN}^1(\mathbb{W},\lambda)$ and as before $\int c\,d\pi=\int\gamma\,d\mu$.

In Proposition 5.6 we will establish a dual problem corresponding to $P_c^{C(\mathbb{R}_+)\times\mathbb{R}_+\times\mathbb{R}}(\mathbb{W},\lambda)$ and Theorem 5.1 will then be a simple consequence. However we need some preparations before we can establish Proposition 5.6.

5.3. **A Non-Adapted (NA) Duality Result.**

We first prove a “non adapted version” of the desired result and afterwards we use the min-max Theorem 5.4 to introduce adaptedness. To this end, put

$$\mathsf{TM}^V(\mathbb{W},\lambda) = \{\pi\in\mathcal{P}(C(\mathbb{R}_+)\times\mathbb{R}_+\times\mathbb{R}) : \mathsf{proj}_{C(\mathbb{R}_+)}(\pi)=\mathbb{W}, \mathsf{proj}_{\mathbb{R}}(\pi)=\lambda, \mathbb{E}_\pi[T]\le V\}, \tag{5.6}$$

and

$$\mathsf{DC}^V_{NA}(c) = \left\{(\varphi,\psi)\in L^\infty(\mathbb{W})\times L^\infty(\lambda) : \exists\alpha\ge 0, \begin{array}{l}\varphi(\omega)+\psi(y)+\alpha(t-V)\ge c(\omega,t,y) \\ \mathbb{W}\text{-a.s. for all } y\in\mathbb{R}, t\ge 0\end{array}\right\}.$$

Note that the set TM^V is compact as a consequence of Prohorov's Theorem.

Proposition 5.3. *Let $c : C(\mathbb{R}_+)\times\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}\cup\{-\infty\}$ be upper semi-continuous and bounded from above. Then*

$$P_c^{NA} = \sup_{\pi\in\mathsf{TM}^V(\mathbb{W},\lambda)}\int c\,d\pi = \inf_{(\varphi,\psi)\in\mathsf{DC}^V_{NA}(c)}\mathbb{W}(\varphi)+\lambda(\psi) = D_c^{NA}. \tag{5.7}$$

Again it is easy to show that $D_c^{NA}\ge P_c^{NA}$. To show the other inequality we first collect some ingredients which will also be useful later on. In particular, we will use the min-max theorem in the following form.

Theorem 5.4 (see e.g. [Str85, Thm. 45.8] or [AH96, Thm. 2.4.1])**.** *Let K,L be convex subsets of vector spaces H_1 resp. H_2, where H_1 is locally convex and let $F : K\times L\to\mathbb{R}$ be given. If*

(1) K is compact,
(2) $F(\cdot,y)$ is continuous and concave on K for every $y\in L$,
(3) $F(x,\cdot)$ is convex on L for every $x\in K$

then

$$\inf_{y\in L}\sup_{x\in K}F(x,y) = \sup_{x\in K}\inf_{y\in L}F(x,y).$$

Lemma 5.5. *If* (5.7) *is valid for a sequence of continuous bounded functions $c_n, n \geq 1$ such that $c_n \downarrow c$ then* (5.7) *applies also to c.*

Proof. To keep track of the different cost functions we write

$$P_{c_n}^{NA} = \sup_{\mathsf{TM}^V(\mathbb{W},\lambda)} \int c_n \, d\pi \quad \text{and} \quad D_{c_n}^{NA} = \inf_{(\varphi,\psi)\in \mathsf{DC}_{NA}^V(c_n)} (\mathbb{E}_{\mathbb{W}}[\varphi] + \mathbb{E}_\lambda[\psi]),$$

where $\mathsf{DC}_{NA}^V(c_n)$ is to remind us on the dependence of the dual constraint set on c_n. P_c^{NA} and D_c^{NA} are defined analogously. We have to prove that $D_c^{NA} \leq P_c^{NA}$. For each k let $\pi_k \in \mathsf{TM}^V(\mathbb{W},\lambda)$ be such that

$$P_{c_k}^{NA} \leq \int c_k \, d\pi_k + \frac{1}{k}.$$

By compactness of $\mathsf{TM}^V(\mathbb{W},\lambda)$ there is a subsequence, still denoted by k, such that $(\pi_k)_k$ converges weakly to some $\pi \in \mathsf{TM}^V(\mathbb{W},\lambda)$. Then by monotone convergence using the monotonicity of the sequence $(c_k)_{k\in\mathbb{N}}$ we have

$$\begin{aligned} P_c^{NA} \geq \int c \, d\pi = \lim_{m\to\infty} \int c_m \, d\pi &= \lim_{m\to\infty}\left(\lim_{k\to\infty} \int c_m \, d\pi_k\right) \\ &\geq \lim_{m\to\infty}\left(\lim_{k\to\infty} \int c_k \, d\pi_k\right) = \lim_{k\to\infty} P_{c_k}^{NA}. \end{aligned}$$

Since, $c_k \geq c$ implies $P_{c_k}^{NA} \geq P_c^{NA}$ and $D_c^{NA} \leq D_{c_k}^{NA}$ this allows us to deduce that

$$D_c^{NA} \leq D_{c_k}^{NA} = P_{c_k}^{NA} \searrow P_c^{NA}.$$

□

Proof of Proposition 5.3. We may assume that c is bounded from above by zero. Hence, by Lemma 5.5 it is sufficient to establish (5.7) for continuous functions whose support satisfies

$$\operatorname{supp} c \subseteq C(\mathbb{R}_+) \times [0,t_0] \times \mathbb{R} \tag{5.8}$$

for some $t_0 \in \mathbb{R}_+$. Put

$$\mathsf{TM}_{t_0}^V(\mathbb{W},\lambda) = \{\pi : \mathsf{proj}_{C(\mathbb{R}_+)}(\pi) = \mathbb{W}, \mathsf{proj}_{\mathbb{R}}(\pi) = \lambda, \mathbb{E}_\pi[T] \leq V, \operatorname{supp}\pi \subseteq C(\mathbb{R}_+) \times [0,t_0] \times \mathbb{R}\},$$

and

$$\mathsf{DC}_{NA,t_0}^V(c) = \left\{(\varphi,\psi) \in L^\infty(\mathbb{W}) \times L^\infty(\lambda) : \exists \alpha \geq 0, \begin{array}{l} \varphi(\omega) + \psi(y) + \alpha(t - V) \geq c(\omega,t,y) \\ \mathbb{W}\text{-a.s. for all } y \in \mathbb{R}, t \leq t_0 \end{array}\right\}.$$

Assume now that c satisfies (5.8) for some $t_0 \geq V$.

We then have

$$\sup_{\pi\in\mathsf{TM}^V(\mathbb{W},\lambda)} \int c \, d\pi = \sup_{\pi\in\mathsf{TM}_{t_0}^V(\mathbb{W},\lambda)} \int c \, d\pi \quad \text{and} \tag{5.9}$$

$$\inf_{(\varphi,\psi)\in\mathsf{DC}_{NA}^V(c)} \mathbb{W}(\varphi) + \lambda(\psi) = \inf_{(\varphi,\psi)\in\mathsf{DC}_{NA,t_0}^V(c)} \mathbb{W}(\varphi) + \lambda(\psi). \tag{5.10}$$

Formally the conditions involving V disappear in $\mathsf{TM}_{t_0}^V(\mathbb{W},\lambda)$ and $\mathsf{DC}_{NA,t_0}^V(c)$ if we put $V = \infty$, we therefore define

$$\mathsf{TM}_{t_0}^\infty(\mathbb{W},\lambda) = \{\pi : \mathsf{proj}_{C(\mathbb{R}_+)}(\pi) = \mathbb{W}, \mathsf{proj}_{\mathbb{R}}(\pi) = \lambda, \operatorname{supp}\pi \subseteq C(\mathbb{R}_+) \times [0,t_0] \times \mathbb{R}\},$$

$$\mathsf{DC}_{NA,t_0}^\infty(c) = \{(\varphi,\psi) \in L^\infty(\mathbb{W}) \times L^\infty(\lambda) : \varphi(\omega) + \psi(y) \geq c(\omega,t,y) \text{ for } t \leq t_0, y \in \mathbb{R}\ \mathbb{W}\text{-a.s.}\}$$

As a consequence of the classical Monge-Kantorovich duality theorem (3.1) we have (see Corollary 3.2)

$$\sup_{\pi\in\mathsf{TM}^{\infty}_{t_0}(\mathbb{W},\lambda)}\int\tilde c\,d\pi=\inf_{(\varphi,\psi)\in\mathsf{DC}^{\infty}_{NA,t_0}(\tilde c)}\mathbb{W}(\varphi)+\lambda(\psi)\tag{5.11}$$

for $\tilde c$ upper semi-continuous and bounded from above.

Using the min-max Theorem 5.4 with the function

$$F(\pi,\alpha)=\int c-\alpha(t-V)\,d\pi$$

for $\pi\in\mathsf{TM}^{\infty}_{t_0}(\mathbb{W},\lambda)$ and $\alpha\geq 0$ we thus obtain

$$\sup_{\pi\in\mathsf{TM}^{V}_{t_0}(\mathbb{W},\lambda)}\int c\,d\pi=\sup_{\pi\in\mathsf{TM}^{\infty}_{t_0}(\mathbb{W},\lambda)}\int c\,d\pi+\inf_{\alpha\geq 0}(-\alpha)\int t-V\,d\pi\tag{5.12}$$

$$=\inf_{\alpha\geq 0}\sup_{\pi\in\mathsf{TM}^{\infty}_{t_0}(\mathbb{W},\lambda)}\int c-\alpha(t-V)\,d\pi\tag{5.13}$$

$$=\inf_{\alpha\geq 0}\inf_{(\varphi,\psi)\in\mathsf{DC}^{\infty}_{NA,t_0}(c-\alpha(t-V))}\mathbb{W}(\varphi)+\lambda(\psi).\tag{5.14}$$

$$=\inf_{(\varphi,\psi)\in\mathsf{DC}^{V}_{NA,t_0}(c)}\mathbb{W}(\varphi)+\lambda(\psi),\tag{5.15}$$

where we have applied (5.11) to the function $\tilde c=c-\alpha(t-V)$ to establish the equality between (5.13) and (5.14).

This concludes the proof. □

5.4. **Introducing Adaptedness.**

We can test "adaptedness" of a measure $\pi\in\mathcal{P}(C(\mathbb{R}_+)\times\mathbb{R}_+\times\mathbb{R})$ by testing it against martingales: Put

$$\mathsf{JOIN}^{V}(\mathbb{W},\lambda)=\mathsf{JOIN}^{1}(\mathbb{W},\lambda)\cap\mathsf{TM}^{V}(\mathbb{W},\lambda).$$

For a continuous and bounded function $f:C(\mathbb{R}_+)\to\mathbb{R}_+$ we consider the martingale f^M defined through $f^M_t=\mathbb{E}[f|\mathcal{F}_t]$ as in (4.7). Then $\pi\in\mathsf{TM}(\mathbb{W},\lambda)$ satisfies $\pi\in\mathsf{JOIN}^1(\mathbb{W},\lambda)$ if and only if for all continuous bounded functions $f:C(\mathbb{R}_+)\to\mathbb{R}$, $g:\mathbb{R}\to\mathbb{R}$

$$\int fg\,d\pi=\int f^M g\,d\pi.\tag{5.16}$$

This follows in complete analogy to Proposition 4.10 and will be crucial for the subsequent argument.

Consider now the following set of dual candidates:

$$\mathsf{DC}^{V}(c)=\left\{(\varphi,\psi):\begin{array}{l}\varphi\text{ is a continuous bounded martingale, }\psi\in L^{\infty}(\lambda),\ \exists\alpha\geq 0,\\ \varphi_t(\omega)+\psi(y)+\alpha(t-V)\geq c(\omega,t,y),\ \mathbb{W}\text{-a.s. for all }y\in\mathbb{R},t\in\mathbb{R}_+\end{array}\right\}.$$

Then we can derive the following, adapted version of Proposition 5.3.

Proposition 5.6. *Let $c:C(\mathbb{R}_+)\times\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}\cup\{-\infty\}$ be upper semi-continuous, predictable (cf. Remark 4.13) and bounded from above. Then,*

$$P_c^{C(\mathbb{R}_+)\times\mathbb{R}_+\times\mathbb{R}}(\mathbb{W},\lambda)=\sup_{\pi\in\mathsf{JOIN}^{V}(\mathbb{W},\lambda)}\int c\,d\pi=\inf_{(\varphi,\psi)\in\mathsf{DC}^{V}(c)}\mathbb{W}(\varphi)+\lambda(\psi)=:D_c^{C(\mathbb{R}_+)\times\mathbb{R}_+\times\mathbb{R}}(\mathbb{W},\lambda).$$

Proof. Let us start with the case that c is continuous and bounded. The general case will follow by approximation, cf. Lemma 5.5. We will use again the notation $f_t^M = \mathbb{E}[f|\mathcal{F}_t]$. We want to use the min-max Theorem 5.4 with the function

$$F(\pi, h) = \int c \, d\pi + \int_{C(\mathbb{R}_+)\times\mathbb{R}_+} \bar{h} \, d\pi$$

for $\pi \in \mathsf{TM}^V(\mathbb{W}, \lambda)$ and

$$h(\omega, y) = \sum_{i=1}^{n} f_i(\omega) g_i(y), \quad \bar{h} = \sum_{i=1}^{n} (f_i - f_i^M) g_i, \tag{5.17}$$

where $n \in \mathbb{N}$, $f_i \in C_b(C(\mathbb{R}_+))$, $g_i \in C_b(\mathbb{R}_+)$.

The set $\mathsf{TM}^V(\mathbb{W}, \lambda)$ is convex and compact by Prohorov's theorem and the set of all $h \in C_b(C(\mathbb{R}_+)) \times C_b(\mathbb{R}_+)$ of the form (5.17) is convex as well. Then we have

$$\begin{aligned}
P_c^{C(\mathbb{R}_+)\times\mathbb{R}_+\times\mathbb{R}} &= \sup_{\pi\in\mathsf{JOIN}^V(\mathbb{W},\lambda)} \int c \, d\pi \\
&= \sup_{\pi\in\mathsf{TM}^V(\mathbb{W},\lambda)} \inf_h \left(\int c \, d\pi + \int_{C(\mathbb{R}_+)\times\mathbb{R}_+} \bar{h} \, d\pi \right) \\
&\overset{Thm5.4}{=} \inf_h \sup_{\pi\in\mathsf{TM}^V(\mathbb{W},\lambda)} \left(\int c + \bar{h} \, d\pi \right) \\
&= \inf_h \inf_{(\varphi,\psi)\in\mathsf{DC}_{NA}^V(c+\bar{h})} (\mathbb{W}(\varphi) + \lambda(\psi)) \,.
\end{aligned}$$

The last equality holds by Proposition 5.3. We set $c_h = c + \bar{h}$. For $(\varphi, \psi) \in \mathsf{DC}_{NA}^V(c_h)$ there is some $\alpha \geq 0$ such that

$$c_h(\omega, t, y) \leq \varphi(\omega) + \psi(y) + \alpha(t - V).$$

Taking conditional expectation w.r.t. $\mathcal{F}_t^0$ we get using predictability of c (cf. Remark 4.13)

$$c_h(\omega, t, y) = \mathbb{E}\Big[c + \textstyle\sum_{i=1}^n (f_i - f_{i,t}^M) g_i \,\Big|\, \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R})\Big](\omega, t, y) \leq \varphi_t^M(\omega) + \psi(y) + \alpha(t - K).$$

This implies that $(\varphi_t, \psi) \in \mathsf{DC}^V(c)$. Because $\mathbb{W}(\varphi_t^M) = \mathbb{W}(\varphi)$ this implies that $\mathsf{DC}_{NA}^V(c_h) \subseteq \mathsf{DC}^V(c)$. Therefore, we have

$$\begin{aligned}
P_c^{C(\mathbb{R}_+)\times\mathbb{R}_+\times\mathbb{R}} &= \inf_{h\in C_b(C)} \inf_{(\varphi,\psi)\in\mathsf{DC}_{NA}^V(c+\bar{h})} (\mathbb{W}(\varphi) + \lambda(\psi)) \\
&\geq \inf_{(\varphi,\psi)\in\mathsf{DC}^V(c)} (W(\varphi) + \lambda(\psi)) = D_c^{C(\mathbb{R}_+)\times\mathbb{R}_+\times\mathbb{R}}.
\end{aligned} \tag{5.18}$$

As usual, the other inequality is straightforward.

□

Proof of Theorem 5.1. We already saw in the beginning of this section that $P_\gamma(\mathbb{W}, \lambda) = P_c^{C(\mathbb{R}_+)\times\mathbb{R}_+\times\mathbb{R}}(\mathbb{W}, \lambda)$ if we set

$$c(\omega, t, y) = \begin{cases} \gamma(\omega, t) & \text{if } \omega(t) = y \\ -\infty & \text{else.} \end{cases}$$

Moreover, as γ was assumed to be upper semi-continuous, also c is upper semi-continuous. Indeed, take any sequence (ω_n, t_n, y_n) converging to (ω, t, y). If $\limsup_n c(\omega_n, t_n, y_n) = -\infty$ there is nothing to prove. On the other hand, if $\limsup_n c(\omega_n, t_n, y_n) > -\infty$ there is a subsequence $(\omega_{n_k}, t_{n_k}, y_{n_k})$ with $\omega_{n_k}(t_{n_k}) = y_{n_k}$ converging to some (ω, t, y). Then, we necessarily have that $\omega(t) = y$ because $|\omega(t)-y| \le |\omega(t)-\omega_{n_k}(t_{n_k})|+|y_{n_k}-y|$. Thus the upper semi-continuity of c follows from the upper semi-continuity of γ.

Hence, by Proposition 5.6 to see that

$$P_\gamma(\mathbb{W},\lambda) \ge D_\gamma(\mathbb{W},\lambda)$$

and it remains to show that $D_c^{C(\mathbb{R}_+)\times\mathbb{R}_+\times\mathbb{R}}(\mathbb{W},\lambda) \ge D_\gamma(\mathbb{W},\lambda)$. A bounded pair (φ,ψ) belongs to $\mathsf{DC}^V(c)$ iff there is $\alpha \ge 0$ such that $\mathbb{W}$-a.s. for all $y \in \mathbb{R}, t \in \mathbb{R}_+$

$$\varphi_t(\omega) + \psi(y) + \alpha(t-V) \ge c(\omega,t,y)$$

which holds iff

$$\varphi_t(\omega) + \psi(\omega(t)) + \alpha(t-V) \ge \gamma(\omega,t).$$

This is trivially equivalent to

$$\left[\varphi_t(\omega) - \alpha(\omega(t)^2-t)\right] + \left[\psi(\omega(t)) + \alpha\omega(t)^2 - \alpha V\right] \ge \gamma(\omega,t). \tag{5.19}$$

The alternative representation in (5.19) is useful to us since $\omega(t)^2 - t$ is a martingale.

Putting

$$\bar\varphi_t(\omega) = \varphi_t(\omega) - \alpha(\omega(t)^2-t) \quad \text{and} \quad \bar\psi(y) = \psi(y) + \alpha y^2 - \alpha V,$$

we have $\bar\varphi_t(\omega) + \bar\psi(\omega(t)) \ge \gamma(\omega,t)$. This means that $(\bar\varphi - \bar\varphi_0, \bar\psi + \bar\varphi_0)$ satisfy the constraint in the dual problem in (5.4). Recalling that V was defined by $V = \int y^2 \lambda(dy)$ we have $\int \bar\psi(y)\,\lambda(dy) = \int \psi(y)\,\lambda(dy)$. Therefore, we can conclude that

$$D_c^{C(\mathbb{R}_+)\times\mathbb{R}_+\times\mathbb{R}}(\mathbb{W},\lambda) \ge D_\gamma(\mathbb{W},\lambda).$$

□

6. Bad Pairs and Closed Stochastic Intervals

In the following, ν will always denote an optimizer of the primal optimization problem (5.3).

The notion of BP given in Definition 2.1 requires that *all* possible extensions (h,u) are considered. In this section we will also consider a weaker notion which is sensitive to the stopping measure ν. To this end we introduce the *conditional randomized stopping time given* (f,s).

Definition 6.1. *Let $\mu \in \mathsf{RST}$ be given and consider the livelyhood function H^μ as in Remark 4.6. The conditional randomized stopping time of μ, given $(f,s) \in S$, denoted by $\mu^{(f,s)}$, is defined to be*

$$\mu_\omega^{(f,s)}([0,t]) := \frac{1}{H(f,s)}\left(1 - H(f \oplus \omega_{\llcorner[0,t]}, s+t)\right), \tag{6.1}$$

if $H(f,s) > 0$ and 0 otherwise.

This is the normalized stopping measure given that we followed the path f up to time s. In other words this is the normalized stopping measure of the "bush" which follows the "stub" (f,s).

Definition 6.2. *The set of* bad pairs relative to ν *is defined by*

$$\mathsf{BP}_\nu = \Bigg\{((f,s),(g,t)) : f(s) = g(t), \tag{6.2}$$

$$\int \gamma(f\oplus\omega_{\llcorner[0,s+r]}, s+r)\,d\nu^{(f,s)}(\omega,r) + \gamma(g,t) < \gamma(f,s) + \int \gamma(g\oplus\omega_{\llcorner[0,t+r]}, t+r)\,d\nu^{(f,s)}(\omega,r)\Bigg\}.$$

The interpretation of BP_ν is that in average it is better to stop at (f, s), chop off the "bush" and transfer it onto the "stub" (g, t).

The following result constitutes an important intermediate step towards Theorem 1.3. In the formulation as well as in the proof we interpret the space $(C(\mathbb{R}_+)\times\mathbb{R}_+)\times(C(\mathbb{R}_+)\times\mathbb{R}_+)$ as a product $X \times Y$ so that we can make sense of the projections proj_X and proj_Y.

Proposition 6.3. *Let ν be a randomized stopping time which maximizes* (5.3). *Then $(Y, \nu) = (C(\mathbb{R}_+) \times \mathbb{R}_+, \nu)$ is a Polish probability space. Assume that $\pi \in \mathsf{JOIN}(\tau, \nu)$ (where τ can be arbitrary) satisfies*

$$H^{\mathsf{proj}_X(\pi)}(f, s) > 0 \quad \Longrightarrow \quad H^\nu(f, s) > 0 \quad \textit{for } (f, s) \in S. \tag{6.3}$$

Then we have $\pi(\mathsf{BP}_\nu) = 0$.

The interpretation of (6.3) is that if a particle has a strictly positive chance to be alive w.r.t. $\mathsf{proj}_X(\pi)$ then the probability that this particle is still alive w.r.t. ν is positive as well.

Proof. Note that, given $\nu' \in \mathsf{RST}(\lambda')$ and $\nu'' \in \mathsf{RST}(\lambda'')$, we have that $(\nu' + \nu'')/2 \in \mathsf{RST}((\lambda' + \lambda'')/2)$. The probabilistic interpretation of this easy fact goes by visualizing the random stopping time $(\nu' + \nu'')/2$ as flipping a coin at time $t = 0$ and subsequently either applying the randomized stopping rule ν' or ν''.

Working towards a contradiction we assume that there is $\pi \in \mathsf{JOIN}(\tau, \nu)$ such that $\pi(\mathsf{BP}_\nu) > 0$. Set $\nu_0 = \nu_1 := \nu$. We then use π to define two modification ν_0^π and ν_1^π of ν such that the following hold true:

(1) The terminal distributions λ_0, λ_1 corresponding to ν_0^π and ν_1^π satisfy $(\lambda_0^\pi+\lambda_1^\pi)/2 = \lambda$.
(2) ν_0^π stops paths earlier than $\nu_0 = \nu$ while ν_1^π stops later than $\nu_1 = \nu$.
(3) The cost of ν_0^π plus the cost of ν_1^π is less than twice the costs of ν, i.e.

$$\int \gamma(\omega_{\llcorner[0,t]}, t)\, d\nu_0(\omega, t) + \int \gamma(\omega_{\llcorner[0,t]}, t)\, d\nu_1(\omega, t) < 2 \int \gamma(\omega_{\llcorner[0,t]}, t)\, d\nu(\omega, t).$$

More formally, (2) asserts that for every $s \geq 0$,

$$(\nu_0^\pi)_\omega[0, s] \geq \nu_\omega[0, s], \text{ a.s.} \tag{6.4}$$

$$\text{and } (\nu_1^\pi)_\omega[0, s] \leq \nu_\omega[0, s], \text{ a.s.,} \tag{6.5}$$

where $\nu_{\omega\in\Omega}, (\nu_0^\pi)_{\omega\in\Omega}, (\nu_1^\pi)_{\omega\in\Omega}$ are disintegrations of $\nu_0, \nu_0^\pi, \nu_1^\pi$ respectively w.r.t. $\mathbb{W}$.

If we are able to construct such a ν_0^π, ν_1^π, then $(\nu_0^\pi + \nu_1^\pi)/2$ is a randomized stopping time in $\mathsf{RST}(\lambda)$ which is strictly better then ν which yields the desired contradiction.

To define ν_0^π, we first consider $p_0 = \mathsf{proj}_X(\pi)$ which is a randomized stopping time. As in Remark 4.6 we can view p_0 as right-continuous decreasing livelyhood-function $H^{p_0} : S \to [0, 1]$ which starts at 1. Possibly p_0 does not decrease to 0 since we allow that particles survive until ∞.

We now define the randomized stopping time ν_0^π as the product

$$H^{\nu_0^\pi}(f, s) := H^{p_0}(f, s) \cdot H^\nu(f, s).$$

The probabilistic interpretation of this definition is that a particle is stopped by ν_0^π if it is stopped by p_0 or stopped by ν, where these events are taken to be conditionally independent given the path $\omega \in C(\mathbb{R}_+)$. Comparing ν_0 and ν_0^π the latter will stop some particles earlier than the first one. We note that this in particular implies that $\mathbb{E}_{\nu_0^\pi}[T] \leq \mathbb{E}_\nu[T] < \infty$.

Let us now turn to the definition of ν_1^π. Set $p_1 := \mathsf{proj}_Y(\pi)$. (Recall that we write $(C(\mathbb{R}_+) \times \mathbb{R}_+) \times (C(\mathbb{R}_+) \times \mathbb{R}_+) = X \times Y$, so that proj_Y denotes the projection on the second coordinate.) Fix an $\mathcal{F}^0$-measurable[10] disintegration $(\nu_\omega)_{\omega\in C(\mathbb{R}_+)}$ of ν. Given $(f,s)\in S$ and $(\omega,t)\in C(\mathbb{R}_+)\times\mathbb{R}_+$ we define a measure on $\mathbb{R}$ with support in $[t,\infty)$ by setting for $A\subseteq[t,\infty)$

$$\nu_{(f,s),(\omega,t)}(A) := H^\nu(f,s)\cdot\nu_{\left(f\oplus\theta_t(\omega)\right)}(A-t+s), \tag{6.6}$$

where $\theta_t(\omega)=(\omega_{s+t}-\omega_t)_{s\geq0}$.

As discussed above, randomized stopping times can be represented either as probability measures on $C(\mathbb{R}_+)\times\mathbb{R}_+$ or as probability measures on S. Formally the tagged random time π is a measures on $(C(\mathbb{R}_+)\times\mathbb{R}_+)\times(C(\mathbb{R}_+)\times\mathbb{R}_+)$. However in defining ν_1^π we consider π as a probability on $S\times(C(\mathbb{R}_+)\times\mathbb{R}_+)$.[11] Define the probability measure ν_1^π on $C(\mathbb{R}_+)\times\mathbb{R}_+$ by

$$\begin{aligned}\nu_1^\pi(B) =&\nu_1(B)-p_1(B)+\\ &\int_{\left((f,s),(\omega,t)\right)\in S\times B}\Big[\nu_{(f,s),(\omega,t)}(B_\omega)+\left(1-\nu_{(f,s),(\omega,t)}(\mathbb{R})\right)\cdot\delta_{(\omega,t)}(B)\Big]\,d\pi((f,s),(\omega,t)),\end{aligned}$$

where $B_\omega=\{t\in B:(\omega,t)\in B\}$.

We then have

(1) $\nu_0^\pi,\nu_1^\pi\in\mathsf{RST}$ and $\mathbb{E}_{\nu_0^\pi}[T],\mathbb{E}_{\nu_1^\pi}[T]<\infty$.
(2) $\nu^\pi=(\nu_0^\pi+\nu_1^\pi)/2\in\mathsf{RST}(\lambda)$.

Finally

$$\begin{aligned}&\int\gamma\,d(\nu^\pi-\nu)=\\ =&\int d\pi((f,s),(g,t))\bigg(H^\nu(f,s)\Big[\int\gamma(f\oplus\omega_{\llcorner[0,s+r]},s+r)\,d\nu^{(f,s)}(\omega,r)+\gamma(g,t)-\\ &\qquad\Big(\gamma(f,s)+\int\gamma(g\oplus\omega_{\llcorner[0,t+r]},t+r)\,d\nu^{(f,s)}(\omega,r)\Big)\Big]\bigg).\end{aligned}$$

This quantity is strictly positive by the definition of bad pairs and Assumption (6.3) □

6.1. **Approximation by particular stopping times.**

Lemma 6.4. *Let τ be a non-randomized stopping time w.r.t. the right-continuous filtration $\mathcal{F}^+$. For any $\varepsilon,\eta>0$ there is an $\mathcal{F}^+$-stopping time ρ such that*

(1) $\rho\leq\tau$
(2) $\mathbb{W}(\tau-\rho\geq\varepsilon)\leq\eta$
(3) $\mathbb{W}(\{\tau=\infty,\rho<\infty\})\leq\eta$
(4) $[\![0,\rho]\!]$ is closed in $C(\mathbb{R}_+)\times\mathbb{R}_+$.

Proof. Fix $\varepsilon,\eta>0$. Assume first that

$$\tau(\omega)=\begin{cases}t & \omega\in A\\ \infty & \text{else}\end{cases},$$

for some $\mathcal{F}_t^0$-measurable set A. By Proposition 4.2, there is a Borel set $A_0\subseteq C([0,t])$ such that $A=A_0\times C((t,\infty))$. In this proof we will often use this kind of identification of $\mathcal{F}_t^0$-measurable events with measurable subsets of $C([0,t])$ without explicitly mentioning

[10] I.e. a disintegration which satisfies that $\nu_\omega(A)=\nu_{\omega'}(A)$ whenever $A\subseteq[0,t]$ and $\omega_{\llcorner[0,t]}=\omega'_{\llcorner[0,t]}$. Formally this is achieved by referring to an $\mathcal{F}^0$-predictable representent of $H^\nu(\omega,t)=1-\nu_\omega([0,t])$.

[11] Recall that we denote by r the natural "projection" from $C(\mathbb{R}_+)\times\mathbb{R}_+$ to S. To represent π as a measure on $S\times(C(\mathbb{R}_+)\times\mathbb{R}_+)$, we may formally set $\tilde\pi:=(r\times\mathrm{Id})(\pi)$.

it. In particular, we will loosely write $\mathbb{W}(D)$ instead of $\mathbb{W}(D \times C((t,\infty)))$ or $\mathbb{W}_{\llcorner C([0,t])}(D)$ for some measurable $D \subseteq C([0,t])$.

By outer regularity of $\mathbb{W}$ there is an open set $O \in C[0,t], O \supseteq A_0$ such that $\mathbb{W}(O \setminus A_0) \le \eta/2$. Moreover, O can be written as a countable union of open sets $O_n, n \ge 1$, where for each n the set O_n is an open sausage corresponding to some continuous function $f_n : [0,t] \to \mathbb{R}$ and some $\eta_n > 0$, i.e. $O_n = \{g : \mathbb{R}_+ \to \mathbb{R}, \sup_{s \le t} |f_n(s) - g(s)| < \eta_n\}$. For all $n \ge 1$ there is $t - \varepsilon \le t_n < t$ such that the open sausage O'_n corresponding to η_n and the function f_n restricted to $[0,t_n]$ satisfies $\mathbb{W}(O'_n \setminus O_n) \le 2^{-(n+1)}\eta$. Put, $O' = \cup_{n\ge1} O'_n$. Then $O \subseteq O'$ and $\mathbb{W}(O' \setminus O) \le \eta/2$ and therefore $\mathbb{W}(O' \setminus A_0) \le \eta$. Set

$$\rho_n(\omega) = \begin{cases} t_n & \omega \in O'_n \\ \infty & \text{else} \end{cases}.$$

Then, $]\!]\rho_n, \infty]\!]$ is open and $[\![0, \rho_n]\!]$ is closed. Put $U = \bigcup_n]\!]\rho_n, \infty]\!]$ and define

$$\rho(\omega) = \inf\{t : (\omega, t) \in U\}.$$

Then, we have

$$[\![0,\rho]\!] = \textstyle\bigcap_n [\![0,\rho_n]\!],$$

which implies that $\rho(\omega) = \inf_n \rho_n(\omega)$. Hence, ρ is an $\mathcal{F}^+$ stopping time and $[\![0,\rho]\!]$ is closed. Moreover, because $t-\varepsilon \le t_n < t$ we have that for all $n \ge 1$ it holds that $t-\varepsilon \le \rho_n(\omega) < \tau(\omega)$. Hence, it also holds that $t - \varepsilon \le \rho(\omega) < \tau(\omega)$. Therefore, we can conclude that

$$\mathbb{W}(\{|\tau - \rho| > \varepsilon\}) = \mathbb{W}(\{\tau = \infty, \rho < \infty\}) = \mathbb{W}(O' \setminus A_0) \le \eta.$$

This proves the Lemma for the case that τ is an $\mathcal{F}^0$-stopping time which only takes the values t and ∞. From here it is straightforward to prove the Lemma for the case where τ takes values in a discrete subset of $\mathbb{R}_+$.

Assume now that τ is an arbitrary $\mathcal{F}^0$-stopping time. Since τ is predictable, there is an $\mathcal{F}^0$-stopping time $\bar\tau$ such that $\bar\tau \le \tau, \mathbb{W}(\tau - \varepsilon/2 < \bar\tau) \ge 1 - \eta/2$. Pick a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ which for any n take only values in some discrete set such that $\tau_n \downarrow \bar\tau$. Put $\varepsilon_n = 2^{-n}\varepsilon/2$ and $\eta_n = 2^{-n}\eta/2$. According to what we have proved above pick ρ_n which are very close (in terms of ε_n, η_n) to the τ_n and satisfy that $]\!]\rho_n, \infty]\!]$ is open. Then set

$$V := \textstyle\bigcup_n]\!]\rho_n, \infty]\!]$$

and

$$\rho := \inf\{t : (\omega, t) \in V\}$$

such that $V =]\!]\rho, \infty]\!]$ is open. Note that $\rho = \inf_n \rho_n$. Hence, by construction $\rho \le \tau$ satisfies the required properties. Indeed, we only have to check that $\mathbb{W}(\tau - \rho \ge \varepsilon) \le \eta$. To this end, one easily checks that

$$\{\tau - \bar\tau < \varepsilon/2\} \cap \bigcap_n \{\tau_n - \rho_n < \varepsilon_n\} \subseteq \{\tau - \rho < \varepsilon\},$$

which directly yields the estimate.

If τ is an $\mathcal{F}^+$-stopping time, it can be represented as a decreasing limit of $\mathcal{F}^0$-stopping times and repeating the above argument yields the result also in this case. $\square$

Corollary 6.5. *Let τ be a non-randomized $\mathcal{F}^+$-stopping time*[12]*. Then there is a sequence of $\mathcal{F}^+$-stopping times τ_n such that*

(1) $\tau_n \uparrow \tau$ $\mathbb{W}$-a.s.

[12] If τ is an $\mathcal{F}^a$-stopping time then the result still applies with a minor modification: we have to allow for an exceptional null set N.

(2) $\mathbb{W}(\{\tau = \infty\} \cap \{\tau_n < \infty\}) \to 0$.
(3) For each n the stochastic interval $[\![0, \tau_n]\!]$ is closed in $C(\mathbb{R}_+) \times \mathbb{R}_+$.

Proof. For each n apply the previous lemma with $\varepsilon_n = \eta_n = 2^{-n}$. □

For two randomized stopping times μ and $\tilde{\mu}$ we have $H^\mu \le H^{\tilde{\mu}}$ iff $\mu([0,t]) \ge \tilde{\mu}([0,t])$ for all t. The last inequality means that μ stops paths before $\tilde{\mu}$. In this case we say that μ *is before* $\tilde{\mu}$.

Corollary 6.6. *Let μ be a randomized stopping time. There exists a sequence of stopping times μ_n such that*

(1) for each n there exist stopping times $\tau_1 \le \ldots \le \tau_k$ and convex coefficients $\alpha_1, \ldots, \alpha_k$ such that

$$H^{\mu_n} \circ r = \sum_{i=1}^k \alpha_i \mathbb{1}_{[\![0,\tau_i[\![} \le \sum_{i=1}^k \alpha_i \mathbb{1}_{[\![0,\tau_i]\!]} \le H^\mu \circ r$$

up to indistinguishability. Moreover the τ_i can be chosen so that $[\![0, \tau_i]\!]$ is closed.
(2) for each n, the stopping time μ_n is before μ, i.e. $H^{\mu_n} \le H^\mu$ and $\mu_n \to \mu$ weakly.

Proof. Fix n and define for $\omega \in C(\mathbb{R}_+)$ and $1 \le i \le 2^n$,

$$\tilde{\tau}_i(\omega) = \inf\left\{s : H^\mu(\omega_{\llcorner[0,s]}) \le \frac{i}{2^n}\right\}.$$

Put $H^n = 2^{-n} \sum_{i=1}^{2^n} \mathbb{1}_{[\![0,\tilde{\tau}_i[\![}$. Then $H^n \le H^{n+1} \le H^\mu$. Indeed, take any (ω, t). Then there is k such that $H^\mu(\omega_{\llcorner[0,t]}) = \alpha \in (k/2^n, (k+1)/2^n]$. This implies that

$$\tilde{\tau}_i(\omega) = \begin{cases} \le t & \text{if } i > k \\ > t & \text{if } i \le k. \end{cases}$$

This in turn yields $H^n(\omega_{\llcorner[0,t]}) = k/2^n < \alpha$.

By Lemma 6.4 there are stopping times $\tau_i < \tilde{\tau}_i$ with $\mathbb{W}(\tilde{\tau}_i - \tau_i > 3^{-n}) \le 3^{-n}$ and such that $[\![0, \tau_i]\!]$ is closed. Defining μ_n by

$$H^{\mu_n} := 2^{-n} \sum_{i=1}^{2^n} \mathbb{1}_{[\![0,\tau_i[\![},$$

$(\mu_n)_{n\ge1}$ is as required. □

In the following we assume that τ is a non-randomized, bounded stopping time such that $[\![0, \tau]\!]$ is closed. Then the set

$$\mathsf{M}_\tau := \{\mu \in \mathsf{M} : \mu(]\!]\tau, \infty]\!]) = 0\} \tag{6.7}$$

is compact as a consequence of Prohorov's theorem. We also let $\mathsf{RST}_\tau = \mathsf{RST} \cap \mathsf{M}_\tau$. Since RST_τ is closed we have the following

Lemma 6.7. *Let τ be a finite stopping time. The set RST_τ is compact in the topology induced by the continuous bounded functions on $C(\mathbb{R}_+) \times \mathbb{R}_+$.*

Recall the definition of joinings in Section 4.5.

We set

$$\mathsf{JOIN}(\tau, \nu) = \left\{\pi \in \mathsf{JOIN}(\mathbb{W}, \nu) : \mathsf{proj}_{C(\mathbb{R}_+)\times\mathbb{R}_+}(\pi) \in \mathsf{RST}_\tau\right\}.$$

Observe:

Lemma 6.8. *Under the above assumptions, the set $\mathsf{JOIN}(\tau, \nu)$ of tagged random times / joinings is compact with respect to the topology coming from the continuous bounded functions on $C(\mathbb{R}_+) \times \mathbb{R}_+ \times \mathbb{R}$.*

7. A Filtered Kellerer-type Lemma and the Principle Of Pointwise Determination

In this section we establish the following result which implies Theorem 1.3 stated in the introduction.

Theorem 7.1. *Assume that $\gamma : S \to \mathbb{R}$ is Borel-measurable, the optimization problem* (5.3) *is well-posed and that $\nu \in \mathsf{RST}(\lambda)$ is an optimizer of $P_\gamma(\lambda)$. There is a Borel set $\Gamma \subseteq S$ such that $\nu(r^{-1}(\Gamma)) = 1$ and there are no bad pairs with respect to Γ, i.e.*

$$\mathsf{BP}_\nu \cap \Gamma^< \times \Gamma = \emptyset,$$

where BP_ν is as in Definition 6.2 and $\Gamma^<$ as in (1.8)*.*

As an intermediate step towards the proof of Theorem 7.1 we will look for two different sets $\Gamma_L \subseteq S$ and $\Gamma_D \subseteq S$ where Γ_L (which roughly corresponds to $\Gamma^<$) represents the "still living pairs", while ν is concentrated on Γ_D which represents the paths which get killed by ν. Here Γ_L is a subset of all (f, s) which lie before the "death"-set Γ_D. The above condition on Γ then corresponds to: for $\big((f, s), (g, t)\big) \in \mathsf{BP}_\nu$, at least one of the following applies:

(1) $(f, s) \notin \Gamma_L$ ((f, s) is not living).
(2) $(g, t) \notin \Gamma_D$ ((g, t) is not dying).

As in (1.9) above, this can equivalently be expressed as

$$\mathsf{BP}_\nu \ \cap \ \Gamma_L \times \Gamma_D = \emptyset. \tag{7.1}$$

Define a (non-randomized) stopping time τ_ν by

$$\tau_\nu(\omega) := \inf\{t : H^\nu \circ r(\omega, t) = 0\}.$$

Using Lemma 6.5 we can pick a sequence $\tau_n, n \geq 1$ of stopping times such that

(1) $\tau_n \uparrow \tau_\nu$.
(2) $\tau_n \leq n$.
(3) $[\![0, \tau_n]\!]$ is closed.
(4) $\tau_n < \inf\{t : \nu_\omega[0, t] \geq 1 - 1/n\}$

Fix n. Then every joining $\pi \in \mathsf{JOIN}(\tau_n, \nu)$ satisfies the assumptions of Proposition 6.3 and hence $\pi(\mathsf{BP}_\nu) = 0$. We write again $X = C(\mathbb{R}_+) \times \mathbb{R}_+ \approx S$. In the proof of Theorem 7.1 we will also specify $Y = S$.

Subsequently we will prove:

Lemma 7.2 (filtered Kellerer Lemma)**.** *Assume that τ is a (non-randomized) bounded $\mathcal{F}^+$-stopping time such that $[\![0, \tau]\!]$ is closed.*

Let $X = C(\mathbb{R}_+) \times \mathbb{R}_+$ and let (Y, ν) be a polish probability space. Consider a ("bad") set $B \subseteq X \times Y$ which is predictable in the sense of Remark 4.13. If $\pi(B) = 0$ for all $\pi \in \mathsf{JOIN}(\tau, \nu)$, then there exist a (non randomized) stopping time κ and a set $N \subseteq Y$ such that $B \subseteq [\![\kappa, \infty]\!] \times Y \ \cup \ X \times N$ and $\mathbb{W}(\kappa < \tau) = \nu(N) = 0$.

Admitting Lemma 7.2 we set $\Gamma_L = [\![0, \kappa[\![$ and $\Gamma_D = Y \setminus N$ to obtain (7.1).

Proof of Theorem 7.1 from Lemma 7.2. Specify $Y = (S, \nu)$. Throughout the proof we will use freely that BP_ν can be viewed as a subset of $(C(\mathbb{R}_+) \times \mathbb{R}_+) \times S$ as well as a subset of $S \times S$. Of course we have to be careful to finally find the desired set Γ within S. We admit Lemma 7.2 and apply it to the stopping times τ_n defined above to find κ_n and $\Gamma_n := S \setminus N_n$ such that

$$\mathsf{BP}_\nu \ \cap \ [\![0, \kappa_n[\![\times \Gamma_n = \emptyset$$

and $\mathbb{W}(\kappa_n < \tau_n) = 0, \nu(\Gamma_n) = 1$. Setting $\tilde\Gamma := \bigcap_n \Gamma_n$ we have $\mathsf{BP}_\nu \cap [\![0,\kappa_n[\![\times \tilde\Gamma = \emptyset$ for all n. With $\kappa := \sup_n \kappa_n$ we have

$$[\![0,\kappa[\![= \textstyle\bigcup_n [\![0,\kappa_n[\![,$$

hence

$$\mathsf{BP}_\nu \cap [\![0,\kappa[\![\times \tilde\Gamma = \emptyset. \tag{7.2}$$

By construction of $(\tau_n)_{n\in\mathbb{N}}$ and $(\kappa_n)_{n\in\mathbb{N}}$ we have $\nu_\omega([0,\kappa(\omega)]) = 1$ almost surely, hence $\nu([\![0,\kappa]\!]) = 1$.

To defined the desired set Γ as a subset of S we need to replace the stochastic interval $[\![0,\kappa[\![$ by a subset of S. To this end, note that (7.2) is equivalent to

$$[\![0,\kappa[\![\cap \operatorname{proj}_X(\mathsf{BP}_\nu \cap S \times \tilde\Gamma) = \emptyset. \tag{7.3}$$

Set $R = \operatorname{proj}_X(\mathsf{BP}_\nu \cap S \times \tilde\Gamma) \subseteq S$ and

$$R^+ = \{(g,t) : \exists (f,s) \in R : (g,t) \text{ extends } (f,s)\} \tag{7.4}$$

$$R^{++} = \{(g,t) : \exists (f,s) \in R : (g,t) \text{ properly extends } (f,s)\}. \tag{7.5}$$

Note that while R, R^+, R^{++} are not necessarily Borel-measurable, they are *analytic* sets and in particular universally measurable. Moreover, since $\nu([\![0,\kappa]\!]) = 1$ we have $\nu(R^{++}) = 0$. Thus also $\nu(\Gamma) = 1$ for $\Gamma = \tilde\Gamma \setminus R^{++}$. Set $L = S \setminus R^{++}$ and $L^- = S \setminus R^+$ and note that $\mathsf{BP}_\nu \cap L^- \times \tilde\Gamma = \emptyset$. But then $L \supseteq \Gamma$ implies that $L^- \supseteq \Gamma^<$ and therefore $\mathsf{BP}_\nu \cap \Gamma^< \times \Gamma = \emptyset$.

Finally we can replace Γ by a K_σ-subset which still has full ν measure. □

It remains to establish Lemma 7.2 which we shall now do.

Important Convention. For the remainder of this section we fix a (finite) non-randomized stopping time τ such that $[\![0,\tau]\!]$ is closed and satisfies $\tau \le t_0$ for some $t_0 \in \mathbb{R}_+$.

7.1. **An auxiliary Optimization Problem.** We fix a Polish probability space (Y,ν) which eventually will be taken to be (S,ν), where ν denotes an optimizer of the primal problem 5.3. We are interested in the following maximization problem

$$P^{\le 1} = P_c^{\le 1}(\mathbb{W}^\tau, \nu) = \sup_{\pi \in \mathsf{JOIN}(\tau,\nu)} \int_{C(\mathbb{R}_+)\times\mathbb{R}_+\times Y} c\, d\pi = \sup_{\pi\in\mathsf{JOIN}(\tau,\nu)} I_c(\pi) \tag{7.6}$$

and its relation to the dual problem

$$D^{\le 1} = D_c^{\le 1}(\mathbb{W}^\tau, \nu) = \inf_{(\varphi,\psi)\in\mathsf{DC}} \left(\mathbb{E}_{\mathbb{W}}[\varphi_\tau] + \mathbb{E}_\nu[\psi]\right), \quad \text{where} \tag{7.7}$$

$$\mathsf{DC} = \{(\varphi,\psi) : \varphi,\psi \ge 0, (\varphi,\psi) \in L^1(\Omega)\times L^1(\nu), c(\omega,t,y) \le \varphi^M_t(\omega) + \psi(y)\ t \le \tau, y \in Y\ \mathbb{W}\text{-a.s.}\}.$$

To indicate the dependence of DC on the cost function c and the stopping time τ we sometimes write $\mathsf{DC}(c)$ or $\mathsf{DC}(c,\tau)$. Note that for integrable φ we always have $\mathbb{E}_{\mathbb{W}}[\varphi] = \mathbb{E}_{\mathbb{W}}[(\varphi_\tau)]$

by optional stopping. Note that this is different from the already established duality results because we allow subprobability measures.

We first establish the easy inequality

Lemma 7.3. *With the above notations and assumptions we have $D^{\le 1} \ge P^{\le 1}$.*

Proof. Take $(\varphi,\psi) \in \mathsf{DC}$ and $\pi \in \mathsf{JOIN}(\tau,\nu)$. Then, by definition of tagged random time we have

$$\mathbb{E}_{\mathbb{W}}\varphi + \mathbb{E}_\nu\psi \ge \int_{C(\mathbb{R}_+)\times\mathbb{R}_+\times Y} \varphi^M_t + \psi\, d\pi \ge \int_{C(\mathbb{R}_+)\times\mathbb{R}_+\times Y} c\, d\pi.$$

The last inequality holds by the dual constraint. □

7.2. **Duality.**

Theorem 7.4. *Let $c : C(\mathbb{R}_+) \times \mathbb{R}_+ \times Y \to \mathbb{R}_+$ be predictable (in the sense of Remark 4.13), upper semi-continuous and bounded from above. Assume that τ is a bounded stopping time such that $[\![0,\tau]\!]$ is closed. Then*

$$P^{\le 1} = \sup_{\pi \in \mathsf{JOIN}(\tau,\nu)} \int_{C(\mathbb{R}_+)\times\mathbb{R}_+\times Y} c\, d\pi = \inf_{(\varphi,\psi)\in\mathsf{DC}} (\mathbb{E}_{\mathbb{W}}[\varphi] + \mathbb{E}_\nu[\psi]) = D^{\le 1}$$

We will first prove a version which applies to not necessarily predictable c. Afterwards, we will use Proposition 4.10 to derive the predictable version. Let us start with

Theorem 7.5. *Let $c : C(\mathbb{R}_+) \times \mathbb{R}_+ \times Y \to \mathbb{R}_+$ be (upper semi-) continuous and bounded from above. Then*

$$P^{\le 1,NA} := \sup_{\pi \in \mathsf{TM}(\tau,\nu)} \int_{C(\mathbb{R}_+)\times\mathbb{R}_+\times Y} c\, d\pi = \inf_{(\varphi,\psi)\in\widetilde{\mathsf{DC}}} (\mathbb{E}_{\mathbb{W}}[\varphi] + \mathbb{E}_\nu[\psi]) =: D^{\le 1,NA},$$

where $\widetilde{\mathsf{DC}} = \{(\varphi,\psi) \ge (0,0) : c((\omega,t),y) \le \varphi(\omega) + \psi(y)$ for all $y \in Y, t \le \tau,\ \mathbb{W}$-a.s.$\}$.

Here the set of all *tagged random measures* is given by

$$\mathsf{TM}(\tau,\nu) := \Big\{\pi \in \mathcal{P}^{\le 1}(C(\mathbb{R}_+)\times\mathbb{R}_+\times Y), \mathsf{proj}_{C(\mathbb{R}_+)\times\mathbb{R}_+}(\pi) \in \mathsf{M}_\tau, \mathsf{proj}_Y(\pi) \le \nu\Big\}.$$

Proof of Theorem 7.5. We reduce the theorem to the classical duality theorem in optimal transport. Put $\bar c(\omega,y) = \sup_{t\le\tau(\omega)} c(\omega,t,y)$. As $[\![0,\tau]\!]$ is closed and bounded $\bar c$ is continuous.

Then the dual constraint set can be written as

$$\widetilde{\mathsf{DC}} = \{(\varphi,\psi) : \bar c(\omega,y) \le \varphi(\omega) + \psi(y)\ \mathbb{W}\text{-a.s., for all } y\}.$$

From the classical duality theorem of optimal transport (3.1) we know that

$$\inf_{(\varphi,\psi)\in\widetilde{\mathsf{DC}}} \mathbb{W}(\varphi) + \nu(\psi) = \sup_{q\in\mathsf{Cpl}(\mathbb{W},\nu)} \int_{\Omega\times Y} \bar c(\omega,y)\, q(d\omega,dy) =: \check P.$$

It remains to show that $\check P = P^{\le 1,NA}$. From the definition of $\bar c$ and TM it is clear that we always have $P^{\le 1,NA} \le \check P$. To prove the other inequality fix $\varepsilon > 0$ and take $q \in \mathsf{Cpl}(\mathbb{W},\nu)$. For any (ω,y) there is $t(\omega,y) \le \tau(\omega)$ such that $c((\omega,t(\omega,y)),y) + \varepsilon \ge \bar c(\omega,y)$. Putting $\pi(d\omega,ds,dy) := q(d\omega,dy)\delta_{t(\omega,y)}(ds) \in \mathsf{TM}$ we get

$$\int_{C(\mathbb{R}_+)\times\mathbb{R}_+\times Y} c((\omega,t),y)\,\pi(d\omega,ds,dy) + \varepsilon \ge \int_{C\times Y} \bar c(\omega,y)\, q(d\omega,dy).$$

This implies that $P^{\le 1,NA} + \varepsilon \ge \check P$. Letting ε go to zero we obtain the claim. □

Remark 7.6. *A consequence of allowing partial transports or sub-probability measures π in the definition of the set* JOIN *is the following: Assume that we are given a cost function c which is non positive. Then, $P^{\le 1}(c) = 0$ as the zero measure is admissible and everything else is worse. Also the value of the dual problem is zero, $D^{\le 1}(c) = 0$, as the constraint is satisfied for $\varphi,\psi \equiv 0$. Similarly, for a general cost function c we have $P^{\le 1}(c) = P^{\le 1}(c \vee 0)$ and also $D^{\le 1}(c) = D^{\le 1}(c \vee 0)$. Hence, the requirement that c is nonnegative is not a restriction.*

Lemma 7.7. *Let $g : C(\mathbb{R}_+)\times\mathbb{R}_+\times Y \to \mathbb{R}_+$ be continuous and assume that $\sup_{\pi\in\mathsf{JOIN}(\tau,\nu)} \int g\, d\pi < \infty$. Then the map*

$$\pi \mapsto \int g\, d\pi$$

is continuous on $\mathsf{JOIN}(\tau,\nu)$ *(w.r.t. to the topology of weak convergence).*

Proof. This is a direct consequence of tightness as the integrals

$$\int_{g\geq R} g \, d\pi \to 0$$

uniformly in π as $R \to 0$ by assumption. ☐

Proof of Theorem 7.4. As c is bounded from above we have $P^{\leq 1} < \infty$. Arguing as in Lemma 5.5, we may assume that the cost function c is continuous.

We will now argue as in Proposition 5.6. I.e. we consider again the functions $h, \bar{h}$ as in (5.17)

and we shall apply Theorem 5.4 to the function

$$F(\pi, h) = \int c \, d\pi + \int_{C(\mathbb{R}_+)\times\mathbb{R}_+} \bar{h} \, d\pi$$

for $\pi \in \mathsf{TM}(\tau, \nu)$. The set $\mathsf{TM}(\tau, \nu)$ is convex and compact by Prohorov's theorem and the set of h under consideration is convex as well. The function F is continuous by Lemma 7.7.

This allows us to deduce

$$\begin{aligned}
P^{\leq 1} &= \sup_{\pi\in\mathsf{JOIN}(\tau,\nu)} \int c \, d\pi \\
&= \sup_{\pi\in\mathsf{TM}(\tau,\nu)} \inf_h \left(\int c \, d\pi + \int_{C(\mathbb{R}_+)\times\mathbb{R}_+} \bar{h} \, d\pi \right) \\
&\overset{Thm5.4}{=} \inf_h \sup_{\pi\in\mathsf{TM}(\tau,\nu)} \left(\int (c + \bar{h}) \, d\pi \right) \\
&= \inf_h \inf_{(\varphi,\psi)\in\widetilde{\mathsf{DC}}(c+\bar{h})} (\mathbb{W}(\varphi) + \nu(\psi))
\end{aligned}$$

The last equality holds by Theorem 7.5. Write

$$c_h(\omega, t, y) = c(\omega, t, y) + \sum_{i=1}^{n} (f_i(\omega) - f^M_{i,t}(\omega)) g(y).$$

For $(\varphi, \psi) \in \widetilde{\mathsf{DC}}(c_h)$ it holds that (see Remark 7.6)

$$c_h(\omega, t, y)_+ = c_h(\omega, t, y) \vee 0 \leq \varphi(\omega) + \psi(y).$$

Taking conditional expectation w.r.t. $\mathcal{F}^0_t \otimes \mathcal{B}(Y)$ we get using the adaptedness (predictability) of c

$$\begin{aligned}
c(\omega, t, y) &= \mathbb{E}[c(\cdot, t, y) + \textstyle\sum_{i=1}^n (f_i(\cdot) - f^M_{i,t}(\cdot)) g(y) | \mathcal{F}_t \otimes \mathcal{B}(Y)](\omega) \\
&\leq \mathbb{E}[(c(\cdot, t, y) + \textstyle\sum_{i=1}^n (f_i(\cdot) - f^M_{i,t}(\cdot)) g(y))_+ | \mathcal{F}_t \otimes \mathcal{B}(Y)](\omega) \leq \varphi_t(\omega) + \psi(y).
\end{aligned}$$

This implies that $(\varphi_t, \psi) \in \mathsf{DC}(c)$. Because $\mathbb{W}(\varphi_t) = \mathbb{W}(\varphi)$ this implies that $\widetilde{\mathsf{DC}}(c_h) \subseteq \mathsf{DC}(c)$. Therefore, we have

$$\begin{aligned}
P^{\leq 1} &= \inf_{f\in C_b(C)} \inf_{(\varphi,\psi)\in\widetilde{\mathsf{DC}}(c+\bar{h})} (\mathbb{W}(\varphi) + \nu(\psi)) \\
&\geq \inf_{(\varphi,\psi)\in\mathsf{DC}(c)} (W(\varphi) + \nu(\psi)) = D^{\leq 1}.
\end{aligned}$$

By Lemma 7.3 we always have $D^{\leq 1} \geq P^{\leq 1}$ and therefore $D^{\leq 1} = P^{\leq 1}$. ☐

Having established the duality we can start drawing conclusions.

Corollary 7.8. *Let $K \subseteq [\![0,\tau]\!] \times Y$ be closed and* predictable *(see Remark 4.13). Then*

$$\sup_{\pi\in\mathsf{JOIN}(\tau,\nu)} \pi(K) \le \inf_{(\kappa,B)\in\mathsf{Cov}(K)} \Big(\mathbb{W}(\kappa<\tau)+\nu(B)\Big) \le 2 \sup_{\pi\in\mathsf{JOIN}(\tau,\nu)} \pi(K),$$

where

$$\mathsf{Cov}(K) = \{\kappa \textit{ is a stopping time}, B \subseteq Y : K \subseteq [\![\kappa,\infty]\!] \times Y \cup (C(\mathbb{R}_+)\times\mathbb{R}_+)\times B\}.$$

Proof. Without loss of generality we may assume that $K = K \cap [\![0,\tau]\!]$. We want to apply the previous theorem with the cost function $c = \mathbb{1}_K$. Clearly, $P^{\le 1}(\mathbb{1}_K) = \sup_{\pi\in\mathsf{JOIN}(\tau,\nu)} \pi(K)$. We have to show that

$$D^{\le 1}(\mathbb{1}_K) \approx \inf_{(\kappa,B)\in\mathsf{Cov}} \Big(\mathbb{W}(\kappa<\infty)+\nu(B)\Big).$$

To this end take $(\varphi,\psi) \in \mathsf{DC}$. As the cost function is $\{0,1\}$-valued, the dual constraint

$$c((\omega,t),y) \le \varphi^\tau_t(\omega) + \psi(y)$$

implies that

$$K \subseteq \{(\omega,t) : \varphi^\tau_t(\omega) \ge 1/2\} \times Y \cup C(\mathbb{R}_+)\times\mathbb{R}_+ \times \{y : \psi(y) \ge 1/2\}.$$

Hence, on the cost of a factor 2 we can replace ψ by the indicator of a set $B \subseteq Y$. We can just take $B = \{\psi \ge 1/2\}$. In particular, given ψ we can safely replace it by $\tilde\psi = 1_B\psi \wedge 1$ because $\tilde\psi$ has smaller expectation and at least as good covering properties as ψ. Obviously, $1/2\nu(B) \le \mathbb{E}_\nu\tilde\psi \le \nu(B)$.

Let us turn our attention to the set $E = \{(\omega,t) : \varphi^\tau_t(\omega) \ge 1\}$ (where we dropped the $1/2$ out of notational convenience on the cost of another factor of 2). Define the stopping time

$$\kappa(\omega) = \inf\{t \ge 0 : \varphi^\tau_t(\omega) \ge 1\}$$

with $\inf(\emptyset) = \infty$ and the martingale $\tilde\varphi_t = \mathbb{E}[\tilde\varphi|\mathcal{F}_t]$ through

$$\tilde\varphi(\omega) = \begin{cases} 1 & \kappa(\omega) \le \tau \\ 0 & \kappa(\omega) = \infty \end{cases}.$$

Then, $\tilde\varphi$ has expectation smaller than or equal to φ^τ and is at least as good in covering K as φ. Indeed, the stopped martingales satisfy $\varphi_\kappa \ge \tilde\varphi_\kappa$. Put $\tilde E = \{\tilde\varphi_t \ge 1\}$, then $\tilde E \supseteq E$. Take $(\omega,t) \in E$. Then, $\kappa(\omega) \le t$ which implies that $(\omega,t) \in \tilde E$. Moreover, $\mathbb{E}_{\mathbb{W}}\tilde\varphi = \mathbb{W}(\kappa<\infty)$. Remembering the factor of 2 we can therefore deduce

$$\inf_{(\kappa,B)\in\mathsf{Cov}(K)} \left(\frac12\mathbb{W}(\kappa<\infty)+\frac12\nu(B)\right) \le \sup_{\pi\in\mathsf{JOIN}(\tau,\kappa)} \pi(K) \le \inf_{(\kappa,B)\in\mathsf{Cov}(K)} \left(\mathbb{W}(\kappa<\infty)+\nu(B)\right).$$

□

7.3. **A Choquet-argument.** We now want to extend the previous result to the more general case of a merely measurable set K.

In the proof we will use Choquet's theorem similarly as in [BLS12]. In the proof we rely on Lemma 6.5 and the Lemma 7.9. Recall that we assume that the stopping time τ is smaller than or equal to some number t_0.

A *simple stopping time* is a right-continuous increasing, $\mathcal{F}^+$-predictable process

$$F : C(\mathbb{R}_+)\times\mathbb{R}_+ \to [0,1]$$

which takes only *finitely many* values and is constant after time t_0. Write SST for the set of all simple stopping times.

Lemma 7.9. *Consider the set of simple coverings of the set K*

$$\mathsf{Cov}_S(K) = \left\{ \begin{array}{l} (\varphi, F) : \varphi : Y \to [0,1], F \in \mathsf{SST}, \\ F(\omega,t) + \varphi(y) \geq \mathbb{1}_K(\omega,t,y)\ \mathbb{W}\text{-a.s. for all } y \in Y, t \leq t_0 \end{array} \right\}.$$

Then

$$D(K) := \inf \left\{ \int \varphi \, d\nu + \int F_{t_0} \, d\mathbb{W} : (\varphi, F) \in \mathsf{Cov}_S(K) \right\}.$$

is a capacity.

Proof. To show that D defines a capacity we have to check the three defining properties of capacities; monotonicity, continuity from below and continuity from above for compact sets. The monotonicity is clear. Let us turn to the continuity from below.

Take an increasing sequence $A_1 \subseteq A_2 \subseteq \ldots \subseteq C(\mathbb{R}_+) \times \mathbb{R}_+ \times Y$ of measurable sets and put $A = \bigcup_n A_n$. For all n there are simple stopping times F_n and measurable functions $\varphi_n : S \to [0,1]$ such that $(\varphi_n, F_n) \in \mathsf{Cov}_R(A_n)$ and

$$\nu(\varphi_n) + \mathbb{W}((F_n)_{t_0}) \leq D(A_n) + \frac{1}{n}.$$

Using a Komlos type lemma we can assume that some appropriate convex combinations of φ_n and F_n converge a.s. to functions φ and F. Let us be a little bit more precise here. By [CS06] there exist convex coefficients $\alpha^n_n, \ldots, \alpha^n_{k_n}, n \geq 1, k_n < \infty$, and full measure subsets $\Omega_1 \subseteq C(\mathbb{R}_+)$, $Y_1 \subseteq Y$ such that with $\tilde{F}_n := \sum_{i=n}^{k_n} \alpha_i^{(n)} F_i$, $\tilde{\varphi}_n := \sum_{i=n}^{k_n} \alpha_i^{(n)} \varphi_i$ we have that for all $\omega \in \Omega_1$ and $y \in Y_1$

$$\lim_{n\to\infty} \tilde{F}_n(\omega,t) =: F(\omega,t) \text{ and } \lim_{n\to\infty} \tilde{\varphi}_n(y) =: \varphi(y) \tag{7.8}$$

exist. Extend these functions to $C(\mathbb{R}_+)$ and Y, resp., through

$$\limsup_{n\to\infty} \tilde{F}_n(\omega,t) =: F(\omega,t) \text{ and } \limsup_{n\to\infty} \tilde{\varphi}_n(y) =: \varphi(y). \tag{7.9}$$

Given $m \leq n$ we have

$$\mathbb{1}_{A_m}(\omega,t,y) \leq \tilde{F}_n(\omega,t) + \tilde{\varphi}_n(y),$$

hence $\mathbb{1}_{A_m}(\omega,t,y) \leq F(\omega,t) + \varphi(y)$ and thus also

$$\mathbb{1}_A(\omega,t,y) \leq F(\omega,t) + \varphi(y).$$

We can then replace F by its right-continuous version which of course does not effect $\mathbb{W}(F_{t_0})$. Given $\varepsilon > 0$, by Corollary 6.6 we can find a simple stopping time $F^\varepsilon \geq F$ such that $\mathbb{W}(F^\varepsilon_{t_0}) - \varepsilon < \mathbb{W}(F_{t_0}) = \lim \mathbb{W}((\tilde{F}_n)_{t_0})$.

Therefore we can conclude

$$D(A) \leq \limsup_n D(A_n) + \tfrac{1}{n} + \varepsilon.$$

To show continuity from above for compact sets, take a sequence $K_1 \supseteq K_2 \supseteq \ldots$ of compact sets in $C(\mathbb{R}_+) \times \mathbb{R}_+ \times Y$ and put $K = \bigcap_n K_n$. Fix $\varepsilon > 0$. Then there is $(\varphi, F) \in \mathsf{Cov}_S(K)$ s.t.

$$\int \varphi \, d\nu + \int F_{t_0} \, d\mathbb{W} \leq D(K) + \varepsilon.$$

Using Corollary 6.5 / Corollary 6.6 and the regularity of the measure ν we can, on the cost of another ε, assume that

$$\varphi = \sum_{i=1}^m a_i \mathbb{1}_{B_i}, \quad F \leq \sum_{j=1}^n b_j \mathbb{1}_{[\![\tau_i,\infty[\![} \leq F^\varepsilon,$$

where $a_i, b_j \geq 0$ and B_i, $]\!]\tau_i, \infty[\![$ are open and $F^\varepsilon \downarrow F$ such that $W(F^\varepsilon_{t_0} - F_{t_0}) \leq \varepsilon/2$. By compactness of the $K_n, n \geq 1$ it follows that there is some N such that also $(\varphi, F^\varepsilon) \in \mathsf{Cov}_S(K_N)$. Hence $D(K_N) \leq D(K) + 2\varepsilon$.

□

Proof of Lemma 7.2. By Lemma 7.9 and Choquet's Theorem it follows that for each $\varepsilon > 0$ there exist a simple stopping time F and a function φ such that $(\varphi, F) \in \mathsf{COV}_S(K)$ and $\nu(\varphi) + \mathbb{W}(F_{t_0}) < \varepsilon$. Set $N = \{\varphi \geq 1/2\}$ and $\kappa := \inf\{t : F_t \geq 1/2\}$. Then $K \subseteq [\![\kappa, \infty[\![\times Y \cup C(\mathbb{R}_+) \times \mathbb{R}_+ \times N$ and $\mathbb{W}(\kappa < \tau) + \nu(N) \leq 2\varepsilon$.

Fix $\eta > 0$ and pick for each k some κ_k, N_k such that $K \subseteq [\![\kappa_k, \infty[\![\times Y \cup C(\mathbb{R}_+) \times \mathbb{R}_+ \times N_k$ and $\mathbb{W}(\kappa_k < \tau) + \nu(N_k) \leq \eta 2^{-k}$. Then

$$K \subseteq \bigcap_j \left([\![\kappa_j, \infty[\![\times Y \cup C(\mathbb{R}_+) \times \mathbb{R}_+ \times \Big(\bigcup_k N_k \Big) \right) = [\![\sup_j \kappa_j, \infty[\![\times Y \cup C(\mathbb{R}_+) \times \mathbb{R}_+ \times \Big(\bigcup_k N_k \Big).$$

This shows that κ can be chosen so that $\mathbb{W}(\kappa < \tau) = 0$. Repeating this argument for the set N we obtain the desired result. □

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