Strong characterizing sequences of countable groups

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Abstract

András Biró and Vera Sós prove that for any subgroup G of $\mathbb{T}$ generated freely by finitely many generators there is a sequence $A \subseteq \mathbb{N}$ such that for all $\beta \in \mathbb{T}$ we have ($\|.\|$ denotes the distance to the nearest integer)

$$\beta \in G \Rightarrow \sum_{n \in A} \|n\beta\| < \infty, \qquad \beta \notin G \Rightarrow \limsup_{n \in A, n \to \infty} \|n\beta\| > 0.$$

We extend this result to arbitrary countable subgroups of $\mathbb{T}$. We also show that not only the sum of norms but the sum of arbitrary small powers of these norms can be kept small. Our proof combines ideas from the above article with new methods, involving a filter characterization of subgroups of $\mathbb{T}$.

Key words: Characterizing sequences, countable subgroups of $\mathbb{T}$, filters

1 Introduction

We study certain subgroups of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and methods to describe them by sequences of positive integers. By $\|.\|$ we denote the distance to the nearest integer. It is easily seen that for any sequence $A \subseteq \mathbb{N}$ the set $\{\beta \in \mathbb{T} : \lim_{n \in A, n \to \infty} \|n\beta\| = 0\}$ is a subgroup of $\mathbb{T}$. It seems natural to ask which subgroups arise in this way. In [1] A. Biró, J.-M. Deshouillers and V. T. Sós show that for any countable group $G < \mathbb{T}$ there is some $A \subseteq \mathbb{N}$ that characterizes G in the above sense.
Another way to connect subsets of $\mathbb{N}$ and $\mathbb{T}$ is to consider the set $\{\beta \in \mathbb{T} : \sum_{n \in A} \|n\beta\| < \infty\}$ which again is a subgroup of $\mathbb{T}$. Following a question of P.

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Liardet A. Biró and V. T. Sós show in [2] that if $1, \alpha_1, \ldots, \alpha_t \in \mathbb{T}$ are linearly independent over the rationals there is a sequence $A \subseteq \mathbb{N}$, that characterizes $\langle \alpha_1, \ldots, \alpha_t \rangle$ simultaneously in both ways. Such a sequence is called a 'strong characterizing sequence' of $\langle \alpha_1, \ldots, \alpha_t \rangle$. Our aim is to find strong characterizing sequences for arbitrary countable subgroups of $\mathbb{T}$. The main result is:

Theorem 1 *Let $G = \{\alpha_t : t \in \mathbb{N}\}$ be a subgroup of $\mathbb{T}$. Then there exists a sequence $A \subseteq \mathbb{N}$, such that for all $\beta \in \mathbb{T}$*

$$\beta \in G \Rightarrow \forall r > 0 \sum_{n \in A} \|n\beta\|^r < \infty, \qquad \beta \notin G \Rightarrow \limsup_{n \in A, n \to \infty} \|n\beta\| \geq 1/6.$$

2 Connecting two methods

In our proof we use the following reformulation of Theorem 1 in [3]:

Proposition 2 *Let G be an arbitrary subgroup of $\mathbb{T}$. Then there is a filter $\mathfrak{F}$ on $\mathbb{N}$ that characterizes G in the sense that that for all $\beta \in \mathbb{T}$*

$$\beta \in G \quad \Longleftrightarrow \quad \mathfrak{F} - \lim_n \|n\beta\| = 0.$$

Here '$\mathfrak{F}-\lim_n \|n\beta\| = 0$' means that for all $\varepsilon > 0$ one has $\{n \in \mathbb{N} : \|n\beta\| \leq \varepsilon\} \in \mathfrak{F}$. The filter-convergence defined in this way is more general than ordinary convergence: For a sequence $A \subseteq \mathbb{N}$ let $\mathfrak{F}(A)$ be the filter consisting of all sets containing $\{k \in A : k \geq n\}$ for some $n \in \mathbb{N}$. Then we have for all $\beta \in \mathbb{T}$

$$\lim_{n \in A, n \to \infty} \|n\beta\| = 0 \quad \Longleftrightarrow \quad \mathfrak{F}(A) - \lim_n \|n\beta\| = 0.$$

The following notation will be useful: Given $\alpha_1, \ldots, \alpha_t \in \mathbb{T}$, $\varepsilon > 0$ and $N \in \mathbb{N}$ the corresponding infinite respectively finite Bohr sets are defined by

$$\begin{aligned} H_\varepsilon(\alpha_1, \ldots, \alpha_t) &:= \{n \in \mathbb{N} : \|n\alpha_1\|, \ldots, \|n\alpha_t\| \leq \varepsilon\}, \\ H_{N,\varepsilon}(\alpha_1, \ldots, \alpha_t) &:= \{n \leq N : \|n\alpha_1\|, \ldots, \|n\alpha_t\| \leq \varepsilon\}. \end{aligned}$$

Using the finite intersection property of filters, one sees that $\mathfrak{F}-\lim_n \|n\beta\| = 0$ for all elements β of some given $G < \mathbb{T}$ implies that for all $\alpha_1, \ldots, \alpha_t \in G$ and $\varepsilon > 0$ $H_\varepsilon(\alpha_1, \ldots, \alpha_t) \in \mathfrak{F}$. For each subgroup $G < \mathbb{T}$ there is a canonical (i.e. smallest) candidate for a filter that characterizes G, namely the filter $\mathfrak{F}_G$ which consists of all sets containing a set $H_\varepsilon(\alpha_1, \ldots, \alpha_t)$ $(\varepsilon > 0, t \in \mathbb{N}, \alpha_1, \ldots, \alpha_t \in G)$.

To illustrate the connections between the number theoretic approach in [1] respectively [2] and the more abstract point of view in [3] we show that the result on the characterization of countable subgroups by sequences of positive integers in [1] implies Proposition 2:

PROOF. Let $G < \mathbb{T}$ be an arbitrary subgroup and let $\mathfrak{F}_G$ be the filter described above. By definition of $\mathfrak{F}_G$ we have $\mathfrak{F}_G - \lim_n \|n\beta\| = 0$ for each $\beta \in G$. Now assume $\mathfrak{F}_G - \lim_n \|n\beta\| = 0$ for some $\beta \in \mathbb{T}$. For $k \in \mathbb{N}$ let $M_k := H_{1/k}(\beta) \in \mathfrak{F}_G$. According to the construction of $\mathfrak{F}_G$, there are sequences $t_1 < t_2 < \dots$ $(t_k \in \mathbb{N})$, $(\alpha_t)_{t\in\mathbb{N}}$ $(\alpha_t \in G)$ and $\varepsilon_1 > \varepsilon_2 > \dots$ $(\varepsilon_k > 0)$ such that $M_k \supseteq H_{\varepsilon_k}(\alpha_1, \dots, \alpha_{t_k})$ for all $k \in \mathbb{N}$.
By the result of A. Biró, J.-M. Deshouillers and V. T. Sós there is a sequence $A \subseteq \mathbb{N}$, such that

$$\{\beta \in \mathbb{T} : \lim_{n\in A, n\to\infty} \|n\beta\| = 0\} = \langle \alpha_t : t \in \mathbb{N} \rangle.$$

In particular we have $\lim_{n\in A,n\to\infty} \|n\alpha_t\| = 0$ for all $t \in \mathbb{N}$. Thus for fixed $m \in \mathbb{N}$ we can find $n_m \in \mathbb{N}$ satisfying $\|n\alpha_t\| \le \varepsilon_m$ for all $n \in A, n \ge n_m$ and for all $t \le t_m$. This implies $\{n \in A : n > n_m\} \subseteq H_{\varepsilon_m}(\alpha_1, \dots, \alpha_{t_m}) \subseteq M_m$, i.e. for all $n \in A, n \ge n_m$ we have $\|n\beta\| \le 1/m$. Since $m \in \mathbb{N}$ was arbitrary this yields $\lim_{n\in A,n\to\infty} \|n\beta\| = 0$ and, as A is a characterizing sequence, $\beta \in \langle \alpha_t : t \in \mathbb{N} \rangle < G$. $\square$

3 Ideas of the proof

The rest of this article focuses on the proof of Theorem 1. The proof splits in several lemmas. Before we state and prove them rigorously, we want to give a short sketch of the strategy of the proof and the informal meaning of the individual lemmas:
Lemma 5 shows how the countable group G may be represented as the limes inferior of certain open subsets V_t of $\mathbb{T}$. These sets may by seen as approximations of G.
Lemma 4 shows that the behaviour of the values $\|n\beta\|$, where n runs in an appropriate finite Bohr set, may decide whether β lies in an approximation V_t of G. Part (1) of the Lemma uses Theorem 2, while part (2) follows easily by a compactness argument similar to the reasoning in [1].
The methods developed so far are powerful enough to prove the existence of sequences that characterize countable groups in the sense of [1]. To provide a strong characterizing sequence we use Lemma 7 to replace a Bohr set H by a somewhat thinner set S that contains the same amount of information but allows in addition to keep the sum $\sum_{n\in S} \|n\alpha\|^r$ $(\alpha \in G, r > 0)$ under control. The proof of Lemma 7 is based on Lemma 6, a deep result on the structure of Bohr sets due to A. Biró and V. T. Sós ([2]).

4 Preparations

The following technical facts will be needed later. The proof is elementary, so we skip it.

Lemma 3 *Let $\alpha, \beta \in \mathbb{T}$ and $n \in \mathbb{N}$.*

(1) Assume $\|\alpha\|, \|2\alpha\|, \ldots, \|n\alpha\| \leq d < 1/3$. Then $\|\alpha\| \leq d/n$.
(2) Assume $\|\beta+2^0\alpha\|, \|\beta+2^1\alpha\|, \ldots, \|\beta+2^n\alpha\| \leq d < 1/6$. Then $\|\alpha\| \leq d/2^{n-2}$.

Given $\alpha_1, \ldots, \alpha_t \in \mathbb{T}$ and $M \in \mathbb{N}$ we define

$$\langle \alpha_1, \ldots, \alpha_t \rangle_M := \{k_1\alpha_1 + \ldots + k_t\alpha_t : |k_1|, \ldots, |k_t| \leq M\}.$$

We further define $\|\beta S\| := \sup\{\|n\beta\| : n \in S\}$ for $\beta \in \mathbb{T}$ and $S \subseteq \mathbb{N}$.

Lemma 4 *Let $\alpha_1, \ldots, \alpha_t \in \mathbb{T}$ and $\varepsilon > 0$.*

(1) There exists some positive integer M such that

$$\|\beta H_\varepsilon(\alpha_1, \ldots, \alpha_t)\| \leq 1/6 \;\Rightarrow\; \beta \in \langle \alpha_1, \ldots, \alpha_t \rangle_M.$$

(2) If $V \supseteq \langle \alpha_1, \ldots, \alpha_t \rangle_M$ is an open subset of $\mathbb{T}$, there exists some positive integer N such that

$$\|\beta H_{N,\varepsilon}(\alpha_1, \ldots, \alpha_t)\| \leq 1/6 \;\Rightarrow\; \beta \in V.$$

PROOF. Throughout the proof we suppress mentioning $\alpha_1, \ldots, \alpha_t$ while notating Bohr sets.

(1) Suppose β satisfies $\|\beta H_\varepsilon\| \leq 1/6$. Let $\mathfrak{F}$ be a filter on $\mathbb{N}$ that characterizes $\langle \alpha_1, \ldots, \alpha_t \rangle$ and let $m \in \mathbb{N}$ be fixed. Of course we have $H_{\varepsilon/m} \in \mathfrak{F}$. For $n \in H_{\varepsilon/m}$ and $k \leq m$ we have $kn \in H_\varepsilon$ and in particular $\|kn\beta\| \leq 1/6$. Since $1/6 < 1/3$ this implies $\|n\beta\| \leq \frac{1}{6m}$ by Lemma 3. Thus we have $\|\beta H_{\varepsilon/m}\| \leq \frac{1}{6m}$ and since m was arbitrary we get $\mathfrak{F} - \lim_n \|n\beta\| = 0$. $\mathfrak{F}$ was assumed to characterize $\langle \alpha_1, \ldots, \alpha_t \rangle$ thus we have $\beta \in \langle \alpha_1, \ldots, \alpha_t \rangle$.
It remains to show that $\{\beta \in \mathbb{T} : \|\beta H_\varepsilon\| \leq 1/6\}$ is finite. The torsion subgroup of $\langle \alpha_1, \ldots, \alpha_t \rangle$ is finite and cyclic, let its order be $q \in \mathbb{N}$. Then $q\langle \alpha_1, \ldots, \alpha_t \rangle$ is torsion free, hence we find some $\gamma_1, \ldots, \gamma_n \in \mathbb{T}$, such that $q\langle \alpha_1, \ldots, \alpha_t \rangle$ is freely generated by $q\gamma_1, \ldots, q\gamma_n$. We have $\langle \alpha_1, \ldots, \alpha_t \rangle = \langle \gamma_1, \ldots, \gamma_n, 1/q \rangle$ and there are uniquely determined $k_{ij} \in \mathbb{Z}$ $(i \leq t, j \leq n)$ and $k_i \in \{0, \ldots, q-1\}$ $(i \leq t)$, such that

$$\alpha_i = \sum_{j=1}^{n} k_{ij}\gamma_j + k_i/q \;(i \leq t).$$

Thus we can find some $\delta > 0$, such that for all $m \in q\mathbb{N}$

$$\|m\gamma_1\|, \ldots, \|m\gamma_n\| \leq \delta \;\Rightarrow \|m\alpha_1\|, \ldots, \|m\alpha_t\| \leq \varepsilon.$$

For each β satisfying $\|\beta H_\varepsilon\| \leq 1/6$ there are uniquely determined $k_j \in \mathbb{Z}$ $(j \leq n)$ and $k \in \{0, \ldots, q-1\}$ such that $\beta = \sum_{j=1}^n k_j\gamma_j + k/q$. If the k_j $(j \leq n)$ don't vanish simultaneously, Kronecker's theorem assures that we can find $m \in q\mathbb{N}$, such that

$$\begin{array}{cccccc} \forall j \leq n & \dfrac{1}{6\sum_{i=1}^n |k_i|} & < \operatorname{sign}(k_j)m\gamma_j < & \dfrac{5}{6\sum_{i=1}^n |k_i|} & \mod 1 \\ \Longrightarrow & \dfrac{1}{6} & < \; m\sum_{i=1}^n k_i\gamma_i \; < & \dfrac{5}{6} & \mod 1, \end{array}$$

i.e. $\|m\beta\| > 1/6$. Thus $\frac{5}{6\sum_{i=1}^n |k_i|} > \delta$. This shows that there are only finitely many choices for the k_i $(i \leq n)$. Thus $\{\beta \in \mathbb{T} : \|\beta H_\varepsilon\| \leq 1/6\}$ is also finite and we can find some $M \in \mathbb{N}$, such that $\{\beta \in \mathbb{T} : \|\beta H_\varepsilon\| \leq 1/6\} \subseteq \langle \alpha_1, \ldots, \alpha_t\rangle_M$.

(2) Let M be as in (1). Then $\langle \alpha_1, \ldots, \alpha_t\rangle_M \subseteq V$ implies

$$\emptyset = V^c \cap \{\beta \in \mathbb{T} : \|\beta H_\varepsilon\| \leq 1/6\} = V^c \cap \bigcap_{n \in H_\varepsilon} \{\beta \in \mathbb{T} : \|n\beta\| \leq 1/6\}.$$

Since $\mathbb{T}$ is compact and all of the above sets are closed, the intersection of finitely many of these sets must be empty, i.e. we can find some $N \in \mathbb{N}$ such that $V^c \cap \bigcap_{n \in H_{N,\varepsilon}} \{\beta \in \mathbb{T} : \|n\beta\| \leq 1/6\} = \emptyset$. Obviously this N is as required.

□

Lemma 5 *Let $G = \{\alpha_t : t \in \mathbb{N}\}$ be a subgroup of $\mathbb{T}$ and let $(M_t)_{t\in\mathbb{N}}$ be a sequence of positive integers. There exists a sequence $(V_t)_{t\in\mathbb{N}}$ of open subsets of T such that*

(i) $V_t \supseteq \langle \alpha_1, \ldots, \alpha_t\rangle_{M_t}$ $(t \in \mathbb{N})$,
(ii) $\bigcup_{k\in\mathbb{N}} \bigcap_{t\geq k} V_t = \liminf_{t\to\infty} V_t = G$.

PROOF. We may assume that $(M_t)_{t\in\mathbb{N}}$ is increasing. We choose a sequence $(\delta_t)_{t\in\mathbb{N}}$ of positive numbers that decreases to 0 and satisfies for all $t \in \mathbb{N}$

(1) $2\delta_t < \min\{\|\alpha - \alpha'\| : \alpha, \alpha' \in \langle \alpha_1, \ldots, \alpha_t\rangle_{M_t}, \alpha \neq \alpha'\}$,

(2) $$\delta_t + \delta_{t+1} < \min\left\{ \|\alpha - \alpha'\| : \begin{array}{l} \alpha \in \langle \alpha_1, \ldots, \alpha_t\rangle_{M_t}, \\ \alpha' \in \langle \alpha_1, \ldots, \alpha_{t+1}\rangle_{M_{t+1}} \setminus \langle \alpha_1, \ldots, \alpha_t\rangle_{M_t} \end{array} \right\}.$$

Using this, we define

$$V_t := \{\beta \in \mathbb{T} : \exists \alpha \in \langle \alpha_1, \ldots, \alpha_t \rangle_{M_t} \ \|\alpha - \beta\| < \delta_t\}.$$

We obviously have $\liminf_{t\to\infty} V_t \supseteq G$. To show the reverse inclusion, assume $\beta \in \liminf_{t\to\infty} V_t$, i.e. $\beta \in V_t$ for all $t \geq t_0$ for some $t_0 \in \mathbb{N}$. By definition of the V_t for all $t \geq t_0$ there is some $\gamma_t \in \langle \alpha_1, \ldots, \alpha_t \rangle_{M_t}$ satisfying $\|\beta - \gamma_t\| < \delta_t$ and (1) shows that this γ_t is uniquely determined. Further $\gamma_t \neq \gamma_{t+1}$ for some $t \geq t_0$ would contradict (2), thus we have $\gamma_{t_0} = \gamma_{t_0+1} = \gamma_{t_0+2} = \cdots$. In particular this shows $\|\beta - \gamma_{t_0}\| = \|\beta - \gamma_t\| < \delta_t \to 0$, hence $\beta = \gamma_{t_0} \in G$. $\square$

From Lemma 1 in [2] one gets:

Lemma 6 *Let $t \in \mathbb{N}$. There exists some constant $C_1 = C_1(t)$, such that for all $\alpha_1, \ldots, \alpha_t \in \mathbb{T}$, positive $\varepsilon \leq 1/C_1$ and positive integers N there are suitable nonzero integers $n_1, \ldots, n_R$ and positive integers $K_1, \ldots, K_R$, $R \leq C_1$ satisfying*

(a) $\sum_{i=1}^{R} K_i \|n_i \alpha_j\| \leq C_1 \varepsilon \quad (1 \leq j \leq t)$

(b) $\sum_{i=1}^{R} K_i |n_i| \leq C_1 N$,

(c) $H_{N,\varepsilon}(\alpha_1, \ldots, \alpha_t) \subseteq \left\{\sum_{i=1}^{R} k_i n_i : 1 \leq k_i \leq K_i\right\}$.

Lemma 7 *Let $t \in \mathbb{N}$. There exists some constant $C_2 = C_2(t)$, such that for all $\alpha_1, \ldots, \alpha_t \in \mathbb{T}$, positive $\varepsilon \leq 1/C_1(t)$, positive $r \leq 1$ and positive integers N and U there is a suitable nonempty finite set S of integers satisfying*

(i) $U < \min S$,

(ii) for all $j \leq t$ we have $\sum_{n\in S} \|n\alpha_j\|^r \leq C_2 \frac{\varepsilon^r}{2^r - 1}$,

(iii) for all $\beta \in \mathbb{T}$ we have $\min\{1/6, \|\beta H_{N,\varepsilon}(\alpha_1, \ldots, \alpha_t)\|\} \leq \|\beta S\|$.

PROOF. Let $\alpha_1, \ldots, \alpha_t \in \mathbb{T}$ and C_1, R, K_i, n_i $(i \leq R)$ as given by Lemma 6. Let $m > U$ be an integer satisfying

$$\|m\alpha_j\|^r \leq \frac{\varepsilon^r}{\lg_2(8C_1^2 N)}$$

for all $j \leq t$ and let

$$S = \{m + 2^l |n_i| : 2^l \leq 8K_i R\}.$$

Clearly S satisfies (i).
For each $j \leq t$ we have

$$\sum_{n\in S} \|n\alpha_j\|^r \leq \mathrm{card}(S)\|m\alpha_j\|^r + \sum_{n\in S} \|(n - m)\alpha_j\|^r.$$

To find an upper bound for the first term, we observe that $K_i \le C_1 N$ implies $\operatorname{card}(S) \le R \lg_2(8C_1NR)$. Thus

$$\operatorname{card}(S)\|m\alpha_j\|^r \le R\lg_2(8C_1NR)\frac{\varepsilon^r}{\lg_2(8C_1^2N)} \le C_1\varepsilon^r.$$

The second term can be estimated by

$$\begin{aligned}\sum_{i=1}^{R}\left(\sum_{l=0}^{\lfloor \lg_2(8K_iR)\rfloor} 2^l\|n_i\alpha_j\|\right)^r &\le \sum_{i=1}^{R}\frac{(2^r)^{\lg_2(8K_iR)+1}-1}{2^r-1}\|n_i\alpha\|^r \\ &< \frac{16^rR^r}{2^r-1}\sum_{i=1}^{R}K_i^r\|n_i\alpha\|^r.\end{aligned}$$

For any $a_1,\dots,a_R$ we have $\frac{1}{R}\sum_{i=1}^{R}a_i^r \le \left(\frac{1}{R}\sum_{i=1}^{R}a_i\right)^r$ by Jensen's inequality. This yields

$$\sum_{n\in S}\|(n-m)\alpha_j\|^r \le \frac{16^rR}{2^r-1}\left(\sum_{i=1}^{R}K_i\|n_i\alpha_j\|\right)^r \le \frac{16^rC_1}{2^r-1}\left(C_1\varepsilon\right)^r.$$

Thus S will satisfy (ii) if we let $C_2 := C_1 + 16C_1^2$.
Finally let $\beta \in \mathbb{T}$ and $d := \|\beta S\|$. We may assume $d < 1/6$. Thus by Lemma 3 for all $i \le R$

$$\|m\beta + 2^l|n_i|\beta\| \le d \ (l \le \lg_2(8K_iR))$$

implies

$$\|n_i\beta\| \le \frac{d}{2^{\lfloor lg_2(8K_iR)\rfloor-2}} \le \frac{d}{K_iR}.$$

By Lemma 6 each $n \in H_{N,\varepsilon}(\alpha_1,\dots,\alpha_t)$ has a representation $n = \sum_{i=1}^{R}k_in_i$ for some integers $k_i, (1 \le i \le R)$ satisfying $1 \le k_i \le K_i$. Using this representation we get

$$\|n\beta\| = \left\|\sum_{i=1}^{R}k_in_i\beta\right\| \le \sum_{i=1}^{R}K_i\|n_i\beta\| \le \sum_{i=1}^{R}K_i\frac{d}{K_iR} = d.$$

Thus S satisfies (iii). □

5 Proof of the Theorem

Finally we are able to give the proof of Theorem 1. Let $(\varepsilon_t)_{t\in\mathbb{N}}$ be a sequence of positive numbers, satisfying $\varepsilon_t < 1/C_1(t)$ and $\sum_{t=1}^{\infty}C_2(t)\frac{\varepsilon_t^{1/t}}{2^{1/t}-1} < \infty$. Combining Lemma 4 and Lemma 5 we find a sequence $(N_t)_{t\in\mathbb{N}}$ of positive integers and a sequence $(V_t)_{t\in\mathbb{N}}$ of open subsets of $\mathbb{T}$, such that:

(1) For all $\beta \in \mathbb{T}$ and for all $t \in \mathbb{N}$ $\|\beta H_{N_t,\varepsilon_t}(\alpha_1,\dots,\alpha_t)\| \le 1/6 \ \Rightarrow\ \beta \in V_t$.

(2) $\bigcup_{k\in\mathbb{N}}\bigcap_{t\geq k} V_t = G$.

Using Lemma 7 we find some sequence $(S_t)_{t\in\mathbb{N}}$ of subsets of $\mathbb{N}$ such that for all $t \in \mathbb{T}$

(i) $\max S_t < \min S_{t+1}$,
(ii) $\sum_{n\in S_t} \|n\alpha_j\|^{1/t} \leq C_2(t)\frac{\varepsilon_t^{1/t}}{2^{1/t}-1}$ $(j \leq t)$,
(iii) for all $\beta \in \mathbb{T}$ $\min\{1/6, \|\beta H_{N_t,\varepsilon_t}(\alpha_1,\ldots,\alpha_t)\|\} \leq \|\beta S_t\|$.

By defining $A := \bigcup_{t\in\mathbb{N}} S_t$ we will in fact get a strong characterizing sequence of G as stated in Theorem 1:
Assume $\beta \in G$ and $r > 0$. Then $\beta = \alpha_{t_0}$ for some $t_0 \in \mathbb{N}$. If we let $m > \max\{t_0, 1/r\}$, we have

$$\sum_{n\in A, n\geq \min S_m} \|n\beta\|^r \leq \sum_{t\geq m}\sum_{n\in S_t} \|n\alpha_{t_0}\|^{1/t} \leq \sum_{t\geq t_0} C_2(t)\frac{\varepsilon_t^{1/t}}{2^{1/t}-1} < \infty.$$

Finally, assume $\beta \notin G$. There exists a sequence $t_1 < t_2 < \ldots$ of positive integers such that $\beta \notin V_{t_k}$ $(k \in \mathbb{N})$. So for each $k \in \mathbb{N}$ we have $\|\beta H_{\varepsilon_{t_k},N_{t_k}}(\alpha_1,\ldots,\alpha_{t_k})\| > 1/6$ and thus can find some $n_k \in S_{t_k}$ satisfying $\|\beta n_k\| \geq 1/6$. This shows $\limsup_{n\in A, n\to\infty} \|n\beta\| \geq 1/6$. $\square$

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