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Some New Results in Multiplicative and Additive Ramsey Theory

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Abstract. There are several notions of largeness that make sense in any semigroup, and others such as the various kinds of density that make sense in sufficiently well behaved semigroups including $(\mathbb{N},+)$ and $(\mathbb{N},\cdot)$. It is known that sets with positive multiplicative density must contain arbitrarily large *geoarithmetic progressions*, that is, sets of the form $\big\{r^j(a+id) : i,j \in \{0,1,\ldots,k\}\big\}$. We establish some combined additive and multiplicative Ramsey Theoretic consequences of known algebraic results in the semigroups $(\beta\mathbb{N},+)$ and $(\beta\mathbb{N},\cdot)$, derive some new algebraic results, and derive consequences of them involving geoarithmetic progressions. For example, we show that in any finite partition of $\mathbb{N}$ there must be, for each k, sets of the form $\big\{b(a+id)^j : i,j \in \{0,1,\ldots,k\}\big\}$ together with d, the arithmetic progression $\big\{a+id : i \in \{0,1,\ldots,k\}\big\}$, and the geometric progression $\big\{bd^j : j \in \{0,1,\ldots,k\}\big\}$ in one cell of the partition.

1. Introduction

Our starting point is the famous theorem of van der Waerden [19] which says that whenever the set $\mathbb{N}$ of positive integers is divided into finitely many classes, one of these classes contains arbitrarily long arithmetic progressions. The corresponding statement about geometric progressions is easily seen to be equivalent via the homomorphisms $b : (\mathbb{N},+) \to (\mathbb{N},\cdot)$ and $\ell : (\mathbb{N}\setminus\{1\},\cdot) \to (\mathbb{N},+)$ where by $b(n) = 2^n$ and $\ell(n)$ is the length of the prime factorization of n.

In 1975, Szemerédi [18], showed that any set with positive upper asymptotic density contains arbitrarily long arithmetic progressions. (An ergodic theoretic proof of Szemerédi's Theorem can be found in [5], [6] or [7].) It has recently been shown [1, Theorem 1.3] that any set having positive multiplicative upper Banach density – the

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notion d^*_m of Definition 2.1 below – must contain substantial combined additive and multiplicative structure; in particular it must contain arbitrarily large *geoarithmetic progressions*, that is, sets of the form $\big\{r^j(a+id) : i,j \in \{0,1,\ldots,k\}\big\}$. (The results in [1] actually require a property weaker than having d^*_m positive.) In Corollary 3.8 we show that one cell of any finite partition of $\mathbb{N}$ must satisfy a stronger conclusion than this.

Another simply stated result from [1] is that any multiplicatively large set contains geometric progressions in which the common ratios form an arithmetic progression, that is a set of the form $\big\{b(a+id)^j : i,j \in \{0,1,\ldots,k\}\big\}$. As a consequence, one cell of any finite partition of $\mathbb{N}$ must satisfy this property. We provide in Corollary 4.6 a new reasonably simple proof of this consequence.

In Sections 3 and 4 we shall be concerned with the question of what sort of combined additive and multiplicative structures can be guaranteed to lie in one cell of a finite partition of $\mathbb{N}$. For example, it was shown in 1975 [10] that there exist sequences $\langle x_n\rangle_{n=1}^\infty$ and $\langle y_n\rangle_{n=1}^\infty$ such that $FS(\langle x_n\rangle_{n=1}^\infty) \cup FP(\langle y_n\rangle_{n=1}^\infty)$ is contained in one cell, where $FS(\langle x_n\rangle_{n=1}^\infty) = \{\sum_{n\in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$, $FP(\langle y_n\rangle_{n=1}^\infty) = \{\prod_{n\in F} y_n : F \in \mathcal{P}_f(\mathbb{N})\}$, and for any set X, $\mathcal{P}_f(X)$ is the set of finite nonempty subsets of X.

In Section 3 we present some combined additive and multiplicative results that are easily obtainable from known algebraic results, but do not seem to have been previously stated. For example, the following is a special case of Corollary 3.7.

1.1 Theorem. *Let $m,k \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^m A_i$. Then there exist $i \in \{1,2,\ldots,m\}$, $a,d,b \in A_i$, and $r \in A_i \setminus \{1\}$ such that*

$$\begin{gathered}\big\{br^s : s \in \{0,1,\ldots,k\}\big\} \cup \big\{a+td : t \in \{0,1,\ldots,k\}\big\} \cup \{rd\} \cup \\ \big\{r(a+td) : t \in \{0,1,\ldots,k\}\big\} \cup \big\{bdr^s : s \in \{0,1,\ldots,k\}\big\} \cup \\ \big\{br^s(a+td) : s,t \in \{0,1,\ldots,k\}\big\} \subseteq A_i\,.\end{gathered}$$

In Section 4 we derive several new algebraic results and new combinatorial consequences thereof.

In particular, we have the following consequence of Corollary 4.4.

1.2 Theorem. *Let $m,k \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^m A_i$. Then there exist $i \in \{1,2,\ldots,m\}$ and $a,d,b \in A_i$ such that*

$$\begin{gathered}\big\{b(a+id)^j : i,j \in \{0,1,\ldots,k\}\big\} \cup \big\{bd^j : j \in \{0,1,\ldots,k\}\big\} \\ \cup \big\{a+di : i \in \{0,1,\ldots,k\}\big\} \subseteq A_i\,.\end{gathered}$$

Consider now the following result, which is a consequence of [1, Theorem 3.13].

1.3 Theorem. *Let* $m, k \in \mathbb{N}$. *For each* $i \in \{0, 1, \ldots, k\}$ *let* $\langle x_{i,t} \rangle_{t=1}^{\infty}$ *and* $\langle y_{i,t} \rangle_{t=1}^{\infty}$ *be sequences in* $\mathbb{N}$. *Let* $\mathbb{N} = \bigcup_{s=1}^{m} A_s$. *Then there exist* $s \in \{1, 2, \ldots, m\}$, $F, G \in \mathcal{P}_f(\mathbb{N})$, *and* $a, b \in \mathbb{N}$ *such that* $\big\{ b(a + \sum_{t \in F} x_{i,t}) \cdot (\prod_{t \in G} y_{j,t}) : i, j \in \{0, 1, \ldots, k\} \big\} \subseteq A_s$.

Notice that a particular consequence of Theorem 1.3 is that one cell of each finite partition of $\mathbb{N}$ must contain arbitrarily long geoarithmetic progressions. Further, the common ratio r can be taken from $FP(\langle y_n \rangle_{n=1}^{\infty})$ for any prescribed $\langle y_n \rangle_{n=1}^{\infty}$ and the additive increment d can be guaranteed to be a multiple of some member of $FS(\langle x_n \rangle_{n=1}^{\infty})$ for any prescribed $\langle x_n \rangle_{n=1}^{\infty}$. To see this, for $i \in \{1, 2, \ldots, k\}$ and $t \in \mathbb{N}$, let $x_{i,t} = i x_t$ and $y_{i,t} = (y_t)^i$. Given F and G as guaranteed by Theorem 1.3, let $d = b \cdot \sum_{t \in F} x_t$ and $r = \prod_{t \in G} y_t$.

We show in Theorem 4.8 that one may take $F = G$ in Theorem 1.3 and in Corollary 4.12 that one may eliminate b (and in particular, that the additive increment for the geoarithmetic progressions described above can be taken from $FS(\langle x_n \rangle_{n=1}^{\infty})$ for any $\langle x_n \rangle_{n=1}^{\infty}$). We show also that one may not simultaneously take $F = G$ and eliminate b. The example of Theorem 4.17 shows also that one cannot eliminate the multiplier b and expect to find configurations if the sort given in Theorem 1.3 in all sets with d_m^* equal to 1.

The following consequence of Corollary 3.8 (or alternatively of Corollary 4.12(c) or Corollary 4.23) says that one can always get the additive increment of a geoarithmetic progression as the initial term of a geometric progression in one cell of a finite partition of $\mathbb{N}$.

1.4 Theorem. *Let* $m \in \mathbb{N}$ *and let* $\mathbb{N} = \bigcup_{s=1}^{m} A_s$. *Then there exist* $s \in \{1, 2, \ldots, m\}$, $a, d \in A_s$ *and* $r \in A_s \setminus \{1\}$ *such that* $\big\{ r^j (a + id) : i, j \in \{0, 1, \ldots, k\} \big\} \cup \big\{ d r^j : j \in \{0, 1, \ldots, k\} \big\} \subseteq A_s$.

In Section 5 we establish some limitations on the algebraic approach and prove a theorem which, for countable commutative semigroups, is even stronger than the powerful Central Sets Theorem. (The Central Sets Theorem for the semigroup $(\mathbb{N}, +)$ is [6, Proposition 8.21].) Central subsets of any semigroup are guaranteed substantial combinatorial structure. See [13, Part III] for numerous examples. Several earlier results in the paper follow immediately from this theorem. However, we prove the earlier results directly instead of stating them as corollaries, because the direct proofs are reasonably simple, while the theorem proved in Section 5 might be considered a little daunting.

2. Preliminaries

We shall be concerned with several notions of largeness, both additive and multiplicative. Among these are various notions of density. The notions $\overline{d}$ and $\overline{d}_m$ defined below are referred to as *upper asymptotic density* and the notions d^* and d^*_m are called *upper Banach density.*

2.1 Definition. Let $A \subseteq \mathbb{N}$, and let $\langle p_n\rangle_{n=1}^\infty$ be the sequence of primes in their natural order.

(a) $\overline{d}(A) = \lim\sup\limits_{n\to\infty} \dfrac{|A\cap\{1,2,\ldots,n\}|}{n}$.

(b) $$\begin{aligned} d^*(A) &= \lim\sup_{n-m\to\infty} \frac{|A\cap\{m+1,m+2,\ldots,n\}|}{n-m} \\ &= \lim_{k\to\infty}\sup\left\{\frac{|A\cap(m+\{1,2,\ldots,n\})|}{n} : m\in\mathbb{N} \text{ and } n\geq k\right\}. \end{aligned}$$

(c) For $n\in\mathbb{N}$, $F_n = \big\{\prod_{i=1}^n {p_i}^{\alpha_i} : \text{for each } i\in\{1,2,\ldots,n\},\ \alpha_i\in\{0,1,\ldots,n\}\big\}$.

(d) $\overline{d}_m(A) = \lim\sup\limits_{n\to\infty} \dfrac{|A\cap F_n|}{|F_n|}$.

(e) $d^*_m(A) = \lim\limits_{k\to\infty}\sup\left\{\dfrac{|A\cap(m\cdot F_n)|}{|F_n|} : m\in\mathbb{N} \text{ and } n\geq k\right\}$.

The sequence $\langle F_n\rangle_{n=1}^\infty$ defined above is a *Følner sequence* in $(\mathbb{N},\cdot)$. The notions $\overline{d}_m$ and d^*_m could be defined in terms of any Følner sequence (with the values depending on the choice of the Følner sequence). See [1, Section 2].

Other notions of largeness with which we shall be concerned originated in the study of topological dynamics and make sense in any semigroup. Four of these, namely *thick*, *syndetic*, *piecewise syndetic* and *IP-set* have simple elementary descriptions and we introduce them now. The fifth, *central* is most simply described in terms of the algebraic structure of βS, which we shall describe shortly. Given a semigroup $(S,\cdot)$, a subset A of S, and $x\in S$, we let $x^{-1}A = \{y\in S : xy\in A\}$.

2.2 Definition. Let $(S,\cdot)$ be a semigroup and let $A\subseteq S$.

(a) A is *thick* if and only if whenever $F\in\mathcal{P}_f(S)$ there exists $x\in S$ such that $Fx\subseteq A$.

(b) A is *syndetic* if and only if there exists $G\in\mathcal{P}_f(S)$ such that $S=\bigcup_{t\in G} t^{-1}A$.

(c) A is *piecewise syndetic* if and only if there exists $G\in\mathcal{P}_f(S)$ such that for every $F\in\mathcal{P}_f(S)$ there exists $x\in S$ such that $Fx\subseteq\bigcup_{t\in G} t^{-1}A$.

(d) A is an *IP-set* if and only if there exists a sequence $\langle x_n\rangle_{n=1}^\infty$ in S such that $FP(\langle x_n\rangle_{n=1}^\infty)\subseteq A$.

Notice that each of thick and syndetic imply piecewise syndetic and thick sets are IP-sets. It is easy to construct examples in $(\mathbb{N},+)$ showing that no other implications among these notions is valid in general.

The following Lemma gives a hint why piecewise syndetic sets will be interesting for our purposes. A family $\mathcal{A}$ of subsets of a set X is *partition regular* provided that whenever X is partitioned into finitely many classes, one of these classes contains a member of $\mathcal{A}$.

2.3 Lemma. *Let $(S,\cdot)$ be a semigroup, let $\mathcal{F}$ be a partition regular family of finite subsets of S, and let A be a piecewise syndetic subset of S. Then there exist $t,x\in S$ and $F\in\mathcal{F}$ such that $tFx\subseteq A$. If $(S,\cdot)$ is commutative, then there exist $t\in S$ and $F\in\mathcal{F}$ such that $tF\subseteq A$.*

Proof. Pick $G\in\mathcal{P}_f(S)$ such that for every $F\in\mathcal{P}_f(S)$ there exists $x\in S$ such that $Fx\subseteq\bigcup_{t\in G}t^{-1}A$. For each $F\in\mathcal{P}_f(S)$ choose $x_F\in S$ such that $Fx_F\subseteq\bigcup_{t\in G}t^{-1}A$ and linearly order G. Let ∞ be a point not in G and let $K=G\cup\{\infty\}$ have the discrete topology. For each $F\in\mathcal{P}_f(S)$ define $\varphi_F\in\times_{s\in S}K$ by $\varphi_F(s)=\min\{t\in G:tsx_F\in A\}$ if $s\in F$ and $\varphi_F(s)=\infty$ if $s\notin F$. Direct $\mathcal{P}_f(S)$ by inclusion and let ψ be a cluster point of the net $\langle\varphi_F\rangle_{F\in\mathcal{P}_f(S)}$ in $\times_{s\in S}K$.

Then $S\subseteq\bigcup_{t\in K}\psi^{-1}[\{t\}]$. Pick $t\in K$ and $F\in\mathcal{F}$ such that $F\subseteq\psi^{-1}[\{t\}]$. Let $U=\{\tau\in\times_{s\in S}K:\text{for all } s\in F,\ \tau(s)=\psi(s)\}$. Then U is a neighborhood of ψ so pick $H\in\mathcal{P}_f(S)$ such that $F\subseteq H$ and $\varphi_H\in U$. Then for all $s\in F$, $\varphi_H(s)=t$ so $t\in G$ and $tFx_H\subseteq A$. □

Notice that if $(S,\cdot)$ is not commutative, then both multipliers in Lemma 2.3 may be required. For example, let S be the free semigroup on the letters a and b. Then $\mathcal{F}=\{bF:F\in\mathcal{P}_f(S)\}$ and $\mathcal{G}=\{Fb:F\in\mathcal{P}_f(S)\}$ are partition regular, aS and Sa are piecewise syndetic, there do not exist $F\in\mathcal{F}$ and $x\in S$ with $Fx\subseteq aS$, and there do not exist $F\in\mathcal{G}$ and $t\in S$ with $tF\subseteq Sa$. (In fact, aS is syndetic in S.)

In Section 3 we shall need to deal with the *columns condition*.

2.4 Definition. Let $u,v\in\mathbb{N}$, let C be a $u\times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{c_1},\vec{c_2},\ldots,\vec{c_v}$ be the columns of C. Let $R=\mathbb{Z}$ or $R=\mathbb{Q}$. The matrix C satisfies the *columns condition* over R if and only if there exist $m\in\mathbb{N}$ and $I_1,I_2,\ldots,I_m$ such that

(1) $\{I_1,I_2,\ldots,I_m\}$ is a partition of $\{1,2,\ldots,v\}$.

(2) $\sum_{i\in I_1}\vec{c_i}=\vec{0}$.

(3) If $m > 1$ and $t \in \{2, 3, \ldots, m\}$, let $J_t = \bigcup_{j=1}^{t-1} I_j$. Then there exist $\langle \delta_{t,i} \rangle_{i \in J_t}$ in R such that $\sum_{i \in I_t} \vec{c_i} = \sum_{i \in J_t} \delta_{t,i} \cdot \vec{c_i}$.

In [17], Rado proved that a $u \times v$ matrix C is *kernel partition regular* over $(\mathbb{N}, +)$ (meaning that whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} A_i$, there exist $i \in \{1, 2, \ldots, r\}$ and $\vec{x} \in {A_i}^v$ such that $C\vec{x} = \vec{0}$) if and only if C satisfies the columns condition over $\mathbb{Q}$.

A $u \times v$ matrix C with entries from $\mathbb{Q}$ is *image partition regular* over $(\mathbb{N}, +)$ if and only if whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} A_i$, there exist $i \in \{1, 2, \ldots, r\}$ and $\vec{x} \in \mathbb{N}^v$ such that all entries of $C\vec{x}$ are in A_i. We shall use the custom of denoting the entries of a matrix by the lower case of the same letter whose upper case denotes the matrix, so that the entry in row i and column j of C is denoted by $c_{i,j}$.

2.5 Definition. Let $u, v \in \mathbb{N}$ and let C be a $u \times v$ matrix with entries from $\mathbb{Q}$.

(a) C is a *first entries matrix* if and only if now row of C is $\vec{0}$ and for all $i, j \in \{1, 2, \ldots, u\}$ and all $k \in \{1, 2, \ldots, v\}$, if $k = \min\{t : c_{i,t} \neq 0\} = \min\{t : c_{j,t} \neq 0\}$, then $c_{i,k} = c_{j,k} > 0$.

(b) The number b is a *first entry* of C if and only if b is the first nonzero entry in some row of C.

Each first entries matrix is image partition regular over $(\mathbb{N}, +)$ and image partition regular matrices can be characterized in terms of first entries matrices. (See [13, Theorem 15.24].)

We now present a brief review of basic facts about $(\beta S, \cdot)$. For additional information and any unfamiliar terminology encountered see [13].

Given a discrete semigroup $(S, \cdot)$ we take the points of the Stone-Čech compactification βS of S to be the ultrafilters on S, the principal ultrafilters being identified with the points of S. Given $A \subseteq S$, $\overline{A} = \{p \in \beta S : A \in p\}$ and the set $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets (and a basis for the closed sets) of βS. Given $p, q \in \beta S$ and $A \subseteq S$, $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$.

With this operation, $(\beta S, \cdot)$ is a compact Hausdorff right topological semigroup with S contained in its topological center. That is, for each $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous and for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x \cdot q$ is continuous. A subset I of a semigroup T is a *left ideal* provided $T \cdot I \subseteq I$, a *right ideal* provided $I \cdot T \subseteq I$, and a *two sided ideal* (or simply an *ideal*) provided it is both a left ideal and a right ideal.

Any compact Hausdorff right topological semigroup T has a smallest two sided ideal $K(T) = \bigcup\{L : L \text{ is a minimal left ideal of } T\} = \bigcup\{R : R \text{ is a minimal right ideal}$

of $T\}$. Given a minimal left ideal L and a minimal right ideal R, $L\cap R$ is a group, and in particular contains an idempotent. An idempotent in $K(T)$ is a *minimal* idempotent. If p and q are idempotents in T we write $p\leq q$ if and only if $pq=qp=p$. An idempotent is minimal with respect to this relation if and only if it is member of the smallest ideal.

A subset of S is an IP-set if and only if it is a member of some idempotent in βS.

2.6 Definition. Let S be a semigroup and let $A\subseteq S$. Then A is *central* if and only if there is a minimal idempotent p of βS such that $A\in p$.

A central set is in particular a piecewise syndetic IP-set. Given a minimal idempotent p and a finite partition of S, one cell must be a member of p, hence at least one cell of any finite partition of S must be central. Central sets are fundamental to the Ramsey Theoretic applications of the algebra of βS.

We shall need the Hales-Jewett Theorem. Given the free semigroup S over an alphabet L, a *variable word* w is a word over $L\cup\{v\}$ in which v occurs, where v is a "variable" not in L. Given a variable word w and $a\in L$, $w(a)$ is the word in S obtained by replacing each occurrence of v by a.

2.7 Theorem (Hales-Jewett). *Let L be a finite alphabet, let S be the free semigroup over L, let $m\in\mathbb{N}$, and let $S=\bigcup_{i=1}^{m}A_i$. Then there exist $i\in\{1,2,\ldots,m\}$ and a variable word w such that $\{w(a):a\in L\}\subseteq A_i$.*

Proof. [9, Theorem 1], or see [8, Theorem 2.3] or [13, Theorem 14.7]. □

Applications which we will use later are the following theorems. These results are well known among afficianados.

2.8 Theorem. *Let $(S,\cdot)$ be a commutative semigroup, let A be a piecewise syndetic subset of S, let $k\in\mathbb{N}$, and for $i\in\{1,2,\ldots,k\}$ let $\langle y_{i,n}\rangle_{i=1}^{\infty}$ be a sequence in S. There exist $F\in\mathcal{P}_f(\mathbb{N})$ and $b\in S$ such that $\{b\}\cup\big\{b\prod_{t\in F}y_{i,t}:i\in\{1,\ldots,k\}\big\}\subseteq A$.*

Proof. By virtue of Lemma 2.3 it is sufficient to show that the family

$$\big\{\{b\}\cup\big\{b\cdot\textstyle\prod_{t\in F}y_{i,t}:i\in\{1,\ldots,k\}\big\}:b\in S,F\in\mathcal{P}_f(\mathbb{N})\big\}$$

is partition regular.

Let $L=\{0,1,\ldots,k\}$ and let T be the free semigroup on the alphabet L. Let $b_0\in S$ be an arbitrary, fixed element. Given a word $w=l_1l_2\cdots l_n$ of length n in S, define $f(w)=b_0\prod_{t\in\{1,2,\ldots,n\},l_t\neq 0}y_{l_t,t}$ if there exists some $t\in\{1,2,\ldots,n\}$ such that $l_t\neq 0$ and $f(w)=b_0$ otherwise.

Consider a partition $\{A_1, A_2, \ldots, A_m\}$ of S. Then $T = \bigcup_{s=1}^{m} f^{-1}[A_s]$ so pick $s \in \{1, 2, \ldots, m\}$ and a variable word $w = l_1 l_2 \cdots l_n$ (with each $l_t \in L \cup \{v\}$) such that $\{w(i) : i \in L\} \subseteq f^{-1}[A_s]$.

Let $F = \{t \in \{1, 2, \ldots, n\} : l_t = v\}$, let $G = \{1, 2, \ldots, n\} \setminus F$ and let $b = f\bigl(w(0)\bigr)$. Then $b \prod_{t \in F} y_{i,t} = f\bigl(w(i)\bigr)$ for $i \in \{1, 2, \ldots, k\}$ and thus $\{b\} \cup \bigl\{b \prod_{t \in F} y_{i,t} : i \in \{1, \ldots, k\}\bigr\} \subseteq A_s$. $\square$

2.9 Corollary. *Let $(S, \cdot)$ be a commutative semigroup, let A be a piecewise syndetic subset of S, let B be an IP-set in S, and let $k \in \mathbb{N}$. There exist $b \in S$ and $r \in B$ such that $\{b, br, br^2, \ldots, br^k\} \subseteq A$. If A is central we may in particular take $A = B$, such that $\{r, b, br, br^2, \ldots, br^k\} \subseteq A$.*

Proof. Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in S such that $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq B$. For $i \in \{1, 2, \ldots, k\}$ and $n \in \mathbb{N}$, let $y_{i,n} = (x_n)^i$. Pick b and F as guaranteed by Theorem 2.8 and let $r = \prod_{t \in F} x_t$.

Any central set is a piecewise syndetic IP-set and thus the in particular statement follows. $\square$

3. New wine from old wineskins

All of the results about the algebraic structure of $\beta\mathbb{N}$ that are used in this section have been known for several years.

There is a long list of configurations which are known to be present in any central subset of $(\mathbb{N}, +)$ and a somewhat shorter, but still lengthy, list of structures which can be found in any central subset of $(\mathbb{N}, \cdot)$. Some of these involve special subsets of $\beta\mathbb{N}$ defined by various notions of density.

3.1 Definition.

(a) $\Delta = \{q \in \beta\mathbb{N} : (\forall A \in q)(\overline{d}(A) > 0)\}$.

(b) $\Delta^* = \{q \in \beta\mathbb{N} : (\forall A \in q)(d^*(A) > 0)\}$.

(c) $\Delta_m = \{q \in \beta\mathbb{N} : (\forall A \in q)(\overline{d}_m(A) > 0)\}$.

(d) $\Delta_m^* = \{q \in \beta\mathbb{N} : (\forall A \in q)(d_m^*(A) > 0)\}$.

We summarize some of the structures guaranteed to be present in any multiplicatively central set first. See [13, Chapter 14] for a formalization of the notion of *tree* in a set as well as the set of successors to a node.

3.2 Theorem. *Let A be a central subset of* $(\mathbb{N},\cdot)$.

- *(a) For any sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ and any $k\in\mathbb{N}$, there exist $b\in\mathbb{N}$ and $r\in FP(\langle x_n\rangle_{n=1}^\infty)$ such that $\{b,br,br^2,\ldots,br^k\}\subseteq A$.*
- *(b) There is a tree T in A such that for any path g through T, $FP(\langle g(n)\rangle_{n=1}^\infty)\subseteq A$ and for every node $f\in T$, the set B_f of successors to f satisfies $d_m^*(B_f)>0$.*
- *(c) If $u,v\in\mathbb{N}$ and C is a $u\times v$ matrix with entries from $\mathbb{Z}$ which satisfies the columns condition over $\mathbb{Z}$, then there exists $\vec{x}\in A^v$ such that for all $i\in\{1,2,\ldots,u\}$, $\prod_{j=1}^v {x_j}^{c_{i,j}}=1$.*
- *(d) If $u,v\in\mathbb{N}$ and C is a $u\times v$ first entries matrix with entries from $\mathbb{Z}$ and all first entries equal to 1, then there exists $\vec{x}$ in $\mathbb{N}^v$ such that for all $i\in\{1,2,\ldots,u\}$, $\prod_{j=1}^v {x_j}^{c_{i,j}}\in A$.*

Proof. (a) Corollary 2.9.

(b) Pick a minimal idempotent q of $(\beta\mathbb{N},\cdot)$ such that $A\in q$. By [13, Theorems 20.5 and 20.6] Δ_m^* is an ideal of $(\beta\mathbb{N},\cdot)$, so $q\in\Delta_m^*$ and [13, Lemma 14.24] applies.

(c) [13, Theorem 15.16(a)].

(d) [13, Lemma 15.14 and Theorem 15.5]. □

The conditions of Theorem 3.2(c) and (d) are stronger than those required for kernel and image partition regularity over $(\mathbb{N},\cdot)$. (And necessarily so. The set $A=\mathbb{N}\setminus\{x^2:x\in\mathbb{N}\}$ is central in $(\mathbb{N},\cdot)$ [13, Exercise 15.1.2], the matrix $\begin{pmatrix}2 & -2 & 1\end{pmatrix}$ is kernel partition regular over $(\mathbb{N},\cdot)$, and the matrix (2) is image partition regular over $(\mathbb{N},\cdot)$. But one cannot get $x,y,z\in A$ with $x^2y^{-2}z=1$ and one cannot get $x\in\mathbb{N}$ with $x^2\in A$.) By contrast, in $(\mathbb{N},+)$, kernel partition regularity of C corresponds to solutions to $C\vec{x}=\vec{0}$ in any central set and image partition regularity of C corresponds to obtaining all entries of $C\vec{x}$ in any central set.

We shall be interested in a property stronger than central for our additive results. By [13, Theorem 6.79], Δ is a compact left ideal of $(\beta\mathbb{N},+)$ so contains a minimal idempotent of $(\beta\mathbb{N},+)$. Consequently, any finite partition of $\mathbb{N}$ will have one cell satisfying the hypothesis of the following theorem.

3.3 Theorem. *Let $A\subseteq\mathbb{N}$ and assume that there is a minimal idempotent q of $(\beta\mathbb{N},+)$ in $\overline{A}\cap\Delta$.*

- *(a) For any sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ and any $k\in\mathbb{N}$, there exist $a\in\mathbb{N}$ and $d\in FS(\langle x_n\rangle_{n=1}^\infty)$ such that $\{a,a+d,\ldots,a+kd\}\subseteq A$.*

(b) There is a tree T in A such that for any path g through T, $FS(\langle g(n)\rangle_{n=1}^{\infty}) \subseteq A$ and for every node $f \in T$, the set B_f of successors to f satisfies $\overline{d}(B_f) > 0$.

(c) If $u, v \in \mathbb{N}$ and C is a $u \times v$ matrix with entries from $\mathbb{Q}$ which is kernel partition regular over $(\mathbb{N}, +)$ (that is C satisfies the columns condition over $\mathbb{Q}$), then there exists $\vec{x} \in A^v$ such that $C\vec{x} = \vec{0}$.

(d) If $u, v \in \mathbb{N}$ and C is a $u \times v$ matrix with entries from $\mathbb{Q}$ which is image partition regular over $(\mathbb{N}, +)$, (in particular if C is a first entries matrix), then there exists $\vec{x}$ in $\mathbb{N}^v$ with all entries of $C\vec{x}$ in A.

(e) Let R be a finite set of polynomials which take integer values at integers and have zero constant term and Let $\langle z_i\rangle_{i=1}^{\infty}$ be a sequence in $\mathbb{Z}$. Then there exists $F \in \mathcal{P}_f(\mathbb{N})$ such that $\{a \in A : \{a + p(\Sigma_{i\in F} z_i) : p \in R\} \subseteq A\}$ is piecewise syndetic.

Proof. (a) Corollary 2.9.

(b) [13, Lemma 14.24].

(c) [13, Theorem 15.16(b)]

(d) [12, Theorem 2.10].

(e) In [3, Theorem C] it was shown that the conclusion follows from the assumption that A is piecewise syndetic. For an algebraic proof see [11, Corollary 3.7]. $\square$

3.4 Lemma. *Let $D = \{q \in \Delta : q \text{ is a minimal idempotent of } (\beta\mathbb{N}, +)\}$. Then $c\ell D$ is a left ideal of $(\beta\mathbb{N}, \cdot)$.*

Proof. We have already observed that $D \neq \emptyset$. Let $r \in c\ell D$. To see that $\beta\mathbb{N} \cdot r \subseteq c\ell R$ it suffices by the continuity of ρ_r in $(\beta\mathbb{N}, \cdot)$ to show that $\mathbb{N} \cdot r \subseteq c\ell D$. So let $x \in \mathbb{N}$ and let $A \in x \cdot r$. Then $x^{-1}A \in r$ so pick $q \in D \cap \overline{x^{-1}A}$. Then $A \in x \cdot q$. By [13, Theorem 6.79] $x \cdot q \in \Delta$. By [12, Lemma 2.1] $x \cdot q$ is a minimal idempotent of $(\beta\mathbb{N}, +)$. $\square$

Plentiful examples of candidates for the sets $\mathcal{G}$ and $\mathcal{H}$ of Theorem 3.5 are provided by Theorems 3.2 and 3.3. Notice in particular that $\mathcal{H}$ could be any family of subsets of $\mathbb{N}$ such that any additively central set must contain a member of $\mathcal{H}$.

3.5 Theorem. *Let $D = \{q \in \Delta : q \text{ is a minimal idempotent of } (\beta\mathbb{N}, +)\}$. Let $\mathcal{G}$ be a set of finite subsets of $\mathbb{N}$ with the property that any multiplicatively central subset of $\mathbb{N}$ contains a member of $\mathcal{G}$ and let $\mathcal{H}$ be a set of (finite or infinite) subsets of $\mathbb{N}$ with the property that, whenever $A \subseteq \mathbb{N}$ and $\overline{A} \cap D \neq \emptyset$, some member of $\mathcal{H}$ is contained in A. Whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} A_i$, there exists $i \in \{1, 2, \ldots, r\}$ such that $\overline{d}(A_i) > 0$, $d_m^*(A_i) > 0$, and there exist $B \in \mathcal{G}$ and $C \in \mathcal{H}$ such that $B \cup C \cup B \cdot C \subseteq A_i$.*

Proof. By Lemma 3.4, D is a left ideal of $(\beta\mathbb{N},\cdot)$ so pick a minimal idempotent q of $(\beta\mathbb{N},\cdot)$ in $c\ell D$. Pick $i\in\{1,2,\ldots,r\}$ such that $A_i\in q$. Since $q\in c\ell D\subseteq\Delta$, $\overline{d}(A_i)>0$. By Theorem 3.2(b), $d^*_m(A_i)>0$. Since $q=q\cdot q$, $\{x\in A_i : x^{-1}A_i\in q\}\in q$. In particular $\{x\in A_i : x^{-1}A_i\in q\}$ is multiplicatively central, so pick $B\in\mathcal{G}$ such that $B\subseteq\{x\in A_i : x^{-1}A_i\in q\}$. Since B is finite, $A_i\cap\bigcap_{x\in B}x^{-1}A_i\in q$ and thus $\overline{(A_i\cap\bigcap_{x\in B}x^{-1}A_i)}\cap D\neq\emptyset$. Pick $C\in\mathcal{H}$ such that $C\subseteq A_i\cap\bigcap_{x\in B}x^{-1}A_i$. $\square$

By adding the requirement that the members of $\mathcal{H}$ be finite, we obtain an infinitary extension of Theorem 3.5 along the lines of the Central Sets Theorem.

3.6 Theorem. *Let* $D=\{q\in\Delta : q\ \text{ is a minimal idempotent of }\ (\beta\mathbb{N},+)\}$. *For each* $n\in\mathbb{N}$, *let* $\mathcal{G}_n$ *be a set of finite subsets of* $\mathbb{N}$ *with the property that any multiplicatively central subset of* $\mathbb{N}$ *contains a member of* $\mathcal{G}_n$ *and let* $\mathcal{H}_n$ *be a set of finite subsets of* $\mathbb{N}$ *with the property that, whenever* $A\subseteq\mathbb{N}$ *and* $\overline{A}\cap D\neq\emptyset$, *some member of* $\mathcal{H}_n$ *is contained in* A. *Whenever* $r\in\mathbb{N}$ *and* $\mathbb{N}=\bigcup_{i=1}^r A_i$, *there exists* $i\in\{1,2,\ldots,r\}$ *such that* $\overline{d}(A_i)>0$, $d^*_m(A)>0$, *and there exist sequences* $\langle B_n\rangle_{n=1}^\infty$ *and* $\langle C_n\rangle_{n=1}^\infty$ *such that* $B_n\in\mathcal{G}_n$ *and* $C_n\in\mathcal{H}_n$ *for each* n *and for any* $F\in\mathcal{P}_f(\mathbb{N})$ *and any* $f\in\times_{n\in F}(B_n\cup C_n\cup B_n\cdot C_n)$, $\prod_{n\in F}f(n)\in A_i$.

Proof. Pick a minimal idempotent q of $(\beta\mathbb{N},\cdot)$ in $c\ell D$ and pick $i\in\{1,2,\ldots,r\}$ such that $A_i\in q$. Then $\overline{d}(A_i)>0$ and $d^*_m(A)>0$. For any $X\in q$, let $X^\star=\{x\in X : x^{-1}X\in q\}$. Then by [13, Lemma 4.14] $X^\star\in q$ and for any $x\in X^\star$, $x^{-1}X^\star\in q$.

Choose $B_1\in\mathcal{G}_1$ such that $B_1\subseteq A_1{}^\star$ and choose $C_1\in\mathcal{H}_1$ such that $C_1\subseteq A_1{}^\star\cap\bigcap_{x\in B_1}x^{-1}A_1{}^\star$.

Inductively, let $n\in\mathbb{N}$ and assume we have chosen $B_t\in\mathcal{G}_t$ and $C_t\in\mathcal{H}_t$ for each $t\in\{1,2,\ldots,n\}$ with the property that for all nonempty $F\subseteq\{1,2,\ldots,n\}$ and all $f\in\times_{t\in F}(B_t\cup C_t\cup B_t\cdot C_t)$, $\prod_{t\in F}f(t)\in A_i{}^\star$. Let

$$X=A_i{}^\star\cap\bigcap\Big\{\big(\textstyle\prod_{t\in F}f(t)\big)^{-1}A_i{}^\star :\ \begin{array}{l}\emptyset\neq F\subseteq\{1,2,\ldots,n\}\text{ and}\\ f\in\times_{t\in F}(B_t\cup C_t\cup B_t\cdot C_t)\Big\}.\end{array}$$

Then X is a finite intersection of members of q so $X\in q$. Pick $B_{n+1}\in\mathcal{G}_{n+1}$ such that $B_{n+1}\subseteq X^\star$. Then $X\cap\bigcap_{x\in B_{n+1}}x^{-1}X\in q$ so pick $C_{n+1}\in\mathcal{H}_{n+1}$ such that $C_{n+1}\subseteq X\cap\bigcap_{x\in B_{n+1}}x^{-1}X$. $\square$

We shall be concerned in the next section with extensions of the following sort of configuration.

3.7 Corollary. *Let* $m,k\in\mathbb{N}$ *and let* $\mathbb{N}=\bigcup_{i=1}^m A_i$. *Then there exist* $i\in\{1,2,\ldots,m\}$,

$a, d, b \in A_i$, and $r \in A_i \setminus \{1\}$ such that $\overline{d}(A_i) > 0$, $d^*_m(A_i) > 0$, and

$$\begin{gathered}\big\{br^s : s \in \{0,1,\ldots,k\}\big\} \cup \big\{a+td : t \in \{0,1,\ldots,k\}\big\} \cup \{rd\} \cup \\ \big\{r(a+td) : t \in \{0,1,\ldots,k\}\big\} \cup \big\{bdr^s : s \in \{0,1,\ldots,k\}\big\} \cup \\ \big\{br^s(a+td) : s,t \in \{0,1,\ldots,k\}\big\} \subseteq A_i\,.\end{gathered}$$

Proof. Let $\mathcal{G} = \Big\{\big\{br^s : s \in \{0,1,\ldots,k\}\big\} \cup \{r\} : b, r \in \mathbb{N}\Big\}$ and let

$$\mathcal{H} = \Big\{\big\{a+td : t \in \{0,1,\ldots,k\}\big\} \cup \{d\} : a, d \in \mathbb{N}\Big\}\,.$$

By applying Theorem 2.9 to $(\mathbb{N}, \cdot)$ and to $(\mathbb{N}, +)$ one concludes that every multiplicatively central set contains a member of $\mathcal{G}$ and that every additively central set contains a member of $\mathcal{H}$. Thus we may apply Theorem 3.6. By assigning 1 to its own cell one may ensure that $r \neq 1$. □

3.8 Corollary. *Let $m, k \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^m A_i$. Then there exist $i \in \{1,2,\ldots,m\}$, $a, d \in A_i$, and $r \in A_i \setminus \{1\}$ such that $\overline{d}(A_i) > 0$, $d^*_m(A_i) > 0$, and*

$$\big\{r^s(a+td) : s,t \in \{0,1,\ldots,k\}\big\} \cup \{dr^s : s \in \{0,1,\ldots,k\}\} \subseteq A_i\,.$$

Proof. Let i, a, b, d, r be as in the proof of Corollary 3.7. Put $a_1 = ab$ and $d_1 = db$. Then $\{a_1\} \cup \{d_1\} \cup \big\{r^s(a_1+td_1) : s,t \in \{0,1,\ldots,k\}\big\} \cup \big\{d_1 r^s : s \in \{0,1,\ldots,k\}\big\} \subseteq A_i$. □

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4. Extensions of geoarithmetic progressions

A *geoarithmetic progression* is a set of the form $\big\{r^j(a+id) : i, j \in \{0,1,\ldots,k\}\big\}$ where $a, d, k \in \mathbb{N}$ and $r \in \mathbb{N} \setminus \{1\}$. We shall be concerned in this section with finding certain generalizations of geoarithmetic progressions in one cell of a finite partition of $\mathbb{N}$.

Our first result in this direction (Corollary 4.3) replaces r in a geometric progression by multiples of members of any partition regular family of finite sets. For that result, one needs to add a multiplier b because one can certainly not expect to find a set of the form $\{r, r^2\}$ for $r > 1$ in one cell of an arbitrary finite partition of $\mathbb{N}$; one may assign the members of $\mathbb{N} \setminus \big\{x^2 : x \in \mathbb{N} \setminus \{1\}\big\}$ to A_1 or A_2 at will, and then assign x^2 to the cell that x is not in, x^4 to the cell x^2 is not in, and so on.

To establish Theorem 4.3 we need the following algebraic result which is of interest in its own right. We let $\omega = \mathbb{N} \cup \{0\}$. The case $(S, +) = (\omega, +)$ of Theorem 4.1 follows from [12, Theorem 2.10]. In any semigroup S, a set $C \subseteq S$ is *central** if and only if for every central subset B of S, $C \cap B \neq \emptyset$. (Equivalently, $S \setminus C$ is not central.) Notice in particular that always S is central* so that if all first entries of a first entries matrix A are equal to 1, the requirement that $1S$ be central* is automatically satisfied.

4.1 Theorem. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ first entries matrix with entries from ω. Let $(S, +)$ be a commutative semigroup with identity 0 and let C be a central subset of S. If for every first entry c of A, cS is central*, then $\{\vec{x} \in S^v : A\vec{x} \in C^u\}$ is central in S^v.*

Proof. Pick a minimal idempotent e of βS such that $C \in e$. Define $\varphi : S^v \to S^u$ by $\varphi(\vec{x}) = A\vec{x}$ and let $\widetilde{\varphi} : \beta(S^v) \to (\beta S)^u$ be its continuous extension. Let $M = \{p \in \beta(S^v) : \widetilde{\varphi}(p) = (e, e, \ldots, e)^T\}$. By [13, Corollary 4.22] $\widetilde{\varphi}$ is a homomorphism, so to see that M is a subsemigroup, it suffices to show that $M \neq \emptyset$.

For each $B \in e$ pick by [13, Theorem 15.5] $\vec{x}_B \in S^v$ such that $\varphi(\vec{x}_B) \in B^u$. Direct e by reverse inclusion and let q be a limit point in $\beta(S^v)$ of the net $\langle \vec{x}_B \rangle_{B \in e}$. Then $q \in M$.

Since M is a compact right topological semigroup, pick a minimal idempotent r of M. We claim that r is minimal in $\beta(S^v)$. To see this, let p be an idempotent of $\beta(S^v)$ such that $p \leq r$. Then $\widetilde{\varphi}(p) \leq \widetilde{\varphi}(r) = (e, e, \ldots, e)^T$ and $(e, e, \ldots, e)^T$ is minimal in $(\beta S)^u$ by [13, Theorem 2.23] so $\widetilde{\varphi}(p) = (e, e, \ldots, e)^T$. Thus $p \in M$ and so $p = r$.

Pick $X \in r$ such that $\widetilde{\varphi}[\,\overline{X}\,] \subseteq (\,\overline{B}\,)^u$. Then $X \subseteq \{\vec{x} \in S^v : A\vec{x} \in B^u\}$. $\square$

4.2 Theorem. *Let $(S, \cdot)$ be a commutative semigroup with identity and let C be a central subset of S. If $\mathcal{F}$ is a partition regular family of finite subsets of S and $k \in \mathbb{N}$, then there exist $b, r \in S$ and $F \in \mathcal{F}$ such that $rF \cup \big\{b(rx)^j : x \in F \text{ and } j \in \{0, 1, \ldots, k\}\big\} \subseteq C$.*

Proof. Let $k \in \mathbb{N}$ and let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & k \end{pmatrix}.$$

Then A is a first entries matrix with all first entries equal to 1 so by Theorem 4.1 $\{(b, r) \in S^2 : \{b, r, br, \ldots, br^k\} \subseteq C\}$ is central in S^2 and is in particular piecewise syndetic. Let $\mathcal{G} = \big\{\{b\} \times F : b \in S \text{ and } F \in \mathcal{F}\big\}$. Then $\mathcal{G}$ is a partition regular family of finite subsets of S^2 so pick by Lemma 2.3 $F \in \mathcal{F}$, $c \in S$, and $(s, r) \in S^2$ such that $(s, r) \cdot (\{c\} \times F) \subseteq \{(b, r) \in S^2 : \{b, r, br, \ldots, br^k\} \subseteq C\}$. Let $b = sc$. $\square$

Notice that, if in the above proof, the matrix A is replaced by a matrix whose set of rows is $\{(0, 0, 1)\} \cup \big\{(0, 1, j) : j \in \{0, 1, \ldots, k\}\big\} \cup \big\{(1, i, j) : i, j \in \{0, 1, \ldots, k\}\big\}$, then the conclusion of Theorem 4.2 becomes "there exist $b, c, r \in S$ and $F \in \mathcal{F}$ such that $rF \cup \big\{b(rx)^j : x \in F \text{ and } j \in \{0,1,\ldots,k\}\big\} \cup \big\{cb^i(rx)^j : x \in F \text{ and } i, j \in \{0,1,\ldots,k\}\big\} \subseteq C$." Of course additional strengthenings can be obtained using first entries matrices with all first entries equal to 1 and additional columns.

4.3 Corollary. *Let $\mathcal{F}$ be a partition regular family of finite subsets of $\mathbb{N}$, let $k \in \mathbb{N}$, and let A be piecewise syndetic in $(\mathbb{N}, \cdot)$. Then there exist $b, r \in \mathbb{N}$ and $F \in \mathcal{F}$ such that $\{b(rx)^j : j \in \{0,1,\ldots,k\}$ and $x \in F\} \subseteq A$.*

Proof. Pick by [13, Theorem 4.43] $t \in \mathbb{N}$ such that $t^{-1}A$ is central in $(\mathbb{N}, \cdot)$. Pick by Theorem 4.2 $c, r \in \mathbb{N}$ such that $rF \cup \big\{c(rx)^j : x \in F \text{ and } j \in \{0,1,\ldots,k\}\big\} \subseteq t^{-1}A$ and let $b = tc$. $\square$

We see now that, given any central subset C of $(\mathbb{N}, \cdot)$ we can get sets of the form $\big\{b(a+id)^j : i, j \in \{0,1,\ldots,k\}\big\}$ together with the multiplier, the increment, and the arithmetic progression in C.

4.4 Corollary. *Let C be a central subset of $(\mathbb{N}, \cdot)$ and let $k \in \mathbb{N}$. There exist $a, b, d \in \mathbb{N}$ such that*

$$\begin{gathered}\big\{b(a+td)^j : t, j \in \{0,1,\ldots,k\}\big\} \cup \big\{bd^j : j \in \{0,1,\ldots,k\}\big\} \\ \cup \big\{a+td : t \in \{0,1,\ldots,k\}\big\} \cup \{d\} \subseteq C\,.\end{gathered}$$

Proof. Let $\mathcal{F} = \big\{\{d, a, a+d, \ldots, a+kd\} : a, d \in \mathbb{N}\big\}$. Pick by Theorem 4.2 $b, r \in S$ and $F \in \mathcal{F}$ such that $rF \cup \big\{b(rx)^j : x \in F \text{ and } j \in \{0,1,\ldots,k\}\big\} \subseteq C$. Pick $c, s \in \mathbb{N}$ such that $F = \{c, s, s+c, \ldots, s+kc\}$. Let $d = rc$ and $a = rs$. $\square$

Again note that if the stronger version of Theorem 4.2 that we mentioned after its proof is used, the conclusion of Corollary 4.4 becomes "There exist $a, b, c, d \in \mathbb{N}$ such that

$$\begin{gathered}\big\{cb^i(a+td)^j : t, i, j \in \{0,1,\ldots,k\}\big\} \cup \big\{cb^id^j : i, j \in \{0,1,\ldots,k\}\big\} \\ \cup \big\{b(a+td)^j : t, j \in \{0,1,\ldots,k\}\big\} \cup \big\{bd^j : j \in \{0,1,\ldots,k\}\big\} \\ \cup \big\{a+td : t \in \{0,1,\ldots,k\}\big\} \cup \{d\} \subseteq C\,."\end{gathered}$$

We remark also that Corollary 4.4 could also be stated in terms of an arbitrary commutative ring with no change in proof.

The following result is stronger than Corollary 4.4. We state it separately because its formulation is more involved and the proof requires more theoretical background.

4.5 Corollary. *Let S be an infinite set with operations $+$ and $\cdot$ such that $(S, +)$ is a commutative semigroup with identity 0, $(S \setminus \{0\}, \cdot)$ is a commutative semigroup with identity 1, and $\cdot$ distributes over $+$. Let C be a central subset of $(S \setminus \{0\}, \cdot)$, let $k \in \mathbb{N}$,* *and let G be a finite subset of $S \setminus \{0\}$. Then there exist $a, b, d \in C$ such that*

$$\big\{b(a+di)^j : i \in G \text{ and } j \in \{0,1,\ldots,k\}\big\} \cup \big\{bd^j : j \in \{0,1,\ldots,k\}\big\} \\ \cup \{a+di : i \in G\} \subseteq C\,.$$

Proof. We observe first that $S\setminus\{0\}$ is central in $(S,+)$. To see this, suppose instead that 0 is a minimal idempotent of $(\beta S,+)$. Then by [13, Theorem 2.9] $\beta S = 0+\beta S = \beta S+0$ is a group and in particular $(S,+)$ is cancellative. But then by [13, Theorem 4.36] $\beta S\setminus S$ is an ideal of $(\beta S,+)$ and so $0 \in \beta S \setminus S$, a contradiction.

Let $\mathcal{F} = \big\{\{a,d\} \cup \{a+dj : j \in G\} : a,d \in S\big\}$. We claim that $\mathcal{F} \cap \mathcal{P}(S \setminus \{0\})$ is partition regular in $S\setminus\{0\}$. So let $r \in \mathbb{N}$ and let $S\setminus\{0\} = \bigcup_{i=1}^{r} D_i$. Pick $i \in \{1,2,\ldots,r\}$ such that D_i is central in $(S,+)$. Let $\langle d_n\rangle_{n=1}^{\infty}$ be a sequence such that $FS(\langle d_n\rangle_{n=1}^{\infty}) \subseteq D_i$. Theorem 2.8 applied to the sequences $\langle jd_n\rangle_{n=1}^{\infty}$ for $j \in G$ yields that there exist $a \in D_i$ and $F \in \mathcal{P}_f(\mathbb{N})$ such that $a + \sum_{t\in F} jd_t \in D_i$ for all $j \in G$. If we let $d = \sum_{t\in F} d_t$ we see that $\{a,d\} \cup \{a+dj : j \in G\} \subseteq D_i$.

Pick by Theorem 4.2 $b, r \in S\setminus\{0\}$ and $F \in \mathcal{F}\cap\mathcal{P}(S\setminus\{0\})$ such that $rF \cup \big\{b(rx)^j : x \in F \text{ and } j \in \{0,1,\ldots,k\}\big\} \subseteq C$. Pick $c, s \in S$ such that $F = \{c,s\} \cup \{s+ic : i \in G\}$. Let $d = rc$ and $a = rs$. Since $a, d \in rF$, we have $a, d \in C$. Also $b = ba^0$ so $b \in C$. □

Suppose that the semigroup S satisfies the hypotheses of Corollary 4.5 and that $0\cdot x = 0$ for every $x \in S$. Then, by [4, Theorem 4.4] first entry matrices over S whose first entries are all 1, can be used to prove Corollary 4.5 as well as a sequence of successively stronger theorems. For example, the theorem stated in the remark following Theorem 4.2 is valid in S if C is any central subset of $(S \setminus \{0\}, \cdot)$, G is any given finite subset of S and $F = \{f\} \cup \{d+tf : t \in G\} \cup \{a+sd+tf : s,t \in G\}$ for some a, d, and f in $S \setminus \{0\}$.

The following corollary is also a consequence of [1, Theorem 3.15].

4.6 Corollary. *Let $k \in \mathbb{N}$, and let A be piecewise syndetic in $(\mathbb{N},\cdot)$. Then there exist $a, b, d \in \mathbb{N}$ such that $\big\{b(a+id)^j : i,j \in \{0,1,\ldots,k\}\big\} \subseteq A$.*

Proof. Pick $t \in \mathbb{N}$ such that $t^{-1}A$ is central and apply Corollary 4.4. □

Now, as we promised in the introduction, we turn our attention to extensions of the following result from [1].

4.7 Theorem. *Let $m, k \in \mathbb{N}$. For each $i \in \{0,1,\ldots,k\}$ let $\langle x_{i,t}\rangle_{t=1}^{\infty}$ and $\langle y_{i,t}\rangle_{t=1}^{\infty}$ be sequences in $\mathbb{N}$. Let $\mathbb{N} = \bigcup_{s=1}^{m} A_s$. Then there exist $s \in \{1,2,\ldots,m\}$, $F, G \in \mathcal{P}_f(\mathbb{N})$, and $a, b \in \mathbb{N}$ such that $\big\{b(a+\sum_{t\in F} x_{i,t}) \cdot (\prod_{t\in G} y_{j,t}) : i,j \in \{0,1,\ldots,k\}\big\} \subseteq A_s$.*

Proof. By [1, Theorem 3.13], every set A with $d^*_m(A) > 0$ contains such a configuration and for some s, $d^*_m(A_s) > 0$. $\square$

We shall show in Theorem 4.8 that one may take $F = G$ in Theorem 4.7 and in Corollary 4.12(a) that the multiplier b may be eliminated. We show in Corollary 4.16, however, that one cannot simultaneously take $F = G$ and eliminate b.

4.8 Theorem. *Let $m, k \in \mathbb{N}$. For each $i \in \{0, 1, \ldots, k\}$ let $\langle x_{i,t}\rangle_{t=1}^\infty$ and $\langle y_{i,t}\rangle_{t=1}^\infty$ be sequences in $\mathbb{N}$. Let $\mathbb{N} = \bigcup_{s=1}^m A_s$. Then there exist $s \in \{1, 2, \ldots, m\}$, $F \in \mathcal{P}_f(\mathbb{N})$, and $a, b \in \mathbb{N}$ such that*

$$\begin{aligned}&\{ba\} \cup \big\{b(a + \textstyle\sum_{t\in F} x_{i,t}) : i \in \{0, 1, \ldots, k\}\big\} \cup\\ &\big\{ba \cdot \textstyle\prod_{t\in F} y_{j,t} : j \in \{0, 1, \ldots, k\}\big\} \cup\\ &\big\{b(a + \textstyle\sum_{t\in F} x_{i,t}) \cdot (\prod_{t\in F} y_{j,t}) : i, j \in \{0, 1, \ldots, k\}\big\} \subseteq A_s\,.\end{aligned}$$

Proof. Let $x_{k+1,t} = 0$ and $y_{k+1,t} = 1$ for all t. Let $A_0 = \{0\}$. Let $L = \{0, 1, \ldots, k + 1\} \times \{0, 1, \ldots, k + 1\}$ and let S be the free semigroup on the alphabet L. Given a word $w = l_1 l_2 \cdots l_n$ of length n in S, define $f(w) = \sum_{t=1}^n x_{\pi_1(l_t),t} \cdot \prod_{t=1}^n y_{\pi_2(l_t),t}$. Then $S = \bigcup_{s=0}^m f^{-1}[A_s]$ so pick by Theorem 2.7, $s \in \{0, 1, \ldots, m\}$ and a variable word $w = l_1 l_2 \cdots l_n$ (with each $l_t \in L \cup \{v\}$) such that $\{w(c) : c \in L\} \subseteq A_s$. Notice that $s \neq 0$.

Let $F = \{t \in \{1, 2, \ldots, n\} : l_t = v\}$ and let $G = \{1, 2, \ldots, n\} \setminus F$. Let $a = \sum_{t\in G} x_{\pi_1(l_t),t}$ and let $b = \prod_{t\in G} y_{\pi_2(l_t),t}$. Then given $i, j \in \{0, 1, \ldots, k+1\}$, $f\big(w\big((i, j)\big)\big) = (a + \sum_{t\in F} x_{i,t}) \cdot b \cdot \prod_{t\in F} y_{j,t}$. $\square$

4.9 Corollary. *Let $k \in \mathbb{N}$. For each $i \in \{0, 1, \ldots, k\}$ let $\langle x_{i,t}\rangle_{t=1}^\infty$ and $\langle y_{i,t}\rangle_{t=1}^\infty$ be sequences in $\mathbb{N}$ and let A be piecewise syndetic in $(\mathbb{N}, \cdot)$. Then there exist $F \in \mathcal{P}_f(\mathbb{N})$ and $a, b \in \mathbb{N}$ such that*

$$\begin{aligned}&\{ba\} \cup \big\{b(a + \textstyle\sum_{t\in F} x_{i,t}) : i \in \{0, 1, \ldots, k\}\big\} \cup\\ &\big\{ba \textstyle\prod_{t\in F} y_{j,t} : j \in \{0, 1, \ldots, k\}\big\} \cup\\ &\big\{b(a + \textstyle\sum_{t\in F} x_{i,t}) \cdot (\prod_{t\in F} y_{j,t}) : i, j \in \{0, 1, \ldots, k\}\big\} \subseteq A\,.\end{aligned}$$

Proof. By Theorem 4.8 the collection of sets H of the form

$$\begin{aligned}H \;=\; &\{ba\} \cup \big\{b(a + \textstyle\sum_{t\in F} x_{i,t}) : i \in \{0, 1, \ldots, k\}\big\} \cup\\ &\big\{ba \textstyle\prod_{t\in F} y_{j,t} : j \in \{0, 1, \ldots, k\}\big\} \cup\\ &\big\{b(a + \textstyle\sum_{t\in F} x_{i,t}) \cdot (\prod_{t\in F} y_{j,t}) : i, j \in \{0, 1, \ldots, k\}\big\}\end{aligned}$$

is partition regular, so by Lemma 2.3 there is some $t \in \mathbb{N}$ and some such H with $tH \subseteq A$. Replacing b by tb yields the desired conclusion. $\square$

4.10 Lemma. *Let $(S, \cdot)$ be a commutative semigroup, let L be a minimal left ideal of $(\beta S, \cdot)$, and let $k \in \mathbb{N}$. Let $\mathcal{F}$ be a family of finite subsets of S such that the family*

$\{bF : F \in \mathcal{F}$ and $b \in S\}$ is partition regular. Let $A \subseteq S$ such that $\overline{A} \cap L \neq \emptyset$. Then there exists $F \in \mathcal{F}$ such that $L \cap \bigcap_{y\in F} \overline{y^{-1}A} \neq \emptyset$.

Proof. Pick $v \in \overline{A} \cap L$. Pick a minimal right ideal R of $(\beta S, \cdot)$ such that $v \in R$ and pick an idempotent $u \in R$. Then $v = uv$ so $B = \{x \in S : x^{-1}A \in v\} \in u$. In particular B is central so pick by Lemma 2.3, some $b \in S$ and $F \in \mathcal{F}$ such that $bF \subseteq B$. So for each $y \in F$, $(by)^{-1}A \in v$. Equivalently for each $y \in F$, $y^{-1}A \in bv$. Since $bv \in L$, we are done. □

We have by Lemma 3.4 that if $D = \{q \in \Delta : q$ is a minimal idempotent of $(\beta\mathbb{N}, +)\}$ then $c\ell D$ is a left ideal of $(\beta\mathbb{N}, \cdot)$ and consequently $c\ell D \cap K(\beta\mathbb{N}, \cdot) \neq \emptyset$. Given any $p \in \beta S$ and any finite partition $\{A_1, \ldots, A_m\}$ there is at least one cell A_i such that $A_i \in p$. Consequently, the partition versions of Theorem 4.11 and Corollary 4.12 are also valid.

4.11 Theorem. *Let $D = \{q \in \Delta : q$ is a minimal idempotent of $(\beta\mathbb{N}, +)\}$ and let A be a subset of $\mathbb{N}$ such that $\overline{A} \cap c\ell D \cap K(\beta\mathbb{N}, \cdot) \neq \emptyset$. Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$ such that the family $\{bF : F \in \mathcal{F}$ and $b \in \mathbb{N}\}$ is partition regular and let $\mathcal{G}$ be a family of subsets of $\mathbb{N}$ such that any set which is central in $(\mathbb{N}, +)$ contains a member of $\mathcal{G}$. Then there exist $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $\overline{d}(\bigcap_{y\in F} y^{-1}A) > 0$, $d^*_m(\bigcap_{y\in F} y^{-1}A) > 0$ and $FG \subseteq A$.*

Proof. Pick a minimal left ideal L of $(\beta\mathbb{N}, \cdot)$ such that $\overline{A} \cap c\ell D \cap L \neq \emptyset$. Since $c\ell D$ is a left ideal of $(\beta\mathbb{N}, \cdot)$, $L \subseteq c\ell D$. Pick, by Lemma 4.10, $F \in \mathcal{F}$ such that $L \cap \bigcap_{y\in F} \overline{y^{-1}A} \neq \emptyset$. Since $L \subseteq K(\beta\mathbb{N}, \cdot) \subseteq \Delta^*_m$ by [13, Theorems 20.5 and 20.6] $d^*_m(\bigcap_{y\in F} y^{-1}A) > 0$. Since $L \subseteq c\ell D$, pick $q \in \Delta$ such that q is a minimal idempotent of $(\beta\mathbb{N}, +)$ and $\bigcap_{y\in F} y^{-1}A \in q$. Then this set is central in $(\mathbb{N}, +)$ so pick $G \in \mathcal{G}$ such that $G \subseteq \bigcap_{y\in F} y^{-1}A$. Since $q \in \Delta$, $\overline{d}(\bigcap_{y\in F} y^{-1}A) > 0$. □

4.12 Corollary. *Let $D = \{q \in \Delta : q$ is a minimal idempotent of $(\beta\mathbb{N}, +)\}$, let A be a subset of $\mathbb{N}$ such that there is a multiplicative idempotent $p \in \overline{A} \cap c\ell D \cap K(\beta\mathbb{N}, \cdot)$, and let $k \in \mathbb{N}$.*

(a) For each $i \in \{1, 2, \ldots, k\}$ let $\langle x_{i,t}\rangle_{t=1}^\infty$ and $\langle y_{i,t}\rangle_{t=1}^\infty$ be sequences in $\mathbb{N}$. Then there exist $H, K \in \mathcal{P}_f(\mathbb{N})$ and $a \in A$ such that $\overline{d}(A \cap \bigcap_{j=1}^k (\prod_{t\in H} y_{j,t})^{-1}A) > 0$, $d^*_m(A \cap \bigcap_{j=1}^k (\prod_{t\in H} y_{j,t})^{-1}A) > 0$, and

$$\big\{a + \textstyle\sum_{t\in K} x_{i,t} : i \in \{1, 2, \ldots, k\}\big\} \cup \big\{a \cdot \prod_{t\in H} y_{j,t} : j \in \{1, 2, \ldots, k\}\big\} \cup \\ \big\{(a + \textstyle\sum_{t\in K} x_{i,t}) \cdot \prod_{t\in H} y_{j,t} : i, j \in \{1, 2, \ldots, k\}\big\} \subseteq A\,.$$

(b) *There exist* $a, r, d \in A$ *such that* $r > 1$, $\overline{d}(\bigcap_{j=0}^{k}(r^j)^{-1}A) > 0$, $d_m^*(\bigcap_{j=0}^{k}(r^j)^{-1}A) > 0$, *and* $\{(a+id)r^j : i, j \in \{0,1,\ldots,k\}\} \cup \{dr^j : j \in \{0,1,\ldots,k\}\} \subseteq A$.

(c) *There exist* $a, r, d \in A$ *such that* $r > 1$, $\overline{d}(\bigcap_{j=1}^{k}(j^r)^{-1}A) > 0$, $d_m^*(\bigcap_{j=0}^{k}(j^r)^{-1}A) > 0$, *and* $\{dj^r : j \in \{1,2,\ldots,k\}\} \cup \{(a+id)j^r : i \in \{0,1,\ldots,k\} \text{ and } j \in \{1,2,\ldots,k\}\} \cup \{a+id : i \in \{0,1,\ldots,k\}\} \subseteq A$.

Proof. Since 1 is not an element of any minimal left ideal of $(\beta\mathbb{N}, \cdot)$, by considering $A \setminus \{1\}$ instead of A we may assume that $1 \notin A$. Let

$$\mathcal{F}_1 = \{\{1\} \cup \{\textstyle\prod_{t\in H} y_{i,t} : i \in \{1,2,\ldots,k\}\} : H \in \mathcal{P}_f(\mathbb{N})\},$$
$$\mathcal{G}_1 = \{\{a\} \cup \{a + \textstyle\sum_{t\in K} x_{i,t} : i \in \{1,2,\ldots,k\} : K \in \mathcal{P}_f(\mathbb{N}) \text{ and } a \in \mathbb{N}\},$$
$$\mathcal{F}_2 = \Big\{\{r^i : i \in \{0,1,\ldots,k\}\} : r \in A\Big\},$$
$$\mathcal{G}_2 = \{\{d\} \cup \{a+id : i \in \{0,1,\ldots,k\}\} : a, d \in \mathbb{N}\},$$

and put $\mathcal{F}_i' = \{bF : b \in \mathbb{N} \text{ and } F \in \mathcal{F}_i\}$ for $i \in \{1,2\}$. By applying Theorem 2.9 and Corollary 2.10 to the semigroup $(\mathbb{N}, \cdot)$ we see that the families $\mathcal{F}_1'$ and $\mathcal{F}_2'$ are partition regular. Similarly by Theorem 2.9 and Corollary 2.10 applied to the semigroup $(\mathbb{N}, +)$, every subset of $\mathbb{N}$ that is central in $(\mathbb{N}, +)$ contains a member of $\mathcal{G}_1$ and a member of $\mathcal{G}_2$. Thus we get (a) by applying Theorem 4.11 to $\mathcal{F}_1$ and $\mathcal{G}_1$ and (b) by applying Theorem 4.11 to $\mathcal{F}_2$ and $\mathcal{G}_2$.

We will prove (c) by using Theorem 4.11 with $\mathcal{F}_1$ and $\mathcal{G}_2$, where we define the sequences $\langle y_{i,n}\rangle_{n=1}^{\infty}$, $i \in \{1,2,\ldots,k\}$ appropriately. Since A is central in $(\mathbb{N}, +)$, choose a sequence $\langle r_n\rangle_{n=1}^{\infty}$ such that $FS(\langle r_n\rangle_{n=1}^{\infty}) \subseteq A$. Using this put $y_{i,n} = i^{r_n}$ for $i \in \{1,2,\ldots,k\}$ and $n \in \mathbb{N}$. By Theorem 4.11 we find $a, d \in A$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that $G = \{d\} \cup \{a+id : i \in \{0,1,\ldots,k\}\}$ and $F = \{1\} \cup \{\prod_{t\in H} y_{j,t} : j \in \{1,2,\ldots,k\}\}$ satisfy the conclusion of Theorem 4.11. Let $r = \sum_{t\in H} r_t \in A$. Then for $j \in \{1,2,\ldots,k\}$, $\prod_{t\in H} y_{j,t} = \prod_{t\in H} j^{r_t} = j^r$. Thus we see that (c) is valid. □

We now turn our attention to showing that one cannot simultaneously let $F = G$ and eliminate the multiplier b in Theorem 4.7.

The following theorem is of interest in its own right. Recall from Corollary 2.9 that when $\mathbb{N}$ is finitely colored, one can find arbitrarily long monochrome arithmetic progressions with increments chosen from any IP-set. This theorem tells us that at least relatively thin sequences cannot replace IP-sets.

4.13 Theorem. *Let* $\langle d_n\rangle_{n=1}^{\infty}$ *be a sequence in* $\mathbb{N}$ *such that for all* $n \in \mathbb{N}$, $3d_n \leq d_{n+1}$. *There exists a partition* $\{A_0, A_1, A_2, A_3\}$ *of* $\mathbb{N}$ *such that there do not exist* $s \in \{0,1,2,3\}$ *and* $a, k \in \mathbb{N}$ *with* $\{a, a+d_k\} \subseteq A_s$.

Proof. For $\alpha \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ we denote by $\|\alpha\|$ the distance to the nearest integer. We will not distinguish strictly between equivalance classes and their representatives in $[0, 1)$.

4.14 Lemma. *There exists* $\alpha \in \mathbb{T}$ *such that* $\|\alpha d_n\| \geq 1/4$ *for each* $n \in \mathbb{N}$.

Proof. For each $n \in \mathbb{N}$ put $R_n = \{\alpha \in \mathbb{T} : \|\alpha d_n\| \geq 1/4\}$. Each R_n consists of intervals of length $\frac{1}{2d_n}$ which are separated by gaps of the same length. Since $d_{n+1} \geq 3d_n$ every interval of R_n is 3 times longer than an interval or a gap of R_{n+1}. Thus any interval of R_n contains an interval of R_{n+1}. This shows that for each $N \in \mathbb{N}$, $\bigcap_{n=1}^{N} R_n \neq \emptyset$. By compactness of $\mathbb{T}$ there exists $\alpha \in \bigcap_{n=1}^{\infty} R_n$. □

Let $\alpha \in \mathbb{T}$ such that $d_n\alpha \in [1/4, 3/4]$ for each $n \in \mathbb{N}$. For $i \in \{0, 1, 2, 3\}$ put $A_i = \{m \in \mathbb{N} : m\alpha \in [i/4, (i + 1)/4)\}$. Then for any $a, k \in \mathbb{N}$ $\alpha(a + d_k) = \alpha a + \beta$ for some $\beta \in [1/4, 3/4]$ and thus αa and $\alpha(a+d_k)$ must not lie in the same quarter of $[0, 1)$. Equivalently there exists no $s \in \{0, 1, 2, 3\}$ such that $\{a, a + d_k\} \subseteq A_s$. □

We remark that Lemma 4.14 is well known. Under the much weaker assumption, that the growth rate of the sequence $\langle d_n \rangle_{n=1}^{\infty}$ is bounded from below by some $q > 1$ B. de Mathan [14] and A. Pollington [15] independently proved that there exists some $\alpha \in \mathbb{T}$ such that $\{\alpha n : n \in \mathbb{N}\}$ is not dense in $\mathbb{T}$. In order to give a self contained proof we have chosen to go with the weaker statement. The loss is that we have to make an additional step to show that any growth rate $q > 1$ is sufficient to avoid monochrome arithmetic progressions with some d_k as increment.

4.15 Corollary. *Let* $q \in \mathbb{R}$ *with* $q > 1$ *and assume that* $\langle d_n \rangle_{n=1}^{\infty}$ *is a sequence in* $\mathbb{N}$ *such that for all* $n \in \mathbb{N}$, $qd_n \leq d_{n+1}$. *There exists a finite partition* $\{A_1, A_2, \ldots, A_r\}$ *of* $\mathbb{N}$ *such that there do not exist* $s \in \{1, 2, \ldots, r\}$ *and* $a, k \in \mathbb{N}$ *with* $\{a, a + d_k\} \subseteq A_s$.

Proof. Pick $m \in \mathbb{N}$ such that $q^m \geq 3$. For $t \in \{0, 1, \ldots, m - 1\}$ and $n \in \mathbb{N}$, let $c_{t,n} = d_{nm-t}$. Given $t \in \{0, 1, \ldots, m\}$ one has that $3c_{t,n} \leq c_{t,n+1}$ for each n so pick by Theorem 4.13 some $\{B_{t,0}, B_{t,1}, B_{t,2}, B_{t,3}\}$ of $\mathbb{N}$ such that there do not exist $s \in \{0, 1, 2, 3\}$ and $a, k \in \mathbb{N}$ with $\{a, a+c_{t,k}\} \subseteq B_{t,i}$. Let $r = 4^m$ and define a partition $\{A_1, A_2, \ldots, A_r\}$ of $\mathbb{N}$ with the property that x and y lie in the same cell of the partition if and only if $x \in B_{t,i} \Leftrightarrow y \in B_{t,i}$ for each $t \in \{0, 1, \ldots, m - 1\}$ and each $i \in \{0, 1, 2, 3\}$. □

4.16 Corollary. *There exist sequences* $\langle x_{0,n} \rangle_{n=1}^{\infty}$, $\langle x_{1,n} \rangle_{n=1}^{\infty}$, *and* $\langle y_n \rangle_{n=1}^{\infty}$ *in* $\mathbb{N}$ *and a partition* $\{A_0, A_1, A_2, A_3\}$ *of* $\mathbb{N}$ *such that there do not exist* $s \in \{0, 1, 2, 3\}$, $F \in \mathcal{P}_f(\mathbb{N})$, *and* $a \in \mathbb{N}$ *with* $\big\{(a + \sum_{n\in F} x_{i,n}) \cdot \prod_{n\in F} y_n : i \in \{0, 1\}\big\} \subseteq A_s$.

Proof. For each $t \in \mathbb{N}$, let $x_{0,t} = 1$, $x_{1,t} = 2$ and $y_{i,t} = 3$. For each $n \in \mathbb{N}$, let $d_n = n3^n$. Pick A_0, A_1, A_2, A_3 as guaranteed by Theorem 4.13. Suppose one has $F \in \mathcal{P}_f(\mathbb{N})$ and $a \in \mathbb{N}$ with $\big\{(a + \sum_{t\in F} x_{i,t}) \cdot \prod_{t\in F} y_t : i \in \{0,1\}\big\} \subseteq A_s$. Let $n = |F|$. Then $(a + \sum_{t\in F} x_{1,t}) \cdot \prod_{t\in F} y_t = d_n + (a + \sum_{t\in F} x_{0,t}) \cdot \prod_{t\in F} y_t$, a contradiction. $\square$

We have just shown that one cannot eliminate the multiplier b from Theorem 4.8. We show now that this multiplier cannot be eliminated from Corollary 4.9. Recall that thick sets in any semigroup are also piecewise syndetic, in fact central.

4.17 Theorem. *There exists a set A which is thick in $(\mathbb{N},\cdot)$ and a sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ with the property that there do not exist $a \in \mathbb{N}$ and $d \in FS(\langle x_n\rangle_{n=1}^\infty)$ with $\{a, a+d\} \subseteq A$.*

Proof. Let $A = \bigcup_{n=1}^\infty \{(3n)!, 2(3n)!, \ldots, n(3n)!\}$ and for each n, let $x_n = (3n+1)!$. Observe that A is thick in $(\mathbb{N},\cdot)$. Let $a \in A$ and let $d \in FS(\langle x_n\rangle_{n=1}^\infty)$. We shall show that $a + d \notin A$. Pick $n \in \mathbb{N}$ and $k \in \{1, 2, \ldots, n\}$ such that $a = k(3n)!$. Pick $F \in \mathcal{P}_f(\mathbb{N})$ such that $d = \sum_{t\in F} x_t$ and let $m = \max F$. Then $(3m+1)! \leq d < (3m+2)!$.

If $m < n$ we have $k(3n)! < a + d < (k+1)(3n)!$ so $a + d \notin A$. If $m \geq n$, then $a < (3m+1)!$ so $(3m+1)! < a + d < (3m+3)!$ and thus $a + d \notin A$. $\square$

It was shown in [1, Theorem 1.3] that the fact that a subset A of $\mathbb{N}$ satisfies $d_m^*(A) > 0$ is enough to guarantee that A contains arbitrarily large geoarithmetic progressions. However, by considering the set $A = \{x \in \mathbb{N} :$ the number of terms in the prime factorization of x is odd$\}$, one sees that the fact that $\overline{d}_m(A) > 0$ is not enough to guarantee geoarithmetic progressions together with the common ratio r, nor together with both b and a.

As is well known among afficianados, geoarithmetic progressions are *strongly partition regular*. That is, for each $m, k \in \mathbb{N}$ there exists $K \in \mathbb{N}$ such that whenever $A, B, D \in \mathbb{N}$, $R \in \mathbb{N}\setminus\{1\}$, and $\big\{BR^s(A+tD) : s, t \in \{0, 1, \ldots, K\}\big\} = \bigcup_{i=1}^m C_i$, there exist $i \in \{1, 2, \ldots, m\}$, $a, b, d \in \mathbb{N}$, and $r \in \mathbb{N}\setminus\{1\}$ such that $\big\{br^s(a+td) : s, t \in \{0, 1, \ldots, k\}\big\} \subseteq A_i$. (The easiest way to see this is to use the Grünwald/Gallai Theorem[1] [8, Theorem 2.8]. Color the pair $(s,t) \in \{0, 1, \ldots, K\} \times \{0, 1, \ldots, K\}$ according to the color of $BR^s(A+tD)$.)

We show now that configurations of the sort produced by Corollary 3.7 are not strongly partition regular.

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[1] This theorem was never published by its author. Its first publication was in [16] where it was referred to as Grünwald's Theorem, Grünwald being the original name of the author. During the period surrounding World War II Grünwald changed his name to Gallai.

4.18 Theorem. *There is a set* $C \subseteq \mathbb{N}$ *such that for each* $k \in \mathbb{N}$ *there exist* $b, a, d \in \mathbb{N}$ *and* $r \in \mathbb{N} \setminus \{1\}$ *such that* $\big\{br^n(a+td) : n, t \in \{0,1,\ldots,k\}\big\} \cup \big\{br^n : n \in \{0,1,\ldots,k\}\big\} \cup \big\{a+td : t \in \{0,1,\ldots,k\}\big\} \subseteq C$ *and there exist sets* A_1 *and* A_2 *such that* $C = A_1 \cup A_2$ *and there do not exist* $i \in \{1,2\}$, $c, a, d \in \mathbb{N}$, *and* $s \in \mathbb{N} \setminus \{1\}$ *such that* $\{cs, cs^2, cs(a+d), cs^2(a+d), cs(a+2d)\} \subseteq A_i$.

Proof. Let $r_1 = 5$. Inductively choose a prime $r_{k+1} > \big(r_k(2k+1)\big)^2$. For each $k \in \mathbb{N}$, let $B_k = \big\{{r_k}^n x : n \in \{1,2,\ldots,k+1\}$ and $x \in \{k+1, k+2, \ldots, 2k+1\}\big\}$ and let $B = \bigcup_{k=1}^{\infty} B_k$.

4.19 Lemma. *If* $a, d \in \mathbb{N}$ *and* $\{a+d, a+2d\} \subseteq B$, *then there exist* $k \in \mathbb{N}$ *and* $n \in \{1,2,\ldots,k+1\}$ *such that* $\{a+d, a+2d\} \subseteq \big\{{r_k}^n x : x \in \{k+1, k+2, \ldots, 2k+1\}\big\}$.

Proof. Pick $k \in \mathbb{N}$, $n \in \{1,2,\ldots,k+1\}$, and $x \in \{k+1, k+2, \ldots, 2k+1\}$ such that $a+d = {r_k}^n x$. Then $a+2d < 2(a+d) = 2{r_k}^n x$. Also $2{r_k}^n x < {r_k}^{n+1}(k+1)$ and $2{r_k}^n x < r_{k+1}(k+2)$. The first member of B larger than ${r_k}^n(2k+1)$ is ${r_k}^{n+1}(k+1)$ (if $n \leq k$) or $r_{k+1}(k+2)$ (if $n = k+1$). Thus there is some $y \in \{x+1, x+2, \ldots, k+1\}$ such that $a+2d = {r_k}^n y$. $\square$

4.20 Lemma. *If* $c \in \mathbb{N}$, $s \in \mathbb{N} \setminus \{1\}$, *and* $\{cs, cs^2\} \subseteq B$, *then there exist* $k \in \mathbb{N}$, $n \in \{0,1,\ldots,k\}$, $t \in \{1,2,\ldots,k+1-n\}$, *and* $y \in \{k+1, k+2, \ldots, 2k+1\}$ *such that* $c = {r_k}^n y$ *and* $s = {r_k}^t$.

Proof. Pick $k \leq m$, $\delta \in \{1,2,\ldots,k+1\}$, $\nu \in \{1,2,\ldots,m+1\}$, $y \in \{k+1, k+2, \ldots, 2k+1\}$, and $z \in \{m+1, m+2, \ldots, 2m+1\}$ such that $cs = {r_k}^\delta y$, and $cs^2 = {r_m}^\nu z$.

Now $s \leq {r_k}^\delta y \leq {r_k}^{k+1}(2k+1)$ and $s = \dfrac{{r_m}^\nu z}{{r_k}^\delta y} > \dfrac{r_m}{{r_k}^{k+1}(2k+1)}$ so

$$r_m < \big({r_k}^{k+1}(2k+1)\big)^2 < r_{k+1}$$

and so $m \leq k$ and thus $m = k$. Therefore $s = {r_k}^{\nu-\delta}\dfrac{z}{y}$. Since r_k is a prime which does not divide y, we must have that y divides z and therefore that $y = z$. Let $t = \nu - \delta$. Since $c{r_k}^{\nu-\delta} = cs = {r_k}^\delta y$ we have $c = {r_k}^{2\delta-\nu} y$. Let $n = 2\delta - \nu$. Since $c = {r_k}^n y$ and $s = {r_k}^t$ we have that $n \geq 0$ and $t \geq 1$. Since $n + t = \delta$ we have that $n + t \leq k+1$. $\square$

To complete the proof of the theorem, let $A_1 = B$, let $A_2 = \big\{{r_k}^n : k \in \mathbb{N}$ and $n \in \{1,2,\ldots,k+1\}\big\}$, and let $C = A_1 \cup A_2$. Given $k \in \mathbb{N}$, let $a = r_k(k+1)$ and let $d = b = r = r_k$. Then for $t, n \in \{0,1,\ldots,k-1\}$ one has $br^n = {r_k}^{n+1} \in A_2$, $a + td = r_k(k+t+1) \in A_1$, and $br^n(a+td) = {r_k}^{n+2}(k+t+1) \in A_1$.

It is trivial that A_2 does not contain $\{cs(a+d), cs(a+2d)\}$ as the latter element is less than twice the former. Suppose we have some $c, a, d \in \mathbb{N}$ and some $s \in \mathbb{N} \setminus \{1\}$ such that

$$\{cs, cs^2, cs(a+d), cs^2(a+d), cs(a+2d)\} \subseteq A_1\,.$$

Pick by Lemma 4.20 some $k \in \mathbb{N}$, $n \in \{0, 1, \ldots, k\}$, $t \in \{1, 2, \ldots, k+1-n\}$, and $y \in \{k+1, k+2, \ldots, 2k+1\}$ such that $c = {r_k}^n y$ and $s = {r_k}^t$. Again invoking Lemma 4.20, pick some $k' \in \mathbb{N}$, $m \in \{0, 1, \ldots, k'\}$, $t' \in \{1, 2, \ldots, k'+1-m\}$, and $z \in \{k'+1, k'+2, \ldots, 2k'+1\}$ such that $c(a+d) = {r_{k'}}^m z$ and $s = {r_{k'}}^{t'}$.

Since ${r_{k'}}^{t'} = s = {r_k}^t$ we have $k = k'$ and $t = t'$. Pick by Lemma 4.19 $k'' \in \mathbb{N}$ and $\nu \in \{1, 2, \ldots, k''+1\}$ such that

$$\{cs(a+d), cs(a+2d)\} \subseteq \big\{{r_{k''}}^\nu w : w \in \{k''+1, k''+2, \ldots, 2k''+1\}\big\}\,.$$

Since $cs(a+d) = {r_k}^{t+m} z$ we have $k'' = k$ and $\nu = t+m$. Since $cs = {r_k}^{t+n} y$ we have $a+d = {r_k}^{m-n}\dfrac{z}{y}$. Since r_k is a prime which does not divide y we have that y divides z so $y = z$ and thus $a+d = {r_k}^{m-n}$.

Pick $w \in \{k+1, k+2, \ldots, 2k+1\}$ such that $cs(a+2d) = {r_k}^{t+m} w$. Then $a+2d = {r_k}^{m-n}\dfrac{w}{y}$ so w divides y and thus $a+2d = {r_k}^{m-n}$. Therefore $d = 0$, a contradiction. $\square$

We now present a general result which is strong enough to establish an extension of Theorem 1.4.

4.21 Theorem. *Let $(S, \cdot)$ be a semigroup, let $\mathcal{F}$ be a set of subsets of S with the property that each central subset of S contains a member of $\mathcal{F}$, let $\mathcal{G}$ be a partition regular family of finite subsets of S, and let A be a central subset of S. Then there exist $F \in \mathcal{F}$, $G \in \mathcal{G}$, and $t \in S$ such that $F \cup tGF \subseteq A$.*

Proof. Pick a minimal idempotent p of βS with $A \in p$. Then by [13, Theorem 4.39] $\{s \in S : s^{-1}A \in p\}$ is syndetic so pick $H \in \mathcal{P}_f(S)$ such that

$$S = \textstyle\bigcup_{t \in H} t^{-1}\{s \in S : s^{-1}A \in p\}\,.$$

Pick $G \in \mathcal{G}$ and $t \in H$ such that $G \subseteq t^{-1}\{s \in S : s^{-1}A \in p\}$. Then for each $s \in G$, $(ts)^{-1}A \in p$ so $A \cap \bigcap_{s \in G}(ts)^{-1}A \in p$. Pick $F \in \mathcal{F}$ such that $F \subseteq A \cap \bigcap_{s \in G}(ts)^{-1}A$. $\square$

4.22 Corollary. *Let $(S, \cdot)$ be a semigroup, let $\mathcal{F}$ and $\mathcal{G}$ be partition regular families of finite subsets of S. Assume that for all $F \in \mathcal{F}$ and all $x, t \in S$, $tFx \in \mathcal{F}$ and let A be a piecewise syndetic subset of S. Then there exist $F \in \mathcal{F}$, $G \in \mathcal{G}$, and $t \in S$ such that $tGF \subseteq A$. If S is commutative, then there exist $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $GF \subseteq A$.*

Proof. Note that by Lemma 2.3 $\mathcal{F}$ has the property that every piecewise syndetic subset of S contains a member of $\mathcal{F}$. In particular every central subset of S contains a member of $\mathcal{F}$. Pick by [13, Theorem 4.43] some $x \in S$ such that $x^{-1}A$ is central. Pick by Theorem 4.21 some $F \in \mathcal{F}$, $G \in \mathcal{G}$, and $t \in S$ such that $F \cup tGF \subseteq x^{-1}A$. Then $(xt)GF \subseteq A$. $\square$

The following corollary extends Theorem 1.4. Recall that any central set is a piecewise syndetic IP set.

4.23 Corollary. *Let A be a piecewise syndetic IP-set in $(\mathbb{N}, \cdot)$ with $1 \notin A$ and let $k \in \mathbb{N}$. Then there exist $a, r, d \in A$ such that*

$$\big\{r^j(a+id) : i, j \in \{0,1,\ldots,k\}\big\} \cup \big\{dr^j : j \in \{0,1,\ldots,k\}\big\} \subseteq A\,.$$

Proof. Let $\mathcal{F} = \Big\{\big\{br^j : j \in \{0,1,\ldots,k\}\big\} : b \in \mathbb{N} \text{ and } r \in A\Big\}$ and let

$$\mathcal{G} = \big\{\{d\} \cup \big\{a+id : i \in \{0,1,\ldots,k\}\big\} : a, d \in \mathbb{N}\big\}\,.$$

By Corollary 2.9, $\mathcal{F}$ and $\mathcal{G}$ are partition regular. And trivially if $F \in \mathcal{F}$ and $t \in \mathbb{N}$, then $tF \in \mathcal{F}$. Pick by Corollary 4.22 some $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $GF \subseteq A$. Pick $b \in \mathbb{N}$ and $r \in A$ such that $F = \big\{br^j : j \in \{0,1,\ldots,k\}\big\}$ and pick $a_1, d_1 \in \mathbb{N}$ such that $G = \{d_1\} \cup \big\{a_1 + id_1 : i \in \{0,1,\ldots,k\}\big\}$. Let $a = a_1 b$ and $d = d_1 b$. $\square$

We see that we can turn the tables somewhat, translating geometric progressions by arithmetic progressions. (Since addition does not distribute over multiplication, we end up with the four variables a, d, b, and r, rather than just the three of Corollary 4.23.)

4.24 Corollary. *Let A be a piecewise syndetic IP-set in $(\mathbb{N}, +)$ and let $k \in \mathbb{N}$. Then there exist $d \in A$, $a, b \in \mathbb{N}$, and $r \in \mathbb{N} \setminus \{1\}$ such that*

$$\big\{a + id + br^j : i, j \in \{0,1,\ldots,k\}\big\} \cup \big\{a + id + r : i \in \{0,1,\ldots,k\}\big\} \subseteq A\,.$$

Proof. Let $\mathcal{F} = \Big\{\big\{a+id : i \in \{0,1,\ldots,k\}\big\} : a \in \mathbb{N} \text{ and } d \in A\Big\}$ and let

$$\mathcal{G} = \big\{\{r\} \cup \big\{br^j : j \in \{0,1,\ldots,k\}\big\} : b \in \mathbb{N} \text{ and } r \in \mathbb{N} \setminus \{1\}\big\}\,.$$

Exactly as in the proof of Corollary 4.23, $\mathcal{F}$ and $\mathcal{G}$ are partition regular and if $F \in \mathcal{F}$ and $t \in \mathbb{N}$, then $t + F \in \mathcal{F}$. Pick by Corollary 4.22 $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $G + F \subseteq A$. Pick $b \in \mathbb{N}$ and $r \in \mathbb{N} \setminus \{1\}$ such that $G = \{r\} \cup \big\{br^j : j \in \{0,1,\ldots,k\}\big\}$. Pick $a \in \mathbb{N}$ and $d \in A$ such that $F = \big\{a + id : i \in \{0,1,\ldots,k\}\big\}$. $\square$

We do not know whether we can require that any of a, b, or r be in A in Corollary 4.24.

5. Algebra in ($\beta\mathbb{N}$, +) and ($\beta\mathbb{N}$,·) – extending the Central Sets Theorem

In attempting to derive results about geoarithmetic progressions, the approach that one might try first after a little experience in deriving Ramsey Theoretic consequences of the algebra of $\beta\mathbb{N}$ would be to choose an appropriate idempotent q in $(\beta\mathbb{N},\cdot)$ and show that if $A\in q$, then there is some r, preferably in A, such that $\bigcap_{s=0}^{k}(r^s)^{-1}A\in q$. We show first that such an approach is doomed to failure.

5.1 Theorem.

(a) For all $q\in\beta\mathbb{N}$, there exists a partition $\{A_0,A_1\}$ of $\mathbb{N}$ such that for all $i\in\{0,1\}$ and all $x\in\mathbb{N}$, $(-x+A_i)\cap(-2x+A_i)\notin q$. In particular there exists $A\in q$ such that for all $x\in\mathbb{N}$, either $-x+A\notin q$ or $-2x+A\notin q$.

(b) There does not exist $q\in\beta\mathbb{N}$ such that for each $A\in q$ there is some $r\in\mathbb{N}\setminus\{1\}$ with $r^{-1}A\in q$ and $(r^2)^{-1}A\in q$.

Proof. (a) Let $q\in\beta\mathbb{N}$. Then $q+\beta\mathbb{N}$ is a right ideal of $(\beta\mathbb{N},+)$ so there is an additive idempotent in $q+\beta\mathbb{N}$. Pick $r\in\beta\mathbb{N}$ such that $q+r$ is an idempotent in $(\beta\mathbb{N},+)$. Then $q+r\in\bigcap_{n=1}^{\infty}c\ell(\mathbb{N}2^n)$ by [13, Lemma 6.6].

Define $f:\mathbb{N}\to\omega$ by $f(n)=\min F$ where $F\in\mathcal{P}_f(\omega)$ and $n=\sum_{t\in F}2^t$. Then f has a continuous extension $\widetilde{f}:\beta\mathbb{N}\to\beta\omega$. For $i\in\{0,1\}$ let $A_i=\{x\in\mathbb{N}:(2\mathbb{N}-i)\in\widetilde{f}(x+r)\}$.

Let $i\in\{0,1\}$ and let $x\in\mathbb{N}$ and suppose that $(-x+A_i)\cap(-2x+A_i)\in q$. Pick $j,k\in\omega$ such that $x=2^j(2k+1)$. Denote addition of z on the left in $\beta\mathbb{N}$ by λ_z and addition of z on the right by ρ_z. Then $\widetilde{f}\circ\lambda_x$ is constantly equal to $f(x)$ and $\widetilde{f}\circ\lambda_{2x}$ is constantly equal to $f(x)+1$ on $\mathbb{N}2^{j+2}$, which is a member of $q+r$. So $\widetilde{f}(x+q+r)=f(x)$ and $\widetilde{f}(2x+q+r)=f(x)+1$. Therefore $\widetilde{f}\circ\lambda_x\circ\rho_r(q)=f(x)$ and $\widetilde{f}\circ\lambda_{2x}\circ\rho_r(q)=f(x)+1$ so

$$\{y\in\mathbb{N}:\widetilde{f}(x+y+r)=f(x)\text{ and }\widetilde{f}(2x+y+r)=f(x)+1\}\in q$$

so pick $y\in(-x+A_i)\cap(-2x+A_i)$ such that $\widetilde{f}(x+y+r)=f(x)$ and $\widetilde{f}(2x+y+r)=f(x)+1$.

Since $x+y\in A_i$, we have that $2\mathbb{N}-i\in\widetilde{f}(x+y+r)=f(x)$ so $f(x)+i\in2\mathbb{N}$. (Recall that we are identifying points of $\mathbb{N}$ with the principle ultrfilters they generate.) Since $2x+y\in A_i$, we have that $2\mathbb{N}-i\in\widetilde{f}(2x+y+r)=f(x)+1$ so $f(x)+i+1\in2\mathbb{N}$, a contradiction.

(b) For $x \in \mathbb{N} \setminus \{1\}$, let $\ell(x)$ be the number of terms in the prime factorization of x. Then ℓ is a homomorphism from $(\mathbb{N} \setminus \{1\}, \cdot)$ onto $(\mathbb{N}, +)$ and so its continuous extension $\widetilde{\ell} : (\beta\mathbb{N} \setminus \{1\}, \cdot) \to (\beta\mathbb{N}, +)$ is also a homomorphism by [13, Corollary 4.22]. $\square$

We know that there exist multiplicative idempotents in the closure of the set of additive idempotents in $\beta\mathbb{N}$. In fact, there exist minimal multiplicative idempotents in the closure of the set of minimal additive idempotents in $\beta\mathbb{N}$, and we used one such in the proof of Theorem 3.6. In particular we know that $c\ell K(\beta\mathbb{N}, +) \cap K(\beta\mathbb{N}, \cdot) \neq \emptyset$. In the following we shall assume that geometric progressions have integer common ratios, though the lemma would remain valid with the more liberal definition.

5.2 Lemma. *Let* $D = \{q \in \beta\mathbb{N} :$ for all $A \in q$, A contains arbitrarily long geometric progressions$\}$. *Then D is a closed two sided ideal of* $(\beta\mathbb{N}, \cdot)$. *In particular* $c\ell K(\beta\mathbb{N}, \cdot) \subseteq D$.

Proof. Trivially D is closed. Let $q \in D$, let $s \in \beta\mathbb{N}$, let $A \in qs$, and let $B \in sq$. Let $n \in \mathbb{N}$. We need to show that A and B contain length n geometric progressions. Now $\{x \in \mathbb{N} : x^{-1}A \in s\} \in q$ so pick $a \in \mathbb{N}$ and $r \in \mathbb{N}\setminus\{1\}$ such that $\{a, ar, ar^2, \ldots, ar^{n-1}\} \subseteq \{x \in \mathbb{N} : x^{-1}A \in s\}$. Then $\bigcap_{t=0}^{n-1}(ar^t)^{-1}A \in s$ so pick $b \in \bigcap_{t=0}^{n-1}(ar^t)^{-1}A$. Then $\{ba, bar, bar^2, \ldots, bar^{n-1}\} \subseteq A$. Also $\{x \in \mathbb{N} : x^{-1}B \in q\} \in s$ so pick $x \in \mathbb{N}$ such that $x^{-1}B \in q$. Pick $c \in \mathbb{N}$ and $d \in \mathbb{N} \setminus \{1\}$ such that $\{c, cd, cd^2, \ldots, cd^{n-1}\} \subseteq x^{-1}B$. Then $\{xc, xcd, xcd^2, \ldots, xcd^{n-1}\} \subseteq B$. $\square$

We see now that there would be interesting Ramsey theoretic consequences of the existence of an additive idempotent in the set D defined above. (Compare the conclusion with those of Theorem 3.6.)

5.3 Theorem. *Let* $D = \{q \in \beta\mathbb{N} :$ for all $A \in q$, A contains arbitrarily long geometric progressions$\}$ *and assume that there exists $q \in D$ such that $q + q = q$. Then whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} A_i$, there exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle H_n \rangle_{n=1}^{\infty}$ such that for each $n \in \mathbb{N}$, H_n is a length n geometric progression and for every $F \in \mathcal{P}_f(\mathbb{N})$, one has $\sum_{n \in F} H_n \subseteq A_i$.*

Proof. Pick $q \in D$ such that $q + q = q$. Given $B \in q$, let $B^\star = \{x \in B : -x + B \in q\}$. Then by [13, Lemma 4.14], whenever $x \in B^\star$ one has $-x + B^\star \in q$.

Pick $i \in \{1, 2, \ldots, r\}$ such that $A_i \in q$. Pick $x \in {A_i}^\star$ and let $H_1 = \{x\}$. Let $n \in \mathbb{N}$ and assume that $\langle H_t \rangle_{t=1}^{n}$ have been chosen so that for any F with $\emptyset \neq F \subseteq \{1, 2, \ldots, n\}$ and any $f \in \times_{t \in F} H_t$, $\sum_{t \in F} f(t) \in {A_i}^\star$. Let

$$B = {A_i}^\star \cap \bigcap\{-\textstyle\sum_{t \in F} f(t) + {A_i}^\star : F \in \mathcal{P}_f(\{1, 2, \ldots, n\}) \text{ and } f \in \times_{t \in F} H_t\}\,.$$

Then $B \in q$ so pick a length $n+1$ geometric progression $H_{n+1} \subseteq B$. $\square$

We now turn our attention to deriving an extension, Theorem 5.8, of the Central Sets Theorem for countable commutative semigroups [13, Theorem 14.11]. The Central Sets Theorem for $(\mathbb{N},+)$ is due to Furstenberg [6, Proposition 8.21]. See [13, Part III] for numerous combinatorial applications of the Central Sets Theorem. Theorem 5.8 has several earlier theorems as immediate corollaries. In particular, it implies a stronger version of Theorem 4.8. To establish this theorem we shall use the notion of *partial semigroup* introduced in [2].

5.4 Definition.

(a) A *partial semigroup* is a set S together with an operation $\cdot$ that maps a subset of $S \times S$ into S and satisfies the associative law $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ in the sense that if either side is defined, then so is the other and they are equal.

(b) Given a partial semigroup $(S, \cdot)$ and $x \in S$, $\varphi(x) = \{y \in S : x \cdot y \text{ is defined}\}$.

(c) Given a partial semigroup $(S, \cdot)$, $x \in S$, and $A \subseteq S$, $x^{-1}A = \{y \in \varphi(x) : x \cdot y \in A\}$.

(d) A partial semigroup $(S, \cdot)$ is *adequate* if and only if for each $F \in \mathcal{P}_f(S)$, $\bigcap_{x \in F} \varphi(x) \neq \emptyset$.

(e) Given an adequate partial semigroup $(S, \cdot)$, $\delta S = \bigcap_{x \in S} c\ell_{\beta S} \varphi(x)$.

5.5 Lemma. *Let* $(S, \cdot)$ *be an adequate partial semigroup and for* $p, q \in \delta S$ *define* $p \cdot q = \big\{A \subseteq S : \{x \in S : x^{-1}A \in q\} \in p\big\}$. *Then, with the relative topology inherited from* βS, $(\delta S, \cdot)$ *is a compact right topological semigroup.*

Proof. [2, Proposition 2.6]. $\square$

5.6 Lemma. *Let* $(S, \cdot)$ *and* $(T, *)$ *be adequate partial semigroups and let* $f : S \xrightarrow{\text{onto}} T$ *have the property that for all* $x \in S$ *and all* $y \in \varphi_S(x)$, $f(y) \in \varphi_T\big(f(x)\big)$ *and* $f(x \cdot y) = f(x) * f(y)$. *Let* $\widetilde{f} : \beta S \to \beta T$ *be the continuous extension of* f. *Then the restriction of* $\widetilde{f}$ *to* δS *is a homomorphism from* $(\delta S, \cdot)$ *to* $(\delta T, *)$.

Proof. [2, Proposition 2.8]. $\square$

5.7 Definition. $\Phi = \{f : \mathbb{N} \to \mathbb{N} : f(n) \leq n \text{ for all } n \in \mathbb{N}\}$.

5.8 Theorem. *Let* $k \in \mathbb{N}$. *For each* $i \in \{1, 2, \ldots, k\}$, *let* E_i *be a countable commutative semigroup with identity* e_i. *For each* $i \in \{1, 2, \ldots, k\}$ *and* $j \in \mathbb{N}$, *let* $\langle z_{i,j,t} \rangle_{t=1}^{\infty}$ *be a sequence in* E_i. *We assume that, for every* $i \in \{1, 2, \ldots, k\}$, $z_{i,1,t} = e_i$ *for every* $t \in \mathbb{N}$, *and that* $\langle z_{i,2,t} \rangle_{t=1}^{\infty}$ *is a sequence which contains every element of* E_i *infinitely often.*

Let ψ be an arbitrary function mapping $E_1 \times E_2 \times \cdots \times E_k$ to a set X and let C_i be a central set in E_i for each $i \in \{1, 2, \ldots, k\}$. Then, for any finite coloring of X, there exist a sequence $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$, a sequence $\langle c_{i,n} \rangle_{n=1}^{\infty}$ in E_i for each $i \in \{1, 2, \ldots, k\}$ and a monochrome subset A of X such that the following statements hold for every $G \in \mathcal{P}_f(\mathbb{N})$, every $i \in \{1, 2, \ldots, k\}$ and all $f_1, f_2, \ldots, f_k \in \Phi$:

(i) $\psi\big(\prod_{n \in G} c_{1,n} \cdot \prod_{t \in H_n} z_{1,f_1(n),t}, \ldots, \prod_{n \in G} c_{k,n} \cdot \prod_{t \in H_n} z_{k,f_k(n),t}\big) \in A$ *and*

(ii) $\prod_{n \in G} c_{i,n} \cdot \prod_{t \in H_n} z_{i,f_i(n),t} \in C_i$.

Proof. Let $L = \mathbb{N}^k$ and let v be a “variable” not in L. A *located word* over L is a function w from a nonempty finite subset D_w of $\mathbb{N}$ to L. Let S_0 be the set of located words over L and let S_1 be the set of located variable words over L, that is the set of words over $L \cup \{v\}$ in which v occurs. Let $S = S_0 \cup S_1$. Given $u, w \in S$, if $\max D_u < \min D_w$, then define $u \cdot w$ by $D_{u \cdot w} = D_u \cup D_w$ and for $t \in D_{u \cdot w}$,

$$(u \cdot w)(t) = \begin{cases} u(t) & \text{if } t \in D_u \\ w(t) & \text{if } t \in D_w\,. \end{cases}$$

With this operation S, S_0, and S_1 are adequate partial semigroups so by Lemma 5.5 δS, δS_0, and δS_1, are compact right topological semigroups. Also δS_1 is an ideal of δS. (The verification of this latter statement is an easy exercise and a good chance for the reader to see whether she has grasped the definition of the operation.) Notice that for $j \in \{1, 2\}$ and $p \in \beta S_j$, one has that $p \in \delta S_j$ if and only if for each $n \in \mathbb{N}$, $\{w \in S_j : \min D_w > n\} \in p$.

For each $a \in L$, define $\theta_a : S \to S_0$ as follows. For $w \in S$, let $D_{\theta_a(w)} = D_w$ and for $t \in D_w$, let

$$\theta_a(w)(t) = \begin{cases} w(t) & \text{if } w(t) \in L \\ a & \text{if } w(t) = v\,. \end{cases}$$

That is, $\theta_a(w)$ is the result of replacing each occurrence of v in w by a. Denote also by θ_a its continuous extension taking βS to βS_0 and notice that θ_a is the identity on S_0 hence also on βS_0.

For each $i \in \{1, 2, \ldots, k\}$, define $g_i : S_0 \to E_i$ by $g_i(w) = \prod_{t \in D_w} z_{i,\pi_i(w(t)),t}$ for each $w \in S_0$. We shall also use g_i to denote the continuous function from βS_0 to βE_i which extends g_i.

We claim that, if $b_i \in E_i$ for each $i \in \{1, 2, \ldots, k\}$ and if $n \in \mathbb{N}$, there exists $w \in S_0$ such that $g_i(w) = b_i$ for every $i \in \{1, 2, \ldots, k\}$ and $\min(D_w) > n$. To see this, observe that we can choose $n_1, n_2, \ldots, n_k$ in $\mathbb{N}$ such that $n < n_1 < n_2 < \ldots < n_k$ and $z_{i,2,n_i} = b_i$ for every $i \in \{1, 2, \ldots, k\}$. We can then define w by putting $D_w = \{n_1, n_2, \ldots, n_k\}$ and

$w(n_i) = (1, 1, \ldots, 1, 2, 1, \ldots, 1)$, with 2 occurring as the i^{th} term in this k-tuple, for each $i \in \{1, 2, \ldots, k\}$.

In particular each $g_i : S_0 \to E_i$ is surjective and so, by Lemma 5.6, The restriction of g_i to δS_0 is a homomorphism to $\delta E_i = \beta E_i$.

For each $i \in \{1, 2, \ldots, k\}$, let p_i be a minimal idempotent in βE_i for which $C_i \in p_i$. We shall first show that we can choose a minimal idempotent $q \in \delta S_0$ and an idempotent $r \in \delta S_1$ such that $q \leq r$, $g_i(q) = p_i$ for every $i \in \{1, 2, \ldots, k\}$, and $\theta_a(r) = q$ for every $a \in L$.

Given $(X_1, X_2, \ldots, X_k, n) \in p_1 \times p_2 \times \cdots \times p_k \times \mathbb{N}$ we choose $w(X_1, X_2, \ldots, X_k, n) \in S_0$ such that $\min(D_{w(X_1,X_2,\ldots,X_k,n)}) > n$ and $g_i\big(w(X_1, X_2, \ldots, X_k, n)\big) \in X_i$ for each $i \in \{1, 2, \ldots, k\}$. We give $p_1 \times p_2 \times \cdots \times p_k \times \mathbb{N}$ a directed set ordering by stating that $(X_1, X_2, \ldots, X_k, n) \prec (X'_1, X'_2, \ldots, X'_k, n')$ if and only if $X'_i \subseteq X_i$ for each $i \in \{1, 2, \ldots, k\}$ and $n < n'$. If x is any limit point of the net $\langle w(X_1, X_2, \ldots, X_k, n)\rangle$ in βS_0, we have $x \in \delta S_0$ and $g_i(x) = p_i$ for every $i \in \{1, 2, \ldots, k\}$. (That $x \in \delta S_0$ follows from the fact that $\min(D_{w(X_1,X_2,\ldots,X_k,n)}) > n$. To see that $g_i(x) = p_i$, let $A \in p_i$ and suppose $g_i(x) \notin \overline{A}$. Pick $B \in x$ such that $g_i[\overline{B}\,] \cap \overline{A} = \emptyset$. Let $X_i = A$ and for $j \neq i$ let $X_j = E_j$. Pick $(X'_1, X'_2, \ldots, X'_k, n') \succ (X_1, X_2, \ldots, X_k, 1)$ such that $w(X'_1, X'_2, \ldots, X'_k, n') \in \overline{B}$. But $g_i\big(w(X'_1, X'_2, \ldots, X'_k, n')\big) \in X'_i \subseteq X_i = A$, a contradiction.)

Let $C = \big\{x \in \delta S_0 : g_i(x) = p_i \text{ for all } i \in \{1, 2, \ldots, k\}\big\}$. We have just seen that C is nonempty, and so it is a compact subsemigroup of δS_0. Let q be a minimal idempotent in C. Then q is minimal in δS_0, because if q' is any idempotent of δS_0 satisfying $q' \leq q$, we have $g_i(q') \leq g_i(q) = p_i$ for every $i \in \{1, 2, \ldots, k\}$. This implies that $g_i(q') = p_i$ for every $i \in \{1, 2, \ldots, k\}$. So $q' \in C$ and thus $q' = q$.

Let r be any idempotent in the intersection of the right ideal $q\delta S_1$ and the left ideal $\delta S_1 q$ of δS_1. Then $r \leq q$. For any $a \in L$, we have $\theta_a(r) \leq \theta_a(q) = q$ and hence $\theta_a(r) = q$.

We define $\gamma : S_0 \to X$ by $\gamma(w) = \psi\big(g_1(w), g_2(w), \ldots, g_k(w)\big)$. We can choose a monochrome subset A of X such that $\gamma^{-1}[A] \in q$. Let $Q = \gamma^{-1}[A] \cap \bigcap_{i=1}^{k} {g_i}^{-1}[C_i]$. Then $Q \in q$. Let $Q^\star = \{w \in Q : w^{-1}Q \in q\}$. Then $Q^\star \in q$ and $w^{-1}Q^\star \in q$ for every $w \in Q^\star$ by [13, Lemma 4.14].

We shall inductively choose a sequence $\langle w_n\rangle_{n=1}^{\infty}$ in S_1 such that for each $n \in \mathbb{N}$,

(a) if $n > 1$, then $\min D_{w_n} > \max D_{w_{n-1}}$ and

(b) for every nonempty $F \subseteq \{1, 2, \ldots, n\}$ and every choice of $a_t \in \{1, 2, \ldots, t\}^k$ for $t \in F$, $\prod_{t \in F} \theta_{a_t}(w_t) \in Q^\star$.

We first choose $w_1 \in S_1$ such that $\theta_a(w_1) \in Q^\star$, where a denotes the k-tuple $(1,1,\ldots,1)$. This is possible because ${\theta_a}^{-1}[Q^\star] \in r$ and so ${\theta_a}^{-1}[Q^\star] \neq \emptyset$. Now let $n \in \mathbb{N}$ and assume that $w_1, w_2, \ldots, w_n$ have been chosen. Let

$$U = \left\{ \prod_{t\in F} \theta_{a_t}(w_t) : \emptyset \neq F \subseteq \{1,2,\ldots,n\} \text{ and for all } t \in F\,,\ a_t \in \{1,2,\ldots,t\}^k \right\}.$$

By our assumption (b), $U \subseteq Q^\star$ so $\bigcap_{u\in U} u^{-1}Q^\star \in q$. We observe that, for any $V \in q$ and any $a \in L$, ${\theta_a}^{-1}[V] \in r$ and that $\{w \in S_1 : \min(D_w) > \max(D_{w_n})\} \in r$. Thus we can choose w_{n+1} such that $\min(D_{w_{n+1}}) > \max(D_{w_n})$, and

$$w_{n+1} \in \bigcap \left\{ {\theta_a}^{-1} \left[Q^\star \cap \bigcap_{u\in U} u^{-1}Q^\star \right] : a \in \{1,2,\ldots,n+1\}^k \right\}.$$

We can now conclude the proof. For each $n \in \mathbb{N}$ and $i \in \{1,2,\ldots,k\}$, let $H_n = \{t \in D_{w_n} : w_n(t) = v\}$ and let $c_{i,n} = \prod_{t\in D_{w_n}\setminus H_n} z_{i,\pi_i(w_n(t)),t}$. Then, if $a \in L$, we have $g_i\big(\theta_a(w_n)\big) = c_{i,n} \cdot \prod_{t\in H_n} z_{i,\pi_i(a),t}$.

Suppose now that $f_1, f_2, \ldots, f_k \in \Phi$ and $G \in \mathcal{P}_f(\mathbb{N})$. For each $n \in \mathbb{N}$, define $a_n \in \{1,2,\ldots,n\}^k$ by $\pi_i(a_n) = f_i(n)$ for each $i \in \{1,2,\ldots,k\}$. Then for each $i \in \{1,2,\ldots,k\}$, we have

$$\begin{aligned}
\prod_{n\in G} c_{i,n} \cdot \prod_{t\in H_n} z_{i,f_i(n),t} &= \prod_{n\in G} c_{i,n} \cdot \prod_{t\in H_n} z_{i,f_i(n),t} \\
&= \prod_{n\in G} g_i\big(\theta_{a_n}(w_n)\big) \\
&= g_i\big(\prod_{n\in G} \theta_{a_n}(w_n)\big)\,.
\end{aligned}$$

Since $\prod_{n\in G} \theta_{a_n}(w_n) \in Q$, $\gamma[Q] \subseteq A$, and $g_i[Q] \subseteq C_i$ for each $i \in \{1,2,\ldots,k\}$ the conclusions of the theorem hold. $\square$

We show now how to derive a simple strengthening of Theorem 4.8 from Theorem 5.8.

5.9 Corollary. *Let $m, k \in \mathbb{N}$. Let C_1 be central in $(\mathbb{N},+)$ and let C_2 be central in $(\mathbb{N},\cdot)$. For each $i \in \{0,1,\ldots,k\}$ let $\langle x_{i,t}\rangle_{t=1}^\infty$ and $\langle y_{i,t}\rangle_{t=1}^\infty$ be sequences in $\mathbb{N}$. Let $\mathbb{N} = \bigcup_{s=1}^m A_s$. Then there exist $s \in \{1,2,\ldots,m\}$, $F \in \mathcal{P}_f(\mathbb{N})$, and $a, b \in \mathbb{N}$ such that*

$$\begin{aligned}
&\{ba\} \cup \big\{b(a + \textstyle\sum_{t\in F} x_{i,t}) : i \in \{0,1,\ldots,k\}\big\} \cup \\
&\big\{ba \cdot \textstyle\prod_{t\in F} y_{j,t} : j \in \{0,1,\ldots,k\}\big\} \cup \\
&\big\{b(a + \textstyle\sum_{t\in F} x_{i,t}) \cdot (\prod_{t\in F} y_{j,t}) : i,j \in \{0,1,\ldots,k\}\big\} \subseteq A_s\,, \\
&\{a\} \cup \big\{a + \textstyle\sum_{t\in F} x_{i,t} : i \in \{0,1,\ldots,k\}\big\} \subseteq C_1\,, \text{ and} \\
&\{b\} \cup \big\{b \cdot \textstyle\prod_{t\in F} y_{j,t} : j \in \{0,1,\ldots,k\}\big\} \subseteq C_2\,.
\end{aligned}$$

Proof. Let $E_1 = (\omega,+)$ and let $E_2 = (\mathbb{N},\cdot)$. Define $\psi : E_1 \times E_2 \to \omega$ by $\psi(a,b) = ab$. For $t \in \mathbb{N}$ let $z_{1,1,t} = 0$ and $z_{2,1,t} = 1$. For $i \in \{1,2\}$ let $\langle z_{i,2,t}\rangle_{t=1}^\infty$ be a sequence which contains every element of E_i infinitely often. For $j \in \{0,1,\ldots,k\}$ and $t \in \mathbb{N}$ let

$z_{1,j+3,t} = x_{j,t}$ and $z_{2,j+3,t} = y_{j,t}$. (For $j > k+3$ we do not care what $z_{1,j,t}$ and $z_{2,j,t}$ are.)

Since $\mathbb{N}$ is an ideal of $(\omega, +)$, C_1 is central in E_1. Pick $\langle H_n\rangle_{n=1}^{\infty}$, $\langle c_{1,n}\rangle_{n=1}^{\infty}$, $\langle c_{2,n}\rangle_{n=1}^{\infty}$, and A as guaranteed by Theorem 5.8. Pick $s \in \{1, 2, \ldots, m\}$ such that $A \subseteq A_s$. Let $n = k+3$. (We choose $n = k+3$ rather than $n = 1$ so that there will be functions f_1 and f_2 in Φ with the properties required of them below.) Let $a = c_{1,n}$, let $b = c_{2,n}$, and let $F = H_n$. If $f_1(n) = 1$, then $c_{1,n} + \sum_{t\in H_n} z_{1,f_1(n),t} = a$. If $f_1(n) = j+3$ for some $j \in \{0, 1, \ldots, k\}$, then $c_{1,n} + \sum_{t\in H_n} z_{1,f_1(n),t} = a + \sum_{t\in F} x_{j,t}$. If $f_2(n) = 1$, then $c_{2,n} \cdot \prod_{t\in H_n} z_{2,f_2(n),t} = b$. If $f_2(n) = j+3$ for some $j \in \{0, 1, \ldots, k\}$, then $c_{2,n} \cdot \prod_{t\in H_n} z_{2,f_2(n),t} = b \cdot \prod_{t\in F} y_{j,t}$. □

We conclude with a simple variation on the proof of Theorem 5.8 which applies in case the semigroups are all the same.

5.10 Theorem. *Let $k \in \mathbb{N}$, let E be a countable commutative semigroup with identity e, let $R_1, R_2, \ldots, R_k$ be IP-sets in E, and let C be a central subset of E. There exist $r_i \in R_i$ and $b_i \in E$ for each $i \in \{1, 2, \ldots, k\}$ such that whenever $f : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, k\}$, $h : \{1, 2, \ldots, k\} \to \{0, 1, \ldots, k\}$, and $\emptyset \neq F \subseteq \{1, 2, \ldots, k\}$, one has $\prod_{i\in F} b_i \cdot (r_{f(i)})^{h(i)} \in C$.*

Proof. Let $L = \{1, 2, \ldots, k^2+k+2\}^k$ and let v, S_0, S_1, S, $\langle D_w\rangle_{w\in S}$, and $\langle \theta_a\rangle_{a\in L}$ be as in the proof of Theorem 5.8. For $j \in \{1, 2, \ldots, k\}$ pick a sequence $\langle x_{j,t}\rangle_{t=1}^{\infty}$ such that $FP(\langle x_{j,t}\rangle_{t=1}^{\infty}) \subseteq R_j$. For $m \in \{0, 1, \ldots, k\}$, $j \in \{1, 2, \ldots, k\}$, and $t \in \mathbb{N}$, let $z_{2+mk+j,t} = (x_{j,t})^m$. Let $z_{1,t} = e$ for each t and let $\langle z_{2,t}\rangle_{t=1}^{\infty}$ be a sequence in E which takes on each member of E infinitely often.

For $i \in \{1, 2, \ldots, k\}$, define $g_i : S_0 \to E$ by $g_i(w) = \prod_{t\in D_w} z_{\pi_i(w(t)),t}$. For $F \in \mathcal{P}_f(\{1, 2, \ldots, k\})$, define $\gamma_F : S_0 \to E$ by $\gamma_F(w) = \prod_{i\in F} g_i(w)$ (so $\gamma_{\{i\}} = g_i$). Denote also by γ_F the continuous extension taking βS_0 to βE.

As in the proof of Theorem 5.8 we see that given any $b_1, b_2, \ldots, b_k \in E$ there is some $w \in S_0$ such that $g_i(w) = b_i$ for each $i \in \{1, 2, \ldots, k\}$. In particular each γ_F is a surjective homomorphism so by Lemma 5.6 the restriction of γ_F to δS_0 is a homomorphism to βE.

Pick a minimal idempotent $p \in \beta E$ such that $C \in p$. We claim that for any $B \in p$ and any $n \in \mathbb{N}$ there exists $w_{B,n} \in S_0$ such that for all $F \in \mathcal{P}_f(\{1, 2, \ldots, k\})$, $\gamma_F(w_{B,n}) \in B$. To see this pick $b_1, b_2, \ldots, b_k$ such that $FP(\langle b_t\rangle_{t=1}^{k}) \subseteq B$, which one may do because p is an idempotent. Pick $w_{B,n}$ such that $g_i(w_{B,n}) = b_i$ for each $i \in \{1, 2, \ldots, k\}$.

Direct $\mathcal{D} = \{(B,n) : B \in p \text{ and } n \in \mathbb{N}\}$ by $(B,n) \prec (B',n')$ if and only if $B' \subseteq B$ and $n < n'$. Let u be a limit point of the net $\langle w_{B,n}\rangle_{(B,n)\in\mathcal{D}}$ in βS_0. We see as in the proof of Theorem 5.8 that $u \in \delta S_0$ and $\gamma_F(u) = p$ for all $F \in \mathcal{P}_f(\{1,2,\ldots,k\})$. Let $J = \{w \in \delta S_0 : \gamma_F(w) = p \text{ for all } F \in \mathcal{P}_f(\{1,2,\ldots,k\})\}$. Then J is a compact subsemigroup of δS_0 since each γ_F is a continuous homomorphism. Pick a minimal idempotent q of J. Given any idempotent $q' \in \delta S_0$ such that $q' \leq q$, for each $F \in \mathcal{P}_f(\{1,2,\ldots,k\})$, $\gamma_F(q') \leq \gamma_F(q) = p$ so $\gamma_F(q') = p$. Thus $q' \in J$ and so $q' = q$. That is, q is minimal in δS_0.

Now we claim that we may choose $w \in S_1$ such that $\gamma_F\big(\theta_a(w)\big) \in C$ for every $a \in L$ and every $F \in \mathcal{P}_f(\{1,2,\ldots,k\})$. To see this, pick an idempotent r in $q\delta S_1 \cap \delta S_1 q$. Then $r \leq q$ so for each $a \in L$, $\theta_a(r) \leq \theta_a(q) = q$ and so $\theta_a(r) = q$ and thus for each $F \in \mathcal{P}_f(\{1,2,\ldots,k\})$, $\gamma_F\big(\theta_a(r)\big) = \gamma_F(q) = p$. Pick $w \in S_1 \cap \bigcap\{(\gamma_F \circ \theta_a)^{-1}[C] : a \in L \text{ and } F \in \mathcal{P}_f(\{1,2,\ldots,k\})\}$.

Let $H = \{t \in D_w : w(t) = v\}$. For $i \in \{1,2,\ldots,k\}$, let $b_i = \prod_{t\in D_w\setminus H} z_{\pi_i(w(t)),t}$ and let $r_i = \prod_{t\in H} x_{i,t}$. Now let $f : \{1,2,\ldots,k\} \to \{1,2,\ldots,k\}$, $h : \{1,2,\ldots,k\} \to \{0,1,\ldots,k\}$, and $\emptyset \neq F \subseteq \{1,2,\ldots,k\}$. Let

$$a = \big(2 + h(1)k + f(1), 2 + h(2)k + f(2), \ldots, 2 + h(k)k + f(k)\big)\,.$$

Then for $i \in F$,

$$\begin{aligned} b_i(r_{f(i)})^{h(i)} &= b_i \cdot \left(\textstyle\prod_{t\in H} (x_{f(i),t})^{h(i)}\right) \\ &= b_i \cdot \textstyle\prod_{t\in H} z_{\pi_i(a),t} \\ &= g_i\big(\theta_a(w)\big) \end{aligned}$$

so $\prod_{i\in F} b_i(r_{f(i)})^{h(i)} = \gamma_F\big(\theta_a(w)\big) \in C$. □

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