The purpose of the following exercise is to discuss the concept of coherent risk measures, which was introduced by Artzner, Delbaen, Eber and Heath in 1997 (Thinking Coherently, RISK, 1997).

**Definition (Coherent Risk measure).** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and let $L^\infty$ denote the set of random variables $\Omega \to \mathbb{R}$, which are bounded $\mathbb{P}$-a.s. A coherent risk measure is a function $\rho : L^\infty \to \mathbb{R}$ such that

1. $X \geq 0 \Rightarrow \rho(X) \leq 0$;
2. $\rho(\lambda X) = \lambda \rho(X)$, $\lambda \geq 0$ (Positive Homogeneity);
3. $\rho(X + k) = \rho(X) - k$, $k \in \mathbb{R}$ (Translation Invariance);
4. $\rho(X + Y) \leq \rho(X) + \rho(Y)$ (Subadditivity).

**Exercise 1.**

a) Discuss properties 1. to 4. in the above definition.

b) Show that a coherent risk measure is monotone, i.e., for $X \geq Y$, we have $\rho(X) \leq \rho(Y)$.

c) Show that VaR is not a coherent risk measure by constructing an example which shows that VaR is not subadditive.

d) We define the expected shortfall of some investment’s profit and losses, $X$, with respect to some fixed level of confidence, $\alpha$, as the expected value of all losses, which exceed the $\alpha$–quantile of $X$, $X(\alpha)$, i.e.:

$$
-\frac{1}{\alpha} \mathbb{E}[X | X \leq X(\alpha)].
$$

Show that the expected shortfall is a coherent risk measure.