Exercises 6 – November 23, 2003

Exercise 1. (European Call Option in an \(n\)-Period Binomial Model)
Consider the binomial model presented in Lecture 6 on November 17. Show that for the 2-period model, the price of a European call equals

\[
C = \frac{p^2 C_{uu} + 2p(1 - p)C_{ud} + (1 - p)^2 C_{dd}}{(1 + i)^2},
\]

where \(p\) and \((1 - p)\) are the risk-neutral probabilities for an up-jump and a down-jump, respectively, of the underlying asset, \(i\) is the risk-free interest rate, and \(C_{uu}\) is the value of the option in the state of the world, where the underlying asset has moved up two times (see page 23 of the lecture notes); i.e.

\[
C_{uu} = \max\{u^2S - K, 0\},
\]

where \(u\) is the relative size of an up-jump in the binomial model, \(S\) is the starting value of the underlying asset and \(K\) is the strike of the call option. \((C_{ud} \text{ and } C_{dd})\) are defined analogously.

Generalize this to \(n\) steps and show that

\[
C = \frac{\sum_{j=0}^{n} p^j(1 - p)^{n-j} \binom{n}{j} \max\{u^j d^{n-j} S - K, 0\}}{(1 + i)^n}.
\]

Exercise 2. (Applying the Black–Scholes PDE to Different Kinds of Derivatives)
Apply the Black–Scholes PDE method to find a closed form solution of the price of

- an interest rate forward,
- an FX forward,
- and an equity forward contract.

(See page 34 of the lecture notes of November 17.)