Exercises 9 – January 19, 2004

Exercise 1.
Reconsider Exercises 1 and 2 from last week.

Exercise 2. (Discussion or Markov chains with finite range — CreditMetrics approach)
Consider some discrete stochastic process $X_n$ which may attain one of 6 states of the world $v_1, \ldots, v_6$. Assume that the conditional probabilities (“transition probabilities")

$$P(X_{n+1} = j | X_n = j) = p_{i,j}, \ i, j = 1, \ldots, 6$$

are independent of the realizations of $X_k$ for $k < i$ (Markov–property).
Consider the matrix $T = (p_{i,j})_{i,j=1}^{6}$ of transition probabilities and assume that the probability distribution for $X_0$ is given by some vector $\pi$; i.e., $P(X_0 = v_i) = \pi_i$. Prove the following assertions:

- $\sum_{j=1}^{6} p_{i,j} = 1$ for $1 \leq i \leq 6$,
- $P(X_m = v_j | X_0 = v_i) = (\pi T^m)_{i,j}$.

Now let $T$ be the matrix

$$
\begin{pmatrix}
\frac{1}{2} & \frac{1}{4} & \frac{13}{100} & \frac{7}{100} & \frac{1}{25} & \frac{1}{100} \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{25} & \frac{1}{100} & \frac{1}{100} \\
\frac{13}{100} & \frac{1}{2} & \frac{1}{2} & \frac{1}{25} & \frac{1}{100} & \frac{1}{100} \\
\frac{7}{100} & \frac{7}{100} & \frac{13}{100} & \frac{1}{4} & \frac{1}{2} & \frac{1}{100} \\
\frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{4} & \frac{1}{2} & \frac{1}{100} \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

Interpret this as the matrix of migration probabilities of credit ratings, where the last rating $v_6$ is default, which is an “absorbing state”; i.e.: $P(X_{m+1} = v_6 | X_m = v_6) = 1$ for all $m$.
Show that — irrespective of the initial distribution $\pi$ —

- $\lim_{m \to \infty} P(X_m = v_i) = 0$ for $i \neq 6$,
- $\lim_{m \to \infty} P(X_m = v_6) = 1$

and generalize this to every matrix of migration probabilities where

- there is an non–vanishing probability for every non–default rating to move to default in the next step,
- default is an absorbing state, i.e., the probability to “move away” from the default state is zero.

Discuss the implication of this fact for the CreditMetrics model for credit risk.