Informations: http://www.mat.univie.ac.at/~mfulmek

DI Andre Horovitz (ERSTE Bank)

2nd Lecture „Risk Management“
We can define a new random variable which takes one from possible values: 

<table>
<thead>
<tr>
<th>ML</th>
<th>FL</th>
<th>MR</th>
<th>FR</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>L</td>
<td>0.1</td>
<td>0.9</td>
<td>R</td>
</tr>
</tbody>
</table>

TU students ranked by [Bivariate Distribution (Illustration)]

• Joint Distributions
If the probability of being male/female is independent of the probability of being left-handed or right-handed, then

\[ P(M^L) = P(M) \cdot P(L) = 0.5 \cdot 0.1 = 0.05 \]

and TOTAL(1) and TOTAL(2) are called „the marginal distributions. „

<table>
<thead>
<tr>
<th></th>
<th>0.00</th>
<th>0.50</th>
<th>0.50</th>
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</thead>
<tbody>
<tr>
<td>TOTAL(2)</td>
<td></td>
<td></td>
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<tr>
<td>0.00</td>
<td>0.90</td>
<td>0.45</td>
<td>R</td>
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<tr>
<td>0.10</td>
<td>0.45</td>
<td>0.05</td>
<td>L</td>
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<tr>
<td>TOTAL(1)</td>
<td></td>
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<td></td>
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<tr>
<td>M</td>
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</tbody>
</table>

\[ \text{Joint Probability Distribution} \]

\[ (T \cap M) \cdot P(T) = 0.5 \cdot 0.1 = 0.05 \]
Key properties

Mean of the sum of two random variables

\[(\lambda) \mathbb{E} \cdot (X) \mathbb{E} = \mathbb{E} (\lambda \cdot X)\]

Mean of the product of two independent random variables

\[(\lambda) \mathbb{E} + (X) \mathbb{E} = (\lambda \mathbb{E} + X \mathbb{E})\]
\[
\begin{align*}
\mathbb{V}_{\mathcal{Z}} & = \mathbb{V}_{\mathcal{Y}} \\
\mathbb{V}_{\mathcal{X} + \mathcal{Y}} & = \mathbb{V}_{\mathcal{X}} + \mathbb{V}_{\mathcal{Y}}
\end{align*}
\]

or more general
\[
\begin{align*}
\mathbb{V}_{\lambda \mathcal{Z}} & = \lambda^2 \mathbb{V}_{\mathcal{Z}} \\
\mathbb{V}_{\lambda \mathcal{Z} + \mathcal{X}} & = \lambda \mathbb{V}_{\mathcal{Z}} + \mathbb{V}_{\mathcal{X}} \\
\mathbb{V}_{\lambda \mathcal{Z} + \mathcal{X} \mathcal{Y}} & = \lambda \mathbb{V}_{\mathcal{Z}} + \mathbb{V}_{\mathcal{X}} \mathbb{V}_{\mathcal{Y}}
\end{align*}
\]
Continuous Random Variables

A continuous random variable is one that can take any value within a given range. Analogous to a “probability distribution” in the case of discrete random variables, we define “probability density functions” (pdf) for continuous random variables.
\[ xp(c) f \approx d \]

Probability of obtaining a value near to a particular value \( c \):

\[ 0 = \int_{p}^{v} xp(x) f \quad (v = x) d \]

\[ \int_{q}^{a} xp(x) f \quad (q \geq x \geq v) d \]
A measure of how likely it is that the random variable will be close to a.

\[ \int_\mathbb{R} (x)f(x) = 1 \]

1. \( D \ni x \in (0, \infty) \)

2. \( \int_D x p(x)f(x) dx = 1 \)

Condition for continuous function \( f(x) \) to be a valid probability density function.
The cumulative distribution function $F(b)$ of the continuous random variable $X$ with probability density function $f(x)$ is:

$$F(b) = \int_{-\infty}^{b} f(x) \, dx = \Pr(X \leq b)$$

The median of the continuous random variable $X$ with probability density function $f(x)$ is the value $M$ such that

$$\Pr(X \leq M) = \frac{1}{2}$$

(Naturally, $F(0)(x) = f(x)$)

$$\int_{-\infty}^{x} f(x) \, dx = \Pr(X \leq x) = F(x)$$

$$\int_{x}^{\infty} f(x) \, dx = \Pr(X > x) = 1 - F(x)$$

The cumulative distribution function $F(q)$ of the continuous random variable $X$ with the probability density function $f(x)$ is:

$$F(q) = \int_{-\infty}^{q} f(x) \, dx = \Pr(X \leq q)$$
The mode of the continuous variable \( X \) with probability density function \( f(x) \) is the value of \( x \) for which \( f(x) \) is a local maximum.

\[
\begin{align*}
E(X) &= \int_{-\infty}^{\infty} x f(x) \, dx \\
\text{Var}(X) &= \int_{-\infty}^{\infty} (x - E(X))^2 f(x) \, dx
\end{align*}
\]
Properties of the Expectation operator for continuous random variables

\[
E(g(X)) = \int_1^\infty g(x) f(x) \, dx
\]

\[
E(g(X) + h(X)) = \int_1^\infty g(x) f(x) \, dx + \int_1^\infty h(x) f(x) \, dx = E(g(X)) + E(h(X))
\]

What about the variance \( \sigma^2 \)?

\[
\sigma^2 = \int_1^\infty (x - \mu)^2 f(x) \, dx = \int_1^\infty (x^2 - 2x\mu + \mu^2) f(x) \, dx
\]

\[
= \mu^2 - \mu^2 = \mu^2
\]

\[
\sigma^2 = \int_1^\infty (x - \mu)^2 f(x) \, dx = \mu^2
\]

\[
((H\gamma) \, \mathbb{E} + ((X)\beta) \, \mathbb{E}) =
\]

\[
\int_1^\infty x p(x) f(x) \gamma \, dx + \int_1^\infty x p(x) f(x) \beta \, dx = ((X)\gamma + (X)\beta) \, \mathbb{E}
\]

\[
\int_1^\infty x p(x) f(x) \beta \, dx = ((X)\beta) \, \mathbb{E}
\]

\* What about the variance \( \sigma^2 \)?

continuous random variables

Properties of the Expectation operator for
The continuous uniform distribution

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

It is a valid pdf since:

$$\int_{-\infty}^{\infty} f(x) \, dx = 1$$

Mean $\equiv \text{Median} = \frac{a+b}{2}$

$$I = \frac{a+b}{x} = \int_{-\infty}^{\infty} \frac{a-b}{x} \, dx$$
\[
\frac{\mathcal{I}}{\mathcal{I} (\mathcal{I} - \nu)} = \frac{\mathcal{I}}{\mathcal{I} (\mathcal{I} - \nu + \mathcal{I} \nu + \mathcal{I} \nu^2)} = \frac{\mathcal{I}}{\mathcal{I} (\mathcal{I} - \nu + \mathcal{I} \nu + \mathcal{I} \nu^2)} = \frac{\mathcal{I}}{\mathcal{I} (\mathcal{I} - \nu + \mathcal{I} \nu + \mathcal{I} \nu^2)} = \left(\frac{\mathcal{I}}{\mathcal{I} (\mathcal{I} - \nu + \mathcal{I} \nu + \mathcal{I} \nu^2)} - \mathcal{I} \right) \frac{\nu}{\mathcal{I}} = \mathcal{I} - \left(\frac{\mathcal{I} \nu}{\mathcal{I}} + \mathcal{I} \nu + \mathcal{I} \nu^2 \right) = [x]_{\mathcal{I}} \nu
\]
For a discrete variable,

\[ M(t) = \sum \frac{X}{P(X = r)} e^{rt} \]

The moment generating function for a discrete variable with pdf \( f(x) \) is

\[ \mathbb{E}(X^n) = \int_{-\infty}^{\infty} x^n f(x) dx \]

The \( s \)th uncorrected moment of a continuous random variable with pdf \( f(x) \) and mean \( \mu \) is

\[ \mathbb{E}(X^s) = \int_{-\infty}^{\infty} x^s f(x) dx \]

The \( s \)th corrected moment of a continuous random variable with \( f(x) \) is

\[ \mathbb{E}(X^s) = \int_{-\infty}^{\infty} \frac{x^s}{\mu} f(x) dx \]

For a discrete variable, the moment generating function

\[ \mathbb{E}(e^{tX}) = \prod_{i=1}^{\infty} \left( 1 + \frac{t}{i} \right) \]
The moment generating function of a random variable $X$ is defined as:

$$E(e^{\lambda X})$$

Notation:

$E$ denotes the expectation.
\[
\cdots + \frac{s_i i^s}{s^i} + \cdots + \frac{\xi_i \xi^i}{\xi^i} + \frac{\zeta_i \zeta^i}{\zeta^i} + n + 1 = (t) W = \left( t \chi \right)^{\vartheta} \]

\[
\cdots + \frac{x \rho(x) f}{x^i} \int_{\infty}^{0} \frac{t}{x^i} + \frac{x \rho(x) f}{x^i} \int_{\infty}^{0} \frac{t}{x^i} + \frac{x \rho(x) f}{x^i} \int_{1}^{0} + (x) f \int_{\infty}^{0} = \left( t \chi \right)^{\vartheta}
\]

So, we can integrate each term separately,

\[
(x) f \int_{\infty}^{0} = \left( t \chi \right)^{\vartheta} = \sum_{i=0}^{\infty} \frac{x \rho(x) f}{x^i} \int_{\infty}^{0} \frac{t}{x^i} + \frac{x \rho(x) f}{x^i} \int_{1}^{0} + (x) f \int_{\infty}^{0} = \left( t \chi \right)^{\vartheta}
\]

\[
\cdots + \frac{i \eta}{\eta(1x)} + \cdots + \frac{i \zeta}{\zeta(1x)} + n + 1 = x \chi
\]

\textbf{Generating the moments}
Examples and Applications

\[
\begin{align*}
= \left( \frac{\lambda}{\theta} - 1 \right) = \frac{\lambda}{\theta} - 1 = \frac{\theta - \lambda}{\lambda} = \frac{\theta}{\lambda} \\
\int_{-\infty}^{\infty} \left[ \frac{\theta - \lambda}{\lambda} \right] = \frac{\theta - \lambda}{\lambda} = \frac{\theta - \lambda}{\lambda} = \frac{\theta - \lambda}{\lambda} \\
\exp x(x - \lambda) - \infty \int_0^\lambda x = x \exp x(x - \lambda) - \infty \int_0^\lambda \text{ MGF: yield MGF:}
\end{align*}
\]

\[
\text{elsewhere}
\]

\[
\begin{cases}
0 & x \in \mathbb{R} \\
(x)^f & x \in \mu
\end{cases}
\]

Take the exponential distribution with the pdf
\[ \frac{t - \chi}{\chi} = \frac{t - \text{id}}{\text{id}} = (t) \mathcal{W} \]

Applying this to the exponential distribution with MGF:

\[ \cdots s^{s} \frac{e}{s^{s}} = 0= t \left( \frac{s^{t} p}{(t) \mathcal{W}^{p}} \right) \cdots , \quad s^{s} \frac{p}{s^{s}} = 0= t \left( \frac{p^{t} p}{(t) \mathcal{W}^{p}} \right) \quad s^{s} \frac{t}{s^{s}} = 0= t \left( \frac{t^{t} p}{(t) \mathcal{W}^{p}} \right) \]

and note that

\[ \cdots + s^{s} \frac{1}{s^{s}} + \cdots + \frac{\gamma^{t}}{\gamma^{t}} \frac{e^{t}}{e^{t}} + \gamma^{t} \frac{\gamma^{t}}{\gamma^{t}} + \frac{\gamma^{t}}{\gamma^{t}} + \frac{t}{t} = (t) \mathcal{W} \]

Consider the expansion of the continuous uniform distribution (can also be derived by traditional means as shown earlier in the case).

\[ \frac{\gamma^{t}}{t} = \frac{\gamma^{t}}{t} - \frac{\gamma^{t}}{t} = \gamma^{t} - \left( \frac{t}{t} \right) \frac{t}{t} = \gamma^{t} - \gamma^{t} \]

Variance:

\[ \frac{\gamma}{t} = \frac{\gamma}{t} \]

Mean:

\[ \frac{\gamma}{t} = \frac{\gamma}{t} \]
\[
\overline{\nu u} = \nu \mathbb{1}_{-u}(\nu + \nu - 1)u \overset{t \to u}{\longrightarrow} \mathcal{N}(\nu + (\nu - 1)u = \frac{\mu}{(t)^{\frac{1}{2}}}
\]

\[
\mathfrak{J}_u \nu \mathcal{P}_{(u)}^{0 \to \nu} = u(q + p)
\]

using the binomial series:

\[
u[\nu + (\nu - 1)] = \infty - u(\nu - 1)u(\nu)(\frac{u}{u})^{0 \to \nu} = (\nu = \mathcal{N}) \mathfrak{J}_u \nu \mathcal{P}_{(u)}^{0 \to \nu} = (\nu)(\mathcal{N})
\]

so,

\[
\{u, \cdots, \nu, 0\} = \nu \mathbb{1}_{-u}(\nu - 1)u(\nu)(\frac{u}{u})^{0 \to \nu} = (\nu = \mathcal{N}) \mathfrak{J}_u
\]

\[
(\nu - 1)\nu u = \text{mean} = \text{variance} = \mu u
\]

already found: mean and variance, now take the binomial distribution \(\mathcal{N}\) and derive once again the result.

\[
(\nu) \quad \frac{\nu \chi}{\mathcal{N}} = 0 \overset{\mathcal{N}}{=} \left(\frac{\nu \mathcal{P}}{(t)^{\frac{1}{2}} \mathcal{P}}\right) : 0 = \mathfrak{J}_u \nu \mathfrak{J}_u \quad \text{so} \quad \frac{\nu(t-\gamma)}{\chi \mathcal{N}} = 0 \overset{\mathcal{N}}{=} \left(\frac{\nu \mathcal{P}}{(t)^{\frac{1}{2}} \mathcal{P}}\right)
\]

\[
(\mathfrak{J}) \quad \frac{\chi}{\mathcal{N}} = 0 \overset{\mathcal{N}}{=} \left(\frac{\nu \mathcal{P}}{(t)^{\frac{1}{2}} \mathcal{P}}\right) : 0 = \mathfrak{J}_u \nu \mathfrak{J}_u \quad \text{so} \quad \frac{\gamma(t-\chi)}{\gamma \mathcal{N}} = \left(\frac{\nu \mathcal{P}}{(t)^{\frac{1}{2}} \mathcal{P}}\right)
\]
\[
\begin{aligned}
I &= \frac{LS^o}{LS^t} \left\{ \frac{(\nu - 1)^{\nu u}}{\nu u - \nu} \right\} = X \\
0 &= \frac{LS^t}{LS^t} \\
(\nu - 1)^{\nu u} &= \frac{\nu u}{\nu u} = \frac{n}{n} \\
\end{aligned}
\]

Let's standardize it.

\[\begin{aligned}
\nu - u(\nu - 1)\nu\left(\frac{u}{u}\right) = (\nu = H) \text{ Binomial distribution (again)}
\end{aligned}\]

The normal distribution.

\[\begin{aligned}
(\nu - 1)^{\nu u} &= \frac{\nu}{\nu} - \frac{\nu}{\nu} + \frac{\nu}{\nu}(1 - u)u = \frac{\nu}{\nu} - \frac{\nu}{\nu} = \frac{\nu}{\nu} \\
\text{so for } \nu - u[\nu - (\nu - 1)]u + \frac{\nu}{\nu} - u[\nu + (\nu - 1)](1 - u)u = \frac{\nu}{\nu} \text{ we obtain that}
\end{aligned}\]
\[ \bar{T} = \frac{x \rho}{2} e^{-\frac{x^2}{2}} \int_{-\infty}^{\infty} \]
A continuous random variable \( X \) has a normal distribution ("normal density") if its PDF is

\[
 f(x) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2}}; \quad x \in \mathbb{R}
\]

It is easy to derive mean and variance (applying partial integration):

\[
\mu = \mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x) \, dx
\]

\[
\sigma^2 = \text{Var}(X) = \mathbb{E}(X^2) - \mu^2 = \int_{-\infty}^{\infty} x^2 f(x) \, dx - \mu^2
\]
Binomial Distribution

Let $R(T) = \log$ the logarithm of the stock return over $n$ periods (of equal length $t$) over $[0, T]$. Let $r(i)$ be the logarithm of the stock return at the end of the $i$-th period. Then $R(T) \sim \text{Binomial}(u, \cdot \cdot \cdot , (\ell,t), \cdot \cdot \cdot , (\ell,t))$.

We assume $u, \cdot \cdot \cdot , (\ell,t), \cdot \cdot \cdot , (\ell,t)$ independent and each $r(j)$ Bernoulli distributed:

$\begin{align*}
I &= b + d, \quad b = (p = (\ell,t)) \cdot \cdot \cdot , (n = (\ell,t)) \cdot \cdot \cdot , (\ell,t) \\
I &= b + d, \quad b = (p = (\ell,t)) \cdot \cdot \cdot , (n = (\ell,t)) \cdot \cdot \cdot , (\ell,t) \\
\end{align*}$

Let $R(T)$ be the logarithm of the stock return over $n$ periods (of equal length $t$) over $[0, T]$. Let $R(T) \sim \text{Binomial}(u, \cdot \cdot \cdot , (\ell,t), \cdot \cdot \cdot , (\ell,t))$.

Binomial Distribution
For big $n$, \( \frac{bdu}{du} \) tends to a Gaussian normal distribution.

\[
\begin{align*}
\gamma_i & = \phi_i (\gamma \boldsymbol{d} + b) \\
\gamma_i & = \phi_i (\gamma \boldsymbol{d} + b) \\
\gamma_i - \mu_i & \sim \mathcal{N}(\phi_i, \sigma^2) \\
\text{var} bdu & = \left( \frac{\partial}{\partial x} \phi_i \right)^2 \\
\text{var} bdu & = \left( \frac{\partial}{\partial x} \phi_i \right)^2 \\
\end{align*}
\]

It follows that:
Proof

Let \( Z = R(T) \)

\[ npq = 1 \]

\[ R(T) \]

Then \( Z(t) = e^{it} + n \ln q + p e^{it} \)

\[ n ! 1 \] and fixed \( t \), \( Z(t) ! e^{2t} \)

Indeed, \( \ln Z(t) = it + q \ln q + p e^{it} \)

Near \( a = 0: \ln(1 + a) = a - \frac{a^2}{2} + \frac{a^2}{f(a)} \) with \( f(a) \to 0 \) as \( a \to 0 \).

With \( x = it + p \):

\[ \ln Z(t) = n px + np(e^x - 1) \]

Then \( \frac{bdu}{}\) and \( \frac{du}{} \) are fixed. For \( t \), \( \infty \sim u \)

Indeed, \( Z = (t)Z \) \( \infty \), \( z \sim e^{-t} \)

Then \( \frac{bd}{du} = (L)H \frac{bd}{du} = \frac{bd}{du} - (L)H = Z \) let 

Proof
Let us prove that

\[ I = \int_0^\infty \frac{\nu/2}{x} \int_0^\infty \frac{x}{d-xe} \, dx \; dx \]
\[ o \psi z^V = x p \left( \frac{\theta}{\eta - x} \right) \int_{\frac{z}{l}}^\infty e^{-t} \]

Also
\[ \mu \psi z^V = x p \int_{z-x}^\infty e^{-t} \mu^z^V = \int_{\frac{x}{l}}^\infty e^{-t} \int \]

It follows that \[ \psi^V = I \]

\[ \psi = \int p \int_{z-x}^\infty e^{-t} \int \psi = \int p \int_{z-x}^\infty e^{-t} \psi \int_{0}^\infty z^I \]
Let $\mathcal{Z}$ be a probability space and $X$ a random variable. Then

\[ \Pr\left( \exists \varepsilon > 0 \text{ s.t. } \left( (X)^{\varepsilon} - X \right) \in \mathcal{E} \right) \leq \frac{\operatorname{var}(X)^{\varepsilon}}{\varepsilon^2} \]
Monte Carlo Simulation

Let $X$ be a random variable with expectation $\mu$ and variance $\sigma^2$. $X$ is associated with an experiment $E$. Let $X_k$ be the random variable associated with the $k$-th repetition of $E$. The repetitions are considered independent, therefore $E(X_k) = E(X) = \mu$, $\text{var}(X_k) = \text{var}(X) = \sigma^2$.

Then 

$$\frac{\sum_{i=1}^{N} X_i}{\sqrt{N}} = (\mu) \text{ var} \iff \eta = (\mu) \text{ var}$$

Let $Y = \sum_{i=1}^{N} X_i / \sqrt{N} = \mu$, \text{var} \iff \eta \sim N(\mu, \sigma^2)$.

Therefore, the independent $X$ associated with the $k$-th repetition of $E$. The repetitions are considered independent with an experiment $E$. Let $X$ be the random variable $X$ is associated with a random variable with expectation $\mu$ and variance $\sigma^2$.
Therefore, from Chebyshev inequality (applied to $X$):

$$P \left( \left| \bar{X} - \mu \right| \geq 2 \sigma \right) \leq \frac{1}{4}$$

In other words,

$$\frac{\sigma}{\bar{X}} \geq \frac{1}{N^{1/2}}$$

If for $\epsilon > 0$, we want that the uncertainty of the estimation $\bar{X}$ is smaller than $\epsilon$, then it is sufficient to choose

$$\epsilon \leq \left| \bar{X} - \frac{1}{N} \right|$$

Therefore, from Chebyshev inequality (applied to $X$):
Monte Carlo Simulation for one underlying variable

Consider an European-style derivative that pays off \( f_T \) at time \( T \). If the derivative price depends only on one underlying stochastic variable (not an interest rate) the value of the derivative at time zero is

\[
(\mathbb{E}^f)_{L_t} e^{-rT} f(T) = f
\]

A large number \( N \) of possible payoffs is generated and \( \mathbb{E}^f \) is estimated as the arithmetic average of them.

Monte Carlo Simulation for one underlying variable