6th Lecture

17th November 2003

Options
Basics

Option contract grants the owner the right but not the obligation to take some action (see previous lectures)

Call - right to purchase underlying at strike price at (european) / until (american) specified maturity

Put - same as call but „sell“ instead of „buy“

Value of option is called „option‘s premium“

Options: at the money, in the money, out of the money
Performance Profile - long call

\[ V_T = \max(S_T - K, 0) \]
Performance Profile - short call
Performance Profile - long put

\[ V_T = \max(K - S_T, 0) \]
Performance Profile - short put
Positions in the underlying instrument

Long underlying

Short underlying

$S_T$
Trading strategies: underlying & call option

Long underl. & short call „covered call“

Short underlying & long call „writing cov. call“
Trading Strategies: underlying & put option

- **Long underl. & long put**
- **Short underl. & short put**
Bull Spreads with calls

Payoff

<table>
<thead>
<tr>
<th>$S_T$</th>
<th>Long call</th>
<th>Short call</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_T &gt; X_2$</td>
<td>$S_T - X_1$</td>
<td>$X_2 - S_T$</td>
<td>$X_2 - X_1$</td>
</tr>
<tr>
<td>$X_1 &lt; S_T &lt; X_2$</td>
<td>$S_T - X_1$</td>
<td>0</td>
<td>$S_T - X_1$</td>
</tr>
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<td>0</td>
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<td>0</td>
</tr>
</tbody>
</table>
Bull spreads with puts

Short put @ X2

Long put @ X1
Bear spreads (with calls)

<table>
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<tr>
<th>$S_T$</th>
<th>long call</th>
<th>short call</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_T &gt; X_2$</td>
<td>$S_T - X_2$</td>
<td>$X_1 - S_T$</td>
<td>$-(X_2 - X_1)$</td>
</tr>
<tr>
<td>$X_1 &lt; S_T &lt; X_2$</td>
<td>0</td>
<td>$X_1 - S_T$</td>
<td>$-(S_T - X_1)$</td>
</tr>
<tr>
<td>$S_T &lt; X_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Bear spreads (with puts)

payoff

<table>
<thead>
<tr>
<th>$S_T$</th>
<th>Long Put</th>
<th>Short Put</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_T&lt;X_1$</td>
<td>$S_T-X_2$</td>
<td>$-(S_T-X_1)$</td>
<td>$X_1-X_2$</td>
</tr>
<tr>
<td>$X_1&lt;S_T&lt;X_2$</td>
<td>$S_T-X_2$</td>
<td>$0$</td>
<td>$S_T-X_2$</td>
</tr>
<tr>
<td>$X_2&lt;S_T$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>
Butterfly spreads (with calls)

### Payoff

<table>
<thead>
<tr>
<th>$S_T$</th>
<th>long call @ X1</th>
<th>long call @ X3</th>
<th>short call @ X2</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_T &lt; X1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$X_1 &lt; S_T &lt; X_2$</td>
<td>$S_T - X_1$</td>
<td>0</td>
<td>0</td>
<td>$S_T - X_1$</td>
</tr>
<tr>
<td>$X_2 &lt; S_T &lt; X_3$</td>
<td>$S_T - X_1$</td>
<td>0</td>
<td>$-2(S_T - X_2)$</td>
<td>$2X_2 - X_1$</td>
</tr>
<tr>
<td>$X_3 &lt; S_T$</td>
<td>$S_T - X_1$</td>
<td>$S_T - X_3$</td>
<td>$-2(S_T - X_2)$</td>
<td>$2X_2 - X_1 - X_3$</td>
</tr>
</tbody>
</table>
Calendar Spreads

Until now, all options had the same expiration date

Calendar spreads combine options with the same strike but different expiry dates

Buy call & sell call
Combinations: straddles, strips, straps...

Straddle: buying a call and a put with same strike price and expiration date

Strips: long one call and two puts with same strike and expiration

Straps: long two calls and one put with same strike and expiration
Combinations: strangles

Position in one call and one put with same expiration date but different strike prices
## Option Pricing - Put / Call Parity Theorem

<table>
<thead>
<tr>
<th>Position</th>
<th>Value at expiration</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$S_T&lt;K$</td>
</tr>
<tr>
<td>Long call (-C)</td>
<td>0</td>
</tr>
<tr>
<td>Short Put (P)</td>
<td>$-(K-S_T)$</td>
</tr>
<tr>
<td>Short stock ($S$)</td>
<td>$-S_T$</td>
</tr>
<tr>
<td>Loan -$K/(1+i)^T$</td>
<td>$K$</td>
</tr>
<tr>
<td>Loan -D (to pay divs.)</td>
<td>0</td>
</tr>
<tr>
<td>-C+P+$S_0-K/(1+i)^T$-D</td>
<td>0</td>
</tr>
</tbody>
</table>
Boundaries of a call option

Call range

C

K

S_T
Option Pricing fundamentals - the binomial model

\begin{align*}
(1+r_u)S &= uS \\
(1+r_d)S &= dS \\
S_0 &\quad \text{Probability} \quad q \\
T=0 &\quad 1-q \\
T=1 &
\end{align*}
Pricing by portfolio replication

Portfolio comprising n shares of underlying and B dollars in risk free debt paying interest i=r-1 replicates call at time T=1

\[ C_u = \max(0, uS - K) \]
\[ C_d = \max(0, dS - K) \]

\[ nuS + Br = C_u \]
\[ ndS + Br = C_d \]

\[ n = \frac{C_u \boxplus C_d}{S(u \boxplus d)} \]
\[ B = \frac{1}{r} \left[ \frac{uC_d \boxplus dC_u}{u \boxplus d} \right] \]

But \( C = nS + B \), so by substituting we obtain

\[ C = \frac{1}{r} \left\{ \left[ \frac{r \boxplus d}{u \boxplus d} \right] C_u + \left[ \frac{u \boxplus r}{u \boxplus d} \right] C_d \right\} \]
Risk neutral probabilities - two period binomial case

\[ p = \frac{r \square d}{u \square d}; (1 \square p) = \frac{u \square r}{u \square d} \]

So C can be written as

\[ C = \frac{1}{\rho} [pC_u + (1 \square p)C_d] \]

Important statement: traders can differ on the probabilities of underlying but still agree on the value of the call option
Two period model as a recombining lattice girder

\[ C_{uu} \]
\[ C \]
\[ C_{ud} = C_{du} \]
\[ C_d \]
\[ C_{dd} \]

**Homework:** Prove using the same algorithm that:

\[ C = \frac{1}{r^2} \left[ p^2 C_{uu} + 2p(1 \Box p) C_{ud} + (1 \Box p)^2 C_{dd} \right] \]

And by extrapolation to \( n \) time steps:

\[ C = \frac{1}{r^n} \sum_{j=0}^{n} \frac{n!}{j!(n \Box j)!} p^j (1 \Box p)^{n-j} \max(0, u^j d^{n-j} S \Box K) \]
Estimating u,d and r

Multiplicative binomial converges to a normal in the variable \( \ln(\frac{S_T}{S}) \)

The characteristics of this probability distribution are given by the first two moments:

\[
\begin{align*}
\mathbb{E}_T &= E[\ln(\frac{S_T}{S})] \\
\mathbb{E}^2_T &= E[(\ln(\frac{S_T}{S}) - \mathbb{E}_T)^2]
\end{align*}
\]

For the binomial to converge to the log-normal, then u,d and q must be chosen such that the mean and variance of the binomial model converge to the mean and variance of the log-normal

To accomplish that, select:

\[
u = e^{\mathbb{E} \sqrt{T/n}}; \quad d = 1/u; \quad q = \frac{1}{2} + \frac{1}{2} \bigg( \mathbb{E} \bigg) \sqrt{\frac{T}{n}}
\]
The Black - Scholes - Merton Framework

Ito’s lemma: Suppose stochastic variable $x$ follows the Ito process:

$$dx = a(x,t)dt + b(x,t)dz,$$

$z$ is a Wiener process, variable $x$ has a drift rate $a$ and variance rate $b^2$, then a function $G$ follows the process:

$$dG = \left( \frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b dz$$

So applying this for the process of stock prices:

$$dS = \left[ \frac{\partial G}{\partial S} S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} S^2 \right] dt + \frac{\partial G}{\partial S} dz$$
Homework 2

Apply Ito’s lemma to $\ln S$ and show that

$$\ln S_T = \ln S + \left( \frac{\sigma^2}{2} \right) (T-t), \sqrt{T-t}$$

Implying that $\ln S_T$ has a lognormal distribution

$$\ln S_T = \ln \left[ \ln S + \left( \frac{\sigma^2}{2} \right) (T-t), \sqrt{T-t} \right]$$
Assumptions of the Black Scholes option pricing model

• The stock price follows a log normal distribution parametrized as in the previous slide

• The short selling of securities with full use of proceeds is permitted

• No transaction costs or taxes. All securities are perfectly divisible

• There are no dividends during the life of the derivative

• There are no riskless arbitrage opportunities

• Security trading is continuous

• The risk free rate $r$ is constant and the same for all maturities (flat term structure)
The Black Scholes Differential Equation

Chose a portfolio of 1 short call option \(f\) and \(\frac{\partial f}{\partial S}\) shares

\[
\begin{align*}
\frac{\partial}{\partial t} f &= \frac{\partial f}{\partial S} \frac{\partial}{\partial S} S \\
\frac{\partial}{\partial t} f &= \frac{\partial f}{\partial S} \frac{\partial}{\partial S} f + \frac{\partial f}{\partial S} \frac{\partial}{\partial S} S
\end{align*}
\]

Applying Ito’s lemma,

\[
\begin{align*}
\frac{\partial}{\partial t} f &= \left( \frac{\partial f}{\partial t} \frac{\partial}{\partial t} f + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \frac{\partial^2}{\partial S^2} f \right) \Delta t
\end{align*}
\]

Due to no arbitrage,

\[
\begin{align*}
\frac{\partial}{\partial t} f &= r \frac{\partial}{\partial t} f \Delta t
\end{align*}
\]

Substituting we obtain:

\[
\left( \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \frac{\partial^2}{\partial S^2} f \right) \Delta t = r \left( f \frac{\partial f}{\partial S} \right) \Delta t
\]
The fundamental PDE

FPDE: \[
\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} = rf
\]

Boundary conditions:

European call option: \( f = \max(S-X,0) \) @ \( t=T \)

European put option: \( f = \max(X-S,0) \) @ \( t=T \)
Sketch of Solution for a European Call Option

Transform:

\[
\begin{align*}
x &= \ln(S/X) \\
\Box &= f/X \\
\Box &= \frac{1}{2} \Box^2 (T \Box t)
\end{align*}
\]

We get:

\[
\frac{\partial \Box}{\partial \Box} = \frac{\partial^2 \Box}{\partial x^2} + (k \Box 1) \frac{\partial \Box}{\partial x} \Box k \Box; \\
k &= \frac{r}{\frac{1}{2} \Box^2}
\]

Note: This is now a forward equation in
Sketch of a solution for a European Call option (cont‘d)

Now by substituting:

\[ u = e^{\frac{1}{2}x + \frac{1}{4}k} \]

And by setting:

\[ u = \frac{1}{2} (k + 1); \quad u = \frac{1}{4} (k + 1)^2 \]

Gives:

\[ \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}; \quad u < x < \; ; u > 0 \]

This is the famous heat diffusion equation, with the following boundary condition:

Now, for a European call the final payoff at maturity is max[S-X,0] so after the first transformation: f(x,0)=max(e^x-1,0) and then,

\[ u(x,0) = \max[e^{\frac{1}{2}(k+1)x}, e^{\frac{1}{2}(k[-1])x}, 0] \]

And the final solution is:

\[ u(x,\Box) = \frac{1}{2\sqrt{\Box}} \int_{\Box}^{\Box} u(s,0) \exp[\frac{1}{4}(x - s)^2 / \Box] ds \]
The Black Scholes solution for a European Call

\[ f(S, t) = SN(d_1) \times Xe^{r(T-t)}N(d_2) \]

Where,

\[ d_1 = \left[ \ln(S/X) + (r + \frac{1}{2}\sigma^2)(T-t) \right] / \sigma \sqrt{T-t} \]

\[ d_2 = d_1 \times \sqrt{T-t} \]

\[ N(z) = \frac{1}{\sqrt{2\pi}} \int \exp(-\frac{1}{2}y^2)dy \]
Synopsis of risk neutral valuation using the equivalent martingale property of asset processes

Model: deterministic \( r \), \( \mathbb{Q} \)

Bond price: \( B_t = \exp(rt) \); stock Price: \( S_t = S_0 \exp \left( \int_0^t \mu dt + \int_0^t \sigma dW_t \right) \)

Find the replicating strategy:

1. Probability measure \( Q \) which converts \( S_t \) into a martingale (apply Girsanov)
2. Form the process \( E_t = E_Q(X_t/F_t) \)
3. Find a previsible process \( _- \), such that \( dE_t = _- dS_t \)

\[
 f_t = B_t E_Q \left( B_F \frac{X}{F_t} \right) = e^{r(T-t)} E_Q \left( \frac{X}{F_t} \right) 
\]

Where \( q \) is the martingale measure for the discounted stock \( B_t^{-1} S_t \)

Detailed derivation in Karatsas, Bingham & Kiessl, Rebonato, Baxter & Rennie
Homework (again!!!) uh!!!

Apply the fundamental PDE and the derivation of the closed form solution for finding the price of

an interest rate forward
an FX forward
An equity forward contract

good luck and be careful with the boundary conditions!
Implied Volatility „smiles“

At the money

Strike Price

Implied Vol
Risk management of options portfolios via hedge parameters
See Chapter 4 from Adritti - Derivatives
A brief dictionary of exotic options

See Nelken, Chapter1