Lecture 8

VAR – Credit Risk
Fundamental Paradigms of Credit Risk Quantification

• **EL** – (Expected loss) Cost of doing business
  
  – Expected default as a percentage of notional within a given period
  – Seen as “insurance premium”
  – Unless default free (U.S., German governments) > 0
  – Depends on default probability (therefore rating class), engagement tenor, expected exposure (at eventual default) and loss given default (assumptions on recovery)
  – \( EL = EDF \times EE \times LGD \)
Fundamental Paradigms of Credit Risk Quantification (cont’d)

• UL (unexpected loss)

- volatility around expected loss
- determined by the probability distribution of expected loss
- unlike market risk, typically not symmetrical and centered around EL
- rarely expressed analytically and very difficult to parametrise
- extremely difficult to back test
- Generically, UL=$\sigma^2(Distr(EL))$
Credit rating systems

• Systems account for both qualitative and quantitative information
• Experience plays a very important role
• We’ll describe the principal rating schemes employed by the two best known rating agencies: Moody’s and Standard&Poor’s as well as a typical internal rating scheme employed by banks
Rating Agencies

- Moody’s (1909) and S&P (1916) are the most respected private and public debt rating agencies.
- S&P is the first agency to have rated mortgage backed bonds (1975), mutual funds (1983) and asset backed securities (1985).
- A credit rating is “an opinion on the future ability and legal obligation of an issuer to make timely payments of principal and interest on a specific fixed income security” - Moody’s.
- Ratings reflect primarily default probabilities and to a second degree severity of loss in case of default (LGD).
- Issue specific credit rating – rating agency distinguishes between long term and short term debt (CPs, CDs & putable bonds).
Moody’s rating analysis of an industrial company

- Rating process includes quantitative, qualitative and legal analyses
- Quantitative analysis is mainly based on the firm’s financial reports
- Qualitative analysis is concerned with management quality, reviews the firm’s competitive situation as well as an assessment of expected growth within the firm’s industry plus the vulnerability to technological changes, regulatory changes, labor relations, etc.)
S&P’S Debt rating process

Ratings categories – see attachments

- Request Rating
  - Assign analytical team
  - Conduct basic research
- Meet issuer
- Rating Committee Meeting
- Issue Rating

Appeals Process
Surveillance
Bank Internal Rating Systems

- Goal is to generate accurate and consistent risk ratings, yet also to allow professional judgment to significantly influence a rating where appropriate.
- A mapping to external agency ratings over borrower EDFs is as good as mandatory.

| Illustrative Risk Rating System |
|---------------------------------
<table>
<thead>
<tr>
<th>Risk</th>
<th>RR</th>
<th>Corresponding Probable S&amp;P or Moody's Ratings</th>
</tr>
</thead>
<tbody>
<tr>
<td>sovereign</td>
<td>0</td>
<td>Not applicable</td>
</tr>
<tr>
<td>Low</td>
<td>1</td>
<td>AAA</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>AA</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>A</td>
</tr>
<tr>
<td>Average</td>
<td>4</td>
<td>BBB+/BBB</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>BBB-</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>BB+/BB</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>BB-</td>
</tr>
<tr>
<td>High</td>
<td>8</td>
<td>B+/B</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>B-</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>CCC+/CCC</td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>CC-</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>In default</td>
</tr>
</tbody>
</table>
Credit VAR - the credit migration approach

• We talked about spreads - they include information on liquidity, credit up or downgrade and default.
• The problem of course is separating these components as they are heavily intercorrelated (market participants often anticipate credit events ahead of their actual happening - therefore they are a priori impacting the market prices, therefore the spreads)
• Simply adding spread risk to downgrade or default risk leads to double counting
• Default is a special (limiting?) case of downgrade - credit quality has deteriorated to the point of the obligor’s inability to meet his contractual obligations
• Therefore one can argue that a credit VAR model should be based on quantifying migration risk
Credit risk models (based on credit migration)

- Credit Metrics (J.P. Morgan 1997) - models the full forward distribution of the values of any bond or loan portfolio (typically one year forward) where changes in value relate only to credit migration.

- Wilson model (Tom Wilson 1997) - allows default probabilities to vary with the credit cycle; default probabilities are functions of macrovariables.

- KMV model - assesses default and migration probabilities based on EDFs, themselves based on "distance from default" - Merton's approach (options pricing analogy) - a variant is the Schmid-Zagst Model (also pricing).

- CreditRisk+ - developed by CSFP bases assessment of default and migration risk on actuarial assumptions (Poisson distribution).

- Reduced form approach - basis for credit derivatives pricing models credit spreads as a stochastic process and makes deterministic assumptions on recoveries.
The Creditmetrics Framework

• Challenges to the market VAR framework:
  • returns are definitely not normally distributed (limited upside against large downside in case of default or downgrading)
  • measuring correlations is far more tricky than in the case of market risk
• In most cases the percentage VAR cannot be estimated from the distribution’s moments
• Creditmetrics estimates correlations from joint probability distributions of equity returns
• No provision for market risk: forward values and exposures are deterministic forward curves; the only uncertainty is credit migration, therefore no correlation between market moves and default likelihood
Creditmetrics framework: 4 building blocs

- BB 1: VAR due to Credit
  - Credit Rating
  - Seniority
  - Credit spreads
  - Standard deviation of Value due to Credit
  - Quality changes for a single exposure
- BB 2: Portfolio VAR due to Credit
- BB 3: Correlations
  - Rating series
  - Equity series
  - Models (e.g., correlations)
- BB 4: Exposures
  - User Portf.
  - Market Vols
  - Exposure Distributions
  - Rating migr. Probabilities
  - Recovery rate in default
  - PV Bond revals
BB1- Step 1 – Specify transition matrix

Rating at Year End (%)

<table>
<thead>
<tr>
<th>Initial rating</th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>B</th>
<th>CCC</th>
<th>Default</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>90,81</td>
<td>8,33</td>
<td>0,68</td>
<td>0,06</td>
<td>0,12</td>
<td>0,00</td>
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<td>AA</td>
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<td>90,65</td>
<td>7,79</td>
<td>0,64</td>
<td>0,06</td>
<td>0,14</td>
<td>0,02</td>
<td>0,00</td>
</tr>
<tr>
<td>A</td>
<td>0,09</td>
<td>2,27</td>
<td>91,05</td>
<td>5,52</td>
<td>0,74</td>
<td>0,26</td>
<td>0,01</td>
<td>0,06</td>
</tr>
<tr>
<td>BBB</td>
<td>0,02</td>
<td>0,33</td>
<td>5,95</td>
<td>86,93</td>
<td>5,30</td>
<td>1,17</td>
<td>1,12</td>
<td>0,18</td>
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<tr>
<td>BB</td>
<td>0,03</td>
<td>0,14</td>
<td>0,67</td>
<td>7,73</td>
<td>80,53</td>
<td>8,84</td>
<td>1,00</td>
<td>1,06</td>
</tr>
<tr>
<td>B</td>
<td>0,00</td>
<td>0,11</td>
<td>0,24</td>
<td>0,43</td>
<td>6,48</td>
<td>83,46</td>
<td>4,07</td>
<td>5,20</td>
</tr>
<tr>
<td>CCC</td>
<td>0,22</td>
<td>0,00</td>
<td>0,22</td>
<td>1,30</td>
<td>2,38</td>
<td>11,24</td>
<td>64,86</td>
<td>19,79</td>
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</table>
BB1-Step 1a: Produce long term average cumulative default rates

<table>
<thead>
<tr>
<th>Term (years)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>7</th>
<th>10</th>
<th>15</th>
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</thead>
<tbody>
<tr>
<td>AAA</td>
<td>0.00</td>
<td>0.00</td>
<td>0.07</td>
<td>0.15</td>
<td>0.24</td>
<td>0.66</td>
<td>1.40</td>
<td>1.40</td>
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<tr>
<td>AA</td>
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<td>0.02</td>
<td>0.12</td>
<td>0.25</td>
<td>0.43</td>
<td>0.89</td>
<td>1.29</td>
<td>1.48</td>
</tr>
<tr>
<td>A</td>
<td>0.06</td>
<td>0.16</td>
<td>0.27</td>
<td>0.44</td>
<td>0.67</td>
<td>1.12</td>
<td>2.17</td>
<td>3.00</td>
</tr>
<tr>
<td>BBB</td>
<td>0.18</td>
<td>0.44</td>
<td>0.72</td>
<td>1.27</td>
<td>1.78</td>
<td>2.99</td>
<td>4.34</td>
<td>4.70</td>
</tr>
<tr>
<td>BB</td>
<td>1.06</td>
<td>3.48</td>
<td>6.12</td>
<td>8.68</td>
<td>10.97</td>
<td>14.46</td>
<td>17.73</td>
<td>19.91</td>
</tr>
<tr>
<td>B</td>
<td>5.20</td>
<td>11.00</td>
<td>15.95</td>
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<td>30.65</td>
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<tr>
<td>CCC</td>
<td>19.79</td>
<td>26.92</td>
<td>31.63</td>
<td>35.97</td>
<td>40.15</td>
<td>42.64</td>
<td>45.10</td>
<td>45.10</td>
</tr>
</tbody>
</table>
BB1 - Steps 2, 3 & 4

• Step2: Specify the credit risk horizon
  • Often one year

• Step 3: Specify the forward pricing model
  – Term structure of zero rates
  – Term structure of credit spreads
  – Account for correlations between the former

• Step4: derive the forward distribution of the changes in portfolio value
BB2 – Credit diversification analysis

- Creditmetrics implements a Monte Carlo simulation to generate a full distribution of the portfolio values at the credit horizon of one year.
- Fundamental stepwise procedure:
  - Derivation of asset return threshold for each rating category.
  - Estimation of the correlations between each pair of obligor’s asset returns.
  - Generation of asset return scenarios according to their joint normal distribution. A standard technique used to generate correlated normal variables is the Cholesky decomposition. Each scenario is characterized by n standardized asset returns, one for each of the n obligors in the portfolio.
  - For each scenario and for each obligor, the standardized asset return is mapped into the corresponding rating, according to the threshold levels derived in Step 1.
  - Repeat the procedure for a large number of times and plot the distribution of portfolio values => derive the percentile VAR.
BB2- Credit VAR and capital charge

- Economic capital (like in market risk) represents the capital cushion that an institution uses to absorb unexpected losses (e.g. resulting from credit migration / defaults)
- $P(c) =$ value of the portfolio in the worst case scenario at the $(1-c)\%$ confidence level
- $FV =$ forward value of the portfolio = $V_0(1+PR)$
- $V_0 =$ current marked to market value of the portfolio
- $PR =$ Promised portfolio return
- $EV =$ Expected value of the portfolio = $V_0(1+ER)$
- $ER =$ Expected return on the portfolio
- $EL =$ Expected Loss = $FV-EV$ (not a contributor to risk capital)

- Capital charge = $EV-P(c)$
BB2- Credit VAR and economic capital charge
BB3 – Estimation of asset correlations

- In Creditmetrics, default correlations are inferred from asset returns correlations, which in turn are proxied by equity returns correlations.
- For a large portfolio of loans and bonds this requires the computation of a very large correlation matrix (for each pair of borrowers).
- In market risk we reduced the dimensionality by using a limited eigenvalue driven dimension. In creditmetrics this is being dealt with by multifactor analysis.
- Thus each borrower is mapped to the countries/ regions/ industries/ other significant clusters likely to determine/ influence performance.
- User specifies industry/ country weights as well as „firm specific risk“ – idiosyncratic to each borrower and therefore not correlated to any other obligor or index.
BB4- Exposures

• In Creditmetrics somewhat misleading since the approach assumes that market risk factors are constant
• Exposure is the term Creditmetrics gives to the use of the forward pricing model applied to each credit rating
• For fixed income securities based products (such as bonds, loans, receivables, lending commitments, LCs, etc.) exposure relates to the future cash flows at risk beyond the one year time horizon
• For derivatives (swaps, FRAs, etc.) exposure is conditional on future interest rates
• For swaps the exposure calculations are opposite to the ones for market risk: if swap is in the money, exposure is positive and negative if the swap is out of the money
Exposure for an interest rate swap – pay floating, receive fixed

Note: bank is at risk only if exposure is positive
Creditmetrics assumptions on swap exposures

- Creditmetrics assumes as given a swap's average exposure, supposedly to be derived from external models.
- Since interest rates are deterministic, the calculation of the forward price distribution relies on an ad hoc procedure:

  - Value of swap in one year, rating \( R \) = Forward risk free value in one year – Expected loss in years one to maturity for the given rating \( R \), where:
  - Expected loss in years one to maturity for the given rating \( R \) = Average exposure from year one to maturity \( X \) Probability of default in years one through maturity for the given rating \( R \) \( X \) (1-recovery rate)

  - The forward risk free value of the swap is calculated by discounting the future net cash flows of the swap, based on the forward curve, using the risk free forward yield curve (gov't bonds)
  - This value is the same for all ratings
  - Probability of default from year one through maturity can be either taken from Moody's or S&P or be derived from the transition matrix
  - Recovery rate is taken from statistical analyses of past behaviors (also provided by rating agencies)
Conditional Transition Probabilities:

CREDITPORTFOLIOVIEW

- CREDITPORTFOLIOVIEW is a multifactor model used to simulate the joint conditional distribution of default and migration probabilities for various rating groups in different industries and for each country, conditional on macroeconomic factors.
- Based on the observation that credit cycles follow (with a lag) business cycles.
- Akin to the ISLM econometric model (variables are unemployment, GDP growth, long term interest rate level, FX rates, Consumption (private sector & gov't expenditures), savings, investment.
- Default probabilities are modeled as logit functions, whereby the independent variable is a country specific index that depends upon current and lagged macroeconomic variables: \( \text{Prob}_{j,t} = \frac{1}{1 + \exp(-Y_{j,t})} \)
- \( \text{Prob}_{j,t} \) is the conditional probability of default in period \( t \), for speculative-grade obligors in country/industry \( j \).
- \( Y_{j,t} \) is the country index value derived from a multifactor model.
To derive the conditional transition matrix, we employ (the unconditional Markovian) transition matrix based on Moody’s or S&P’s historical data $\Phi M$

Transitional probabilities are „unconditional“ – means they are historical averages over periods covering several business cycles (in practice > 20 years)

So

$$\frac{SDP_t}{\Phi SDP} > 1 \quad \text{In a recession}$$

$$\frac{SDP_t}{\Phi SDP} < 1 \quad \text{In an economic expansion}$$

SDP$_t$ simulated default probability for a speculative grade obligor;

$\Phi SDP$ unconditional default probability based on historical averages
CREDITPORTFOLIOVIEW (cont’d)

• Transition matrix $M$ conditional on the state of the economy: $M_t = M(\text{Prob}_{j,t}/\Phi \text{ SDP})$

  • Adjustment consists in shifting the probability mass into downgraded and defaulted states when the ratio $\text{Prob}_{j,t}/\Phi \text{ SDP} > 1$, and vice versa

  • Multiperiod transition matrixes are generated over simulations over the full time horizon: $t=1,\ldots,T$, yielding $M_T = \prod_{t=1,\ldots,T} M(\text{Prob}_{j,t}/\Phi \text{ SDP})$

  • The same can be obtained also via Monte Carlo simulations which converge to the distribution of conditional default probabilities

  • KMV (as will be seen later) accomplishes the same by simulating conditional default probabilities (and migrations) based on microeconomic factors
Default Prediction – the econometric Model

• Start with the logit function:
  \[ \text{Prob}_{j,t} = (1 + \exp(-Y_{j,t}))^{-1} \]
• Note the logit functions ensure that probabilities are btw. 0 and 1
• Specify the multifactor model (index):
  \[
  Y_{j,t} = \beta_{j,0} + \beta_{j,1} X_{j,1,t} + \ldots + \beta_{j,m} X_{j,m,t} + \nu_{j,t}
  \]

  \[ Y_{j,t} \text{ index value period } t, \text{ country/industry } j \]
  \[ \beta_{j,m} \text{ are coefficients to be estimated} \]
  \[ X_{j,m,t} \text{ are period } t \text{ values of the macroeconomic variables} \]
  \[ \nu_{j,t} \text{ is the error term – independent of } X \text{ and normally distributed} \]

Each macroeconomic variable is assumed to follow a univariate, autoregressive model of order 2 (AR2):

\[
X_{j,i,t} = \gamma_{j,i,0} + \gamma_{j,i,1} X_{j,i,t-1} + \gamma_{j,i,2} X_{j,i,t-2} + e_{j,i,t}
\]

\[ e_{j,i,t} \approx N(0, \sigma_{e_{j,i}}); e_t \approx N(0, \Sigma_e) \]

\[ e_t \text{ vector of stacked error terms } e_{j,i,t} \text{ of the } jXi \text{ AR2 equations} \]

\[ \Sigma_e \text{ is the } (jxi)(jxi) \text{ covariance matrix of the error terms } e_t \]
The econometric model (cont’d)

• Now, to calibrate the default probability model, we need to solve the system:

\[
Prob_{j,t} = \frac{1}{1 + e^{-\gamma_j t}}
\]

\[
Y_{j,t} = \beta_{j,0} + \beta_{j,1} X_{j,1,t} + \ldots + \beta_{j,m} X_{j,m,t} + v_{j,t}
\]

\[
X_{j,i,t} = \gamma_{j,0} + \gamma_{j,1} X_{j,i-1,t} + \gamma_{j,2} X_{j,i-2,t} + e_{j,i,t}
\]

Where the vector of innovations \( E_t \) is

\[
E_t = \begin{bmatrix} v_t \\ e_t \end{bmatrix} \approx N(0, \Sigma)
\]

\[
\Sigma = \begin{bmatrix} \Sigma_{v,v} & \Sigma_{v,e} \\ \Sigma_{e,v} & \Sigma_{e,e} \end{bmatrix}
\]

\( \Sigma_{v,e} \) and \( \Sigma_{e,v} \) denote the cross correlation matrices.

Once the system is calibrated one can simulate the distribution of defaults by cholesky decomposing \( \Sigma = AA' \)
Merton’s model (as applied in CreditMetrics)

- The firm’s assets are assumed to follow a geometric brownian motion:

\[ V_t = V_0 \exp \left\{ \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma \sqrt{t} Z_t \right\} \]

\[ Z_t \approx N(0,1) \]

\( \mu \) & \( \sigma^2 \) mean and variance of the instantaneous rate of return on the firm’s assets; \( V_t \) is log-normally distributed; \( E(V_t) = V_0 \exp(\mu t) \); The dynamics of \( V(t) \) are described by \( dV_t/V_t = \mu dt + \sigma dW_t \); \( W_t \) standard brownian motion; \( \sqrt{t}Z_t = W_t - W_0 \)

\( \sim N(0,t) \)

Firm is only capitalized by equity \( S_t \) and one zero bond \( B_t \) with face value \( F \)

Value of firm’s assets, \( V_t = S_t + B_t \); \( \text{Prob(Def)} = \text{Prob}[V_t \leq V_{\text{Def}}] \); Therefore default occurs when:

\[
\text{Prob}(\text{Def}) = \text{Prob} \left[ \frac{\ln \left( \frac{V_{\text{Def}}}{V_0} \right) - (\mu - \frac{\sigma^2}{2}) t}{\sigma \sqrt{t}} > Z_t \right] = N(-d_2)
\]

\[
r = \frac{\ln \left( \frac{V_t}{V_0} \right) - (\mu - \frac{\sigma^2}{2}) t}{\sigma \sqrt{t}} \approx N(0,1)
\]
Merton’s model (cont’d)

- So $Z_{CCC}$ is simply the threshold point in the standard normal distribution $N(0,1)$ corresponding to a cumulative probability $\text{Prob}_{\text{Def}}$
- $V_{\text{DEF}}$ corresponds to $Z_{CCC} = -d_2$,

\[
d_2 = \frac{\ln\left(\frac{V_0}{V_{\text{DEF}}}\right) + (\mu - \frac{\sigma^2}{2})t}{\sigma \sqrt{t}}
\]

$d_2$ is called „distance to default“

Denote $r_{BB}$ and $r_A$ instantaneous RoR of assets rated BB and a respectively; $\rho$ the instantaneous correlation btw. $r_A$ and $r_{BB}$; the normalized log-returns of both assets follow a joint normal distribution:

\[
f(r_{BB}, r_A; \rho) = \frac{1}{2\pi \sqrt{1-\rho^2}} \exp\left\{ \frac{-1}{2(1-\rho^2)} \left[ r_{BB}^2 - 2 \rho r_{BB} r_A + r_A^2 \right]\right\}
\]

The default correlation is:

\[
corr / DEF_A, DEF_{BB} = \frac{\text{Pr ob}(DEF_A, DEF_{BB}) - \text{Pr ob}(DEF_A) \text{Pr ob}(DEF_{BB})}{\sqrt{\text{Pr ob}(DEF_A)[1 - \text{Pr ob}(DEF_A) \text{Pr ob}(DEF_{BB})][1 - \text{Pr ob}(DEF_{BB})]}}
\]
Merton’s Model (cont’d)

• So the joint probability of both obligors defaulting would be:

\[
\begin{align*}
\text{Prob}(DEF_A, DEF_{BB}) &= \text{Prob}[V_A \leq V_{DEF_A}, V_{BB} \leq V_{DEF_{BB}}] = N_2(-d_A^A, -d_{BB}^B, \rho) \\
\text{Prob}(DEF_A, DEF_{BB}) &= \text{Prob}[r_A \leq d_A^A, r_{BB} \leq d_{BB}^B] = N_2(-d_A^A, -d_{BB}^B, \rho)
\end{align*}
\]

\(N_2\) is the standard bivariate normal distribution with correlation \(\rho\) among the 2 variables.
The options approach to measuring default risk

• Weakness of the CreditMetrics approach: it relies on ratings transition probabilities that are based on average historical frequencies of defaults and credit migration

• Thus, accuracy of the CreditMetrics approach depends on two critical assumptions:
  – All firms within the same rating class have the same default rate and the same spread term structure even at different recovery rates
  – The actual default rate is equal to the historical average

• KMV have shown that historical average default rates and transition probabilities can differ significantly from the actuals

• Likewise, substantial differences can exist within the same rating class and the overlap in default probability ranges can be quite large (BBB and AA bonds can exhibit the same default probability)
Structural approach (option based)

- Economic value of default is presented as a put option on the value of the firm’s assets
- Firstly presented by Merton (1974); builds on the limited liability rule which allows shareholders to default on their obligations while surrendering the firm’s assets to various stakeholders
- The firm’s liabilities are seen as contingent claims issued against the firm’s assets, with the payoffs to the various debt holders completely specified by seniority and the other covenants (remember Miller-Modigliani?)
- Default occurs at debt maturity whenever the firm’s asset value falls short of debt value at that time
- The loss rate is endogenously determined and depends on the asset value, volatility and the risk free interest rate for the debt maturity
Merton’s structural model (1974)

- Assumes a simple capital structure with all debt represented by one zero coupon bond
- We will derive the loss rates endogenously, together with the default probability
- Risky asset $V$, equity $S$, one zero bond $B$ maturing at $T$ and face value (incl. Accrued interest) $F$
- Default risk on the loan to the firm is tantamount to the firm’s assets $V_T$ falling below the obligations to the debt holders $F$
- Credit risk exists as long as $\text{prob}(V_T < F) > 0$
- This naturally implies that at $t=0$, $B_0 < Fe^{-rT}; y_T > r_f$, where $\pi_T = y_T - r_f$ is the default spread which compensates the bond holder for taking the default risk
- In frictionless markets, no taxes, no bankruptcy, $V_0 \equiv S_0 + B_0$
Merton’s structural model (1974) (cont’d)

- Leverage ratio \( LR = \frac{Fe^{-rT}}{V_0} \)
- Volatility of the assets \( Ror = \sigma \)
- Assumptions: the loan is the only debt instrument; the only other source of financing is equity
- In this case the credit value equals the value of a put option on the firm’s assets \( V \), at the strike \( F \), maturing at \( T \)
- Thus the put option can convert the risky debt into a risk free debt eliminating the credit default risk
- Using Black Scholes (remember?), \( P_0 = -N(-d_1)V_0 + Fe^{-rT}N(-d_2) \)
  \( d_1 = \frac{\ln(V_0/F) + (r + 1/2 \sigma^2)T}{\sigma T^{1/2}} \); \( d_2 = d_1 - \sigma T^{1/2} \)
- Note that the cost of credit risk is a homogenous function of LR (means that it stays constant for a scale expansion of LR)
Merton’s structural model (1974) (cont’d)

- We can now derive the yield to maturity of the corporate discount bond \( y_T \)

\[
y_T = -\frac{\ln \frac{B_0}{F}}{T} = -\frac{\ln \frac{F e^{rT} - P_0}{F}}{T}
\]

\[
\pi_T = y_T - r = -\frac{1}{T} \ln \left( N(d_2) + \frac{V_0}{F e^{rT}} N(-d_1) \right)
\]

Notes: the default spread can be computed as a function of the leverage ratio, the asset’s volatility and the maturity

When \( r \) increases, \( \rho_T \) declines, as \( \frac{\partial \pi_T}{\partial r} < 0 \)
Probability of default, conditional expected recovery and default spread

- From the put value $P_0$, one can extract the loan's default probability.
- $N(d_2)$ is the probability that the firm's value at $T$ will be higher than $F$, hence the default probability is $1-N(d_2)=N(-d_2)$.
- $P_0$ is the premium the company pays on credit insurance.
- We can write: $P_0 = \frac{-N(-d_1)}{N(-d_2)}V_0 + Fe^{-rT}N(-d_2)$.
- $\operatorname{abs}\{\frac{N(-d_1)}{N(-d_2)}V_0\}$ is the expected discount recovery of the loan conditional of $V_T \leq F$; it represents the risk neutral expected payment to the bank in the case where the firm can't pay the full obligation at time $T$.
- $\frac{-N(-d_1)}{N(-d_2)}V_0 + Fe^{-rT}$ represents the expected shortfall (Present valued) conditional on the firm being bankrupt at time $T$.
- $N(-d_2)$ as seen is the probability of default.
From the put premium we can compute the expected loss in the event of default at time $T$: $EL_T$

$EL_T = \text{probability of default} \times \text{loss in case of default}$

$EL_T = N(-d_2)F - N(-d_1)V_0e^{rT} = F - N(d_2)V_0e^{rT} = F(1 - N(d_2) - N(-d_1)/LR)$

Expected payoff from the corporate debt at maturity is $F - EL_T$

The expected cost of default (expressed in yield) is:

$$
\frac{1}{T}\ln\left(\frac{F}{F - EL_T}\right) = \frac{1}{T}\ln\left(\frac{F(N(d_2) + N(-d_1)\frac{V_0}{Fe^{rT}})}{F}\right) = \pi_T
$$

Consistent with the result derived on page 35!
Credit risk as a function of Equity value

• We showed so far that we can model credit risk as a function of a firm's total assets, $V$
• If both the firm's debt and equity are traded it is easy to add the two market values to derive the firm's fair value of assets
• In most cases (definitely in Europe loans aren't traded, hence we can only observe equity prices)
• Question becomes if default risk can be hedged by trading shares and derivatives of the firm's equity
• Equity itself is a contingent claim on the firm's assets (remember Miller – Modigliani ?); $S=VN(d_1)-Fe^{-rTN(d_2)}$
• We can synthetically create a put by selling short $N(-d_1)$ units of the firm's assets, and buying $Fe^{-rTN(-d_2)}$ units of government bonds maturing at time $T$, with face value $F$
• If one sells short $N(-d_1)/N(-d_2)$ units of stock S, one effectively creates a short position in the firm's assets of $N(-d_1)$ units, since:

$$-\frac{N(-d_1)}{N(d_1)}S = -VN(-d_1) + Fe^{-rT}N(d_2)\frac{N(-d_1)}{N(d_1)}$$
Credit risk as a function of Equity value (cont’d)

• Thus if V is not directly observed, one can create a put option dynamically by shorting the appropriate number of shares.

• Remember that equity itself reflects the default risk, and seen as an option (contingent claim) its instantaneous volatility is: \( \sigma_S = \eta_{S,V} \sigma \), where \( \eta_{S,V} = N(d_1)V/S \) is the instantaneous elasticity of equity with respect to the firm’s value: \( \eta_{S,V} = (\frac{\partial S}{\partial V})(V/S) \) and \( \eta_{S,V} \geq 1 \).

• Note that since \( \sigma_S \) is stochastic (changing with V) we cannot apply the conventional Black Scholes model to value puts on S.

• In practice when dealing with long term options, the estimated \( \sigma_S \) estimated in the second bullet point doesn’t change widely from day to day, therefore practitioners use \( \sigma_S = \eta_{S,V} \sigma \) as an estimator of volatility (constant and deterministic) with the Black Scholes model, even when the underlying instrument doesn’t follow a stationary log-normal distribution.
The KMV Approach

• Derives the distribution of default probabilities based on the Merton (1974) model
• Probability of default is therefore a function of the firm‘s capital structure, volatility of asset returns and the current asset value
• EDFs are firm specific and can be mapped onto any rating system to derive the equivalent rating of the borrower
• EDFs are in KMV „cardinal rankings“ of the obligors as opposed to „ordinal rankings“ employed by rating agencies
• Contrary to CreditMetrics, KMV don‘t rely on transition probabilities (the‘re embedded in the EDFs)
• Assumes that the firm is financed by equity, $S_t$ and one zero bond maturing at $T$ with face value $F$ and current market value $B_t$
• The firm‘s balance sheet=> $V_t\equiv B_t(F)+S_t$; $V_t$ follows a geometric brownian motion
KMV – default occurs at maturity of debt if $V_T < F$

$E(V_T) = V_0 e^{\mu T}$

$V_T = V_0 \exp\{[\mu - \frac{\sigma^2}{2}] T + \sigma \sqrt{T} Z_T\}$

Asset Value

Probability of default

$V_T$

$V_0$

$F$

$T$

Time
KMV - Derivation of default probabilities

Three stages:

1. Estimation of the market value and volatility of the firm's assets
2. Calculation of the distance to default (an index measure of default risk)
3. Scaling the distance to default to actual probabilities (using a default database)
The Model by Schmid & Zagst [2000]
Modelling of the stochastic Processes under the Martingale measure Q

• Dynamic of term structure (non-defaultable Short Rate)

\[ dr(t) = [\theta_r(t) - a_r \cdot r(t)]dt + \sigma_r dW_r(t), \quad t \in [0, T^*], \quad a_r > 0, \sigma_r > 0 \]

• Dynamic of enterprise quality (Index of uncertainty)

\[ du(t) = [\theta_u - a_u \cdot u(t)]dt + \sigma_u \sqrt{u(t)} dW_u(t), \quad t \in [0, T^*], \quad \theta_u \geq 0, a_u > 0, \sigma_u > 0 \]

• Dynamic of spreads (Short Rate Spreads)

\[ ds(t) = [b_s \cdot u(t) - a_s \cdot s(t)]dt + \sigma_s \sqrt{s(t)} dW_s(t), \quad t \in [0, T^*], \quad b_s > 0, a_s > 0, \sigma_s > 0 \]

• The Wiener Processes \( W_r, W_u \) are \( W_s \) uncorrelated
The Model by Schmid & Zagst [2000]
Valuation of Non-Defaultable Bonds

Theoreme (Hull & White [1990], Hull [1997]).
Under weak regularity conditions the price $P^*(r,t,T)$ of a non-defaultable zero bond with maturity $T$ at time stamp $t$ is given by

$$P^*(r,t,T) = A^*(t,T) \cdot e^{-B^*(t,T)r}$$

where $B^*(t,T) = \frac{1}{a_r} \left( 1 - e^{-a_r(T-t)} \right)$ and

$$\ln A^*(t,T) = \ln \left( \frac{P^*(r,0,T)}{P^*(r,0,t)} \right) - B^*(t,T) \cdot \frac{\partial \ln P^*(r,0,t)}{\partial t} - \frac{\sigma_r^2}{4a_r^3} \cdot \left( e^{-a_rT} - e^{-a_rT} \right)^2 \cdot (e^{2a_rT} - 1).$$
The Model by Schmid & Zagst [2000]
Valuation of Non-Defaultable Bonds

Comments

• The non-defaultable zero rate at time $t$ for a maturity time $T$ is given by

$$R(t, T) = -\frac{1}{T-t} \cdot \ln P(t, T) = a(t, T) + b(T-t) \cdot r(t)$$  \hspace{1cm} (1)

with

$$a(t, T) = -\frac{\ln A(t, T)}{T-t} \quad \text{and} \quad b(T-t) = \frac{B(t, T)}{T-t}.$$

• We assume that

$$\text{Cov} \left( d\hat{W}_r(t), d\hat{W}_s(t) \right) = \text{Cov} \left( d\hat{W}_r(t), d\hat{W}_u(t) \right)$$

$$= \text{Cov} \left( d\hat{W}_s(t), d\hat{W}_u(t) \right) = 0.$$
The Model by Schmid & Zagst [2000]
Valuation of Defaultable Bonds

**Theorem (Defaultable Zero Bond, Schmid & Zagst [2000]).**
The arbitragefree price \( P(r,s,u,t,T) \) of a defaultable zero bond at time stamp \( t \) is given by

\[
P(r,s,u,t,T) = A(t,T) \cdot e^{-\beta^*(t,T)r - C(t,T)s - D(t,T)u}
\]

where

\[
C(t,T) = \frac{1 - e^{-\delta_s^{(T-t)}}}{K_s^1 - K_s^2 \cdot e^{-\delta_s^{(T-t)}}}
\]

with \( \delta_s = \sqrt{a_s^2 + 2 \cdot \sigma_s^2} \)

and \( K_s^k = \frac{1}{2} \cdot (a_s - (-1)^k \cdot \delta_s), k \in \{1,2\} \)

\[
D(t,T) = -\frac{2 \cdot v'(t,T)}{\sigma_u^2 \cdot v(t,T)}
\]

with \( v(t,T) \) difficult

\[
A(t,T) = A^*(t,T) \cdot e^{-S(t,T)}
\]

with \( S(t,T) = -\frac{2 \cdot \theta_u}{\sigma_u^2} \cdot \ln \left| \frac{v(T,T)}{v(t,T)} \right| \).
The Model by Schmid & Zagst [2000]
Valuation of Defaultable Bonds

Comments

• The defaultable zero rate, at time $t$ for a maturity time $T$ is given by

$$R^d(t, T) = -\frac{\ln A(t, T)}{T - t} - \frac{2\theta_u}{\sigma_u^2 \cdot (T - t)} \cdot \ln \left| \frac{v(T, T)}{v(t, T)} \right| + \frac{B(t, T)}{T - t} \cdot r(t)$$

$$+ \frac{C^d(t, T)}{T - t} \cdot s(t) + \frac{D^d(t, T)}{T - t} \cdot u(t).$$

• The credit spread at time $t$ for a maturity time $T$ can be calculated as

$$S(t, T) = R^d(t, T) - R(t, T)$$

$$= a^d(T - t) + c(T - t) \cdot s(t) + d(T - t) \cdot u(t) \quad \text{(2)}$$

with

$$a^d(T - t) = -\frac{2\theta_u}{\sigma_u^2} \cdot \ln \left| \frac{v(T, T)}{v(t, T)} \right|, \quad c(T - t) = \frac{C^d(t, T)}{T - t}, \quad d(T - t) = \frac{D^d(t, T)}{T - t}. $$