ENUMERATION OF PERMUTATIONS CONTAINING A PRESCRIBED NUMBER OF OCCURRENCES OF A PATTERN OF LENGTH THREE

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ABSTRACT. We consider the problem of enumerating the permutations containing exactly k occurrences of a pattern of length 3. This enumeration has received a lot of interest recently, and there are a lot of known results. This paper presents an alternative approach to the problem, which yields a proof for a formula which so far only was conjectured (by Noonan and Zeilberger). This approach is based on bijections from permutations to certain lattice paths with "jumps", which were first considered by Krattenthaler.

1. INTRODUCTION

1.1. **Patterns in permutations.** We consider the group $\mathcal{S}_n$ of permutations of the set $\{1,\ldots,n\}$. Let τ be an arbitrary permutation in $\mathcal{S}_k$ ($k \geq 2$, in order to avoid trivial cases). We say that some permutation $\rho = (\rho_1,\ldots,\rho_n) \in \mathcal{S}_n$ *contains* the *pattern* τ, if there exists a subword $\sigma = (\rho_{i_1},\ldots,\rho_{i_k})$ in ρ, such that the entries of the subword σ appear in the same "relative order" as the entries of the permutation τ. More formally stated, the subword σ can be transformed into the permutation τ by the following construction. Replace the smallest element in σ by 1, the second–smallest by 2, the third–smallest by 3, and so on.

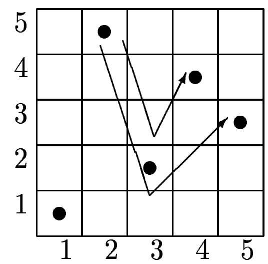
If $\rho \in \mathcal{S}_n$ does not contain pattern τ, we say that ρ is τ*–avoiding*.

For example, the permutation $\rho = (1,5,2,4,3)$ contains the pattern $\tau = (3,1,2)$, since the subword $\sigma = (\rho_2,\rho_3,\rho_4) = (5,2,4)$ shows the same relative ordering of elements as τ. First comes the largest element (i.e., 5), then the smallest (i.e., 2), and then the second–largest (i.e., 4).

The enumeration of the set $\mathcal{S}_n(\tau, r)$ of permutations from $\mathcal{S}_n$, which contain the fixed pattern τ precisely r times, has recently received considerable interest [1, 2, 3, 4, 5, 6, 7, 8, 9, 13, 11, 15, 14, 12, 16, 17, 18, 19, 20, 23].

Date: April 25, 2002.

FIGURE 1. Graphical illustration. Permutation $(1, 5, 2, 4, 3)$ contains exactly 2 $(3, 1, 2)$–patterns.



In our above example $\rho = (1, 5, 2, 4, 3)$, the pattern $(3, 1, 2)$ occurs exactly twice, since the subword $\sigma = (\rho_2, \rho_3, \rho_5) = (5, 2, 3)$, too, shows the same relative ordering as $(3, 1, 2)$. Hence $(1, 5, 2, 4, 3)$ belongs to $\mathcal{S}_5((3, 1, 2),\, 2)$.

We shall denote the cardinality of $\mathcal{S}_n(\tau,\, r)$ by $s_n(\tau,\, r)$, where we set, by convention, $s_0(\tau,\, 0) = s_1(\tau,\, 0) = 1$ and $s_0(\tau,\, r) = s_1(\tau,\, r) = 0$ for $r > 0$ (recall that τ is of length at least 2). We denote the generating function of these numbers by $F_{\tau,\, r}(x) := \sum_{n=0}^{\infty} s_n(\tau,\, r)\, x^n$

1.2. **Outline of this paper.** In this paper, we consider patterns of length 3. It is well known that there are only two patterns in $\mathcal{S}_3$ which are essentially different, namely $(3, 2, 1)$ and $(3, 1, 2)$. (We shall give a simple argument for this fact right below). For both of these patterns, we make use of a bijective construction which basically "translates" permutations into certain "generalized" Dyck–paths, where "jumps" are allowed. This construction was first considered by Krattenthaler [9, Lemma ϕ and Lemma Ψ]. Using the well–known generating function for Dyck–paths, we are able to obtain the generating functions $F_{(3,1,2),\, r}(x)$ and $F_{(3,2,1),\, r}(x)$ for $r = 0, 1, 2$ in a uniform way. Especially, for $r = 0$ we thus obtain a very simple "graphical" proof of the formulas for $s_n((3, 1, 2),\, 0)$ and $s_n((3, 2, 1),\, 0)$ (which, of course, are very well known, see [22, Exercises 6.19.ee and 6.19.ff]). The derivation of $F_{(3,2,1),\, 2}(x)$ seems to be new (it was conjectured by Noonan and Zeilberger [17]).

In Section 2 we present the respective formulas. In Section 3 we introduce the bijective construction and show how the generating functions can be derived.

1.3. **Graphical approach to permutation patterns.** "Graphical arguments" play a key role in our presentation. We introduce the *permutation graph* $G(\rho)$ (see [21, p. 71]) as a simple way for visualizing some permutation $\rho \in \mathcal{S}_n$:

$$G(\rho) := \{(1, \rho_1), \dots, (n, \rho_n)\} \subseteq \{1, \dots, n\} \times \{1, \dots, n\}.$$

Figure 1 illustrates this concept. The set $\{1, \dots, n\} \times \{1, \dots, n\}$ is visualized by a rectangular grid of square cells, the elements (i, ρ_i) of the graph are indicated by dots in the respective cell. Note that the two $(3, 1, 2)$–patterns in $(1, 5, 2, 4, 3)$ appear as "hook–like configurations", as indicated by the arrows in the picture.

Clearly, $G(\rho)$ is just another way to denote the permutation ρ. Hence, in the following we shall not make much difference between a permutation and its graph.

We call the "horizontal line" $\{(1, i), \dots, (n, i)\}$ the i–th row of the graph, and the "vertical line" $\{(j, 1), \dots, (j, n)\}$ the j–th column of the graph. Note that the rows of $G(\rho)$ correspond to the rows of the *permutation matrix* of ρ in reverse order. Clearly, each row and each column of the graph of some permutations contains precisely one dot.

A simple geometric argument now shows that, when talking about patterns of length 3, we may restrict our attention to 2 "essentially different" patterns, namely $(3, 2, 1)$ and $(3, 1, 2)$. Given the permutation graph of an arbitrary permutation τ, consider the rotations by $0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$. Clearly, every pattern contained in τ is rotated accordingly, so the numbers $s_n(\tau,\, r)$ are the same as the numbers $s_n(\tau',\, r)$, where τ' denotes the permutation obtained by rotating the graph of τ. Therefore, we only have to consider the different *orbits* of the permutation graphs of $\mathcal{S}_3$ under the action of this rotation group. It is easy to see that there are precisely two of them, one containing $(3, 1, 2)$ (see Figure 2), the other containing $(3, 2, 1)$ (see Figure 3).

2. Results and conjectures

2.1. **Formulas for 0,1, and 2 occurrences of pattern τ.** In the following, we summarize the formulas and generating functions we are going to prove later. We start with a very well known fact:

$$s_n((3, 1, 2),\, 0) = s_n((3, 2, 1),\, 0) = \frac{1}{n+1}\binom{2n}{n}. \tag{1}$$

These are the Catalan numbers, which go like this: 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, ... The corresponding generating function is

$$F_{(3,1,2),\,0}(x) = F_{(3,2,1),\,0}(x) = \frac{1}{2x} - \frac{1}{2x}\sqrt{1 - 4x}. \tag{2}$$

FIGURE 2. Graphical illustration of the orbit of $(3,1,2)$ under rotations of the permutation graph

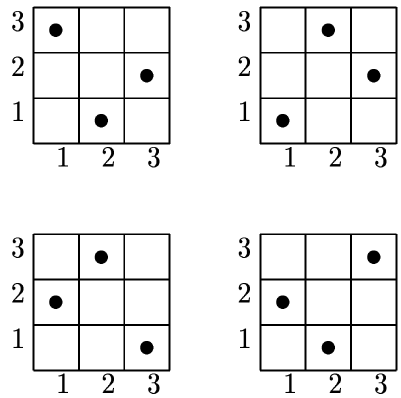
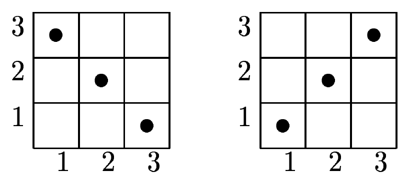


FIGURE 3. Graphical illustration of the orbit of $(3,2,1)$ under rotations of the permutation graph



The next formula was first proved by Bóna [7]:

$$s_n((3,1,2),\,1) = \binom{2n-3}{n-3}. \tag{3}$$

The numbers go like this: 0, 0, 0, 1, 5, 21, 84, 330, 1287, 5005, 19448, 75582, 293930, ... The corresponding generating function is

$$F_{(3,1,2),\,1}(x) = \frac{x-1}{2} - \frac{3x-1}{2}(1-4x)^{-1/2}. \tag{4}$$

The next formula was first proved by Mansour and Vainshtein [11]:

$$s_n((3,1,2),\,2) = \binom{2n-6}{n-4}\frac{n^3+17n^2-80n+80}{2n(n-1)}, \tag{5}$$

The numbers go like 0, 0, 0, 0, 4, 23, 107, 464, 1950, 8063, 33033, 134576, 546312, ... The corresponding generating function is

$$F_{(3,1,2),\,2}(x) = \frac{x^2+3x-2}{2} + \frac{2x^4-4x^3+29x^2-15x+2}{2}(1-4x)^{-3/2}. \quad (6)$$

Remark 1. *Mansour and Vainshtein* [11, Corollary 3] *are able to compute the numbers $s_n((3,1,2),\ r)$ and generating functions $F_{(3,1,2),\,r}(x)$ for r up to 6.*

The next formula was first proved by Noonan [16]:

$$s_n((3,2,1),\ 1) = \frac{3}{n}\binom{2n}{n-3}. \quad (7)$$

These numbers go like 0, 0, 0, 1, 6, 27, 110, 429, 1638, 6188, 23256, 87210, 326876, ... The corresponding generating function is

$$F_{(3,2,1),\,1}(x) = -\frac{2x^3-9x^2+6x-1}{2x^3} - \frac{\sqrt{1-4x}(3x^2-4x+1)}{2x^3}. \quad (8)$$

The next formula was conjectured by Noonan and Zeilberger [17]; to the best of my knowledge the first proof is contained in this paper:

$$s_n((3,2,1),\ 2) = \frac{59n^2+117n+100}{2n(2n-1)(n+5)}\binom{2n}{n-4}. \quad (9)$$

These numbers go like 0, 0, 0, 0, 3, 24, 133, 635, 2807, 11864, 48756, 196707, 783750, ... The corresponding generating function is

$$F_{(3,2,1),\,2}(x) = -\frac{5x^5-7x^4+17x^3-20x^2+8x-1}{2x^5} + \frac{\sqrt{1-4x}(x^5-3x^4+5x^3-10x^2+6x-1)}{2x^5}. \quad (10)$$

Remark 2. *Toufik Mansour* [10] *informed me that he found a proof of formula 9, too.*

2.2. **Further conjectures.** We state the following conjectures concerning generating functions.

Conjecture 3. *The generating function $F_{(3,2,1),\,3}(x)$ is*

$$F_{(3,2,1),\,3}(x) = -\frac{(-1+10x-33x^2+32x^3+31x^4-70x^5+35x^6-2x^8)}{2x^7} + \frac{\sqrt{1-4x}(-1+8x-19x^2+6x^3+27x^4-28x^5+7x^6+2x^7)}{2x^7}. \quad (11)$$

Conjecture 4. *The generating function $F_{(3,2,1),4}(x)$ is*

$$F_{(3,2,1),4}(x) = \frac{-1}{2x^9}\big[(-1+12x-50x^2+65x^3+107x^4-437x^5+588x^6 - 492x^7+314x^8-108x^9+3x^{10}) + \sqrt{1-4x}(-1+10x-32x^2+17x^3+107x^4-245x^5+256x^6-192x^7+102x^8 - 18x^9-x^{10})\big]. \quad (12)$$

2.3. **General form of the formulas.** By inspection of the formulas listed above, one is immediately led to conjecture a simple general form for the generating functions $F_{(3,1,2),r}(x)$ and $F_{(3,2,1),2}(x)$. The "$(3,1,2)$–part" was proved by Bóna [5, Proposition 1 and Lemma 3].

Theorem 5 (Bóna)**.** *The generating functions $F_{(3,1,2),r}(x)$ are rational functions in x and $\sqrt{1-4x}$. Moreover, when written in smallest terms, the denominator of $F_{(3,1,2),r}(x)$ is equal to $\left(\sqrt{1-4x}\right)^{2r-1}$.*

For the "$(3,2,1)$–part", we state the following conjecture:

Conjecture 6. *The generating functions $F_{(3,2,1),r}(x)$ is of the form*

$$\frac{1}{2x^{2r+1}}\left(P_r\left(x\right)+\sqrt{1-4x}\,Q_r\left(x\right)\right),$$

where P_r and Q_r are polynomials.

Remark 7. *Toufik Mansour* [10] *informed me that he found a proof for the general form of the generating function $F_{(3,2,1),r}(x)$.*

3. The proofs

3.1. **The case of pattern–avoidance.** Let us start with a simple proof of (1). Recall that the Catalan numbers C_n arise in the enumeration of Dyck paths, which are paths in the integer lattice consisting of "up–steps" $(1,1)$ and "down–steps" $(1,-1)$, which start in $(0,0)$ and end at $(2n,0)$, such that the path never goes below the horizontal axis. More precisely, we have

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \#\text{ of Dyck–paths from } (0,0) \text{ to } (2n,0).$$

So (1) will follow, if we can establish a bijection between permutations in $\mathcal{S}_n$ which do avoid the pattern $(3,2,1)$ or $(3,1,2)$, respectively, and Dyck–paths from $(0,0)$ to $(2n,0)$. Denote the set of such paths by D_n.

In the following, τ will always denote one of the patterns $(3,1,2)$ or $(3,2,1)$. If some assertion A is valid for both patterns, we shall simply say "A is valid for τ" instead of "A is valid for $(3,1,2)$ and for $(3,2,1)$". For an entry ρ_i in some permutation $\rho = (\rho_1, \ldots, \rho_n) \in \mathcal{S}_n$, we call the entries ρ_j with $j < i$ the entries *to the left*, and with $j \leq i$ the entries *weakly to the left*, and likewise for entries (weakly) to the right.

Definition 8. *Let $\rho = (\rho_1, \ldots, \rho_n) \in \mathcal{S}_n$ be an arbitrary permutation. An entry ρ_i is called a* left–to–right maximum, *if ρ_i is greater than all entries to the left of it; i.e., if $\rho_i > \rho_j$ for all $j < i$. Entries which are* not *left–to–right maxima are called* remaining entries.

Observation 9. *By definition, for a subword $(\sigma_1, \sigma_2, \sigma_3)$ in ρ which establishes an occurrence of the pattern τ, neither entry σ_2 nor σ_3 can be a left–to–right maximum in ρ. On the other hand, the entry σ_1 is either a left–to–right maximum itself, or there is a left–to–right maximum ρ_k to the left of σ_1, such that $(\rho_k, \sigma_2, \sigma_3)$ gives another τ–occurrence. In fact, this is the case for* all *left–to–right maxima ρ_k weakly to the left of σ_2, such that $\rho_k > \max(\sigma_2, \sigma_3)$.*

To establish our desired bijections, we first consider a single mapping

$$\psi : \mathcal{S}_n \to D_n,$$

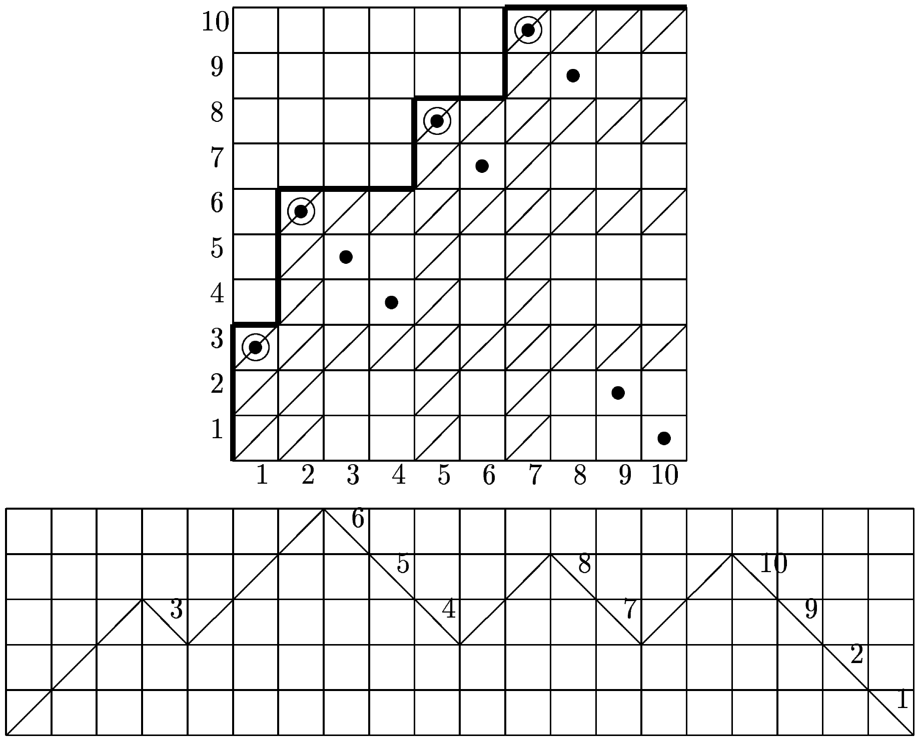
which will turn out to yield both bijections for $\tau = (3,2,1)$ and $\tau = (3,1,2)$ by simply restricting to $\mathcal{S}_n(\tau, 0)$. ψ is defined by the following "graphical" construction: For an arbitrary permutation $\rho \in \mathcal{S}_n$,

- draw the permutation graph of ρ and mark the left–to–right maxima (i.e., the points (i, ρ_i) with $\rho_i > \rho_j$ for all $j < i$),
- consider the points $(x, y) \in \{1, \ldots, n\} \times \{1, \ldots, n\}$, which lie below and to the right of some left–to–right maximum (i.e., $x \geq i$ and $y \leq \rho_i$ for some left–to–right maximum (i, ρ_i)),
- rotate the "polygonal figure" formed by these points by $-\frac{\pi}{4}$ and consider its upper boundary, which appears as some path p from $(0,0)$ to $(2n, 0)$,
- set $\psi(\rho) = p$.

Figure 4 gives an illustration of this construction. The left–to–right maxima are marked by circles, the boundary line of the polygonal figure is shown as bold line, and the corresponding path is drawn below. Note that the entries of the permutation correspond to the down–steps of the path (this correspondence is indicated by the labels in Figure 4); the left–to–right maxima of the permutation correspond to the "peaks" of the path. In the following, we will more or less identify the i–th entry of ρ and the i–th down–step of $\psi(\rho)$.

We have to show that the resulting path $p = \psi(\rho)$ is actually in D_n, i.e., p does not go below the horizontal axis. For convenience, we introduce some notation concerning paths.

Figure 4. Illustration of the mapping ψ from permutations to Dyck–paths. (Note: The permutation $(3,6,5,4,8,2,10,9,7,1)$ is $(3,1,2)$–avoiding.)



Definition 10. *A down–step from $(x, h+1)$ to $(x+1, h)$ is said to have (or be of) height h. By abuse of notation, we will also say that the entry of the permutation corresponding to such down–step has height h.*

The i–th down–step in $\psi(\rho)$ corresponds to the i–th entry of ρ. There must exist left–to–right maxima ρ_j weakly to the left of this entry (i.e., $j \leq i$). We call the right–most of these the preceding left–to–right maximum *with respect to the down–step; it corresponds to the peak weakly to the left of the down–step.*

So what we have to show is that all down–steps have nonnegative heights. First note that the height of a left–to–right maximum ρ_m is exactly the number of entries to the right which are smaller; i.e.,

$$\#\{j : m < j \leq n \text{ and } \rho_j < \rho_m\},$$

which clearly is nonnegative. More generally, let h_k be the height of the k–th down–step, and let ρ_m be the preceding left–to–right maximum of height h_m. It is easy to see that $h_k = h_m - (k - m)$, which is nonnegative, too, since $(k - m)$ cannot exceed h_m. Hence $p = \psi(\rho)$ is indeed in D_n for all $\rho \in \mathcal{S}_n$.

Next we show that the mapping ψ, restricted to $\mathcal{S}_n(\tau, 0)$, is a bijection, for $\tau = (3, 2, 1)$ as well as for $\tau = (3, 1, 2)$. Thus (1) will follow immediately. Given an arbitrary Dyck–path p in D_n, we will construct a permutation $\rho \in \mathcal{S}_n$ such that

- $\psi(\rho) = p$,
- ρ is the *unique* τ–avoiding permutation in $\psi^{-1}(p)$.

We start by rotating path p by $\frac{\pi}{4}$, thus obtaining a "configuration of left–to–right maxima", which we may view as an "incomplete permutation graph".

While there are empty rows and columns in this "incomplete graph", repeat the following step for the case $\tau = (3, 2, 1)$:

> Consider the *left–most* column j, which does not yet contain a dot, and the *lowest* row i, which does not yet contain a dot, and put a dot into the cell (i, j).

Observe that a permutation obtained from this construction is $(3, 2, 1)$-avoiding. Figure 5 illustrates this construction for the "configuration of left–to–right maxima" from Figure 4.

For the case $\tau = (3, 1, 2)$, repeat the following step:

> Consider the left–most column j, which does not yet contain a dot, and the *highest* row i, which does not yet contain a dot *and* which lies *below* the preceding left–to–right maximum, and put a dot into the cell (i, j).

Observe that a permutation obtained from this construction is $(3, 1, 2)$-avoiding, as is illustrated by Figure 4

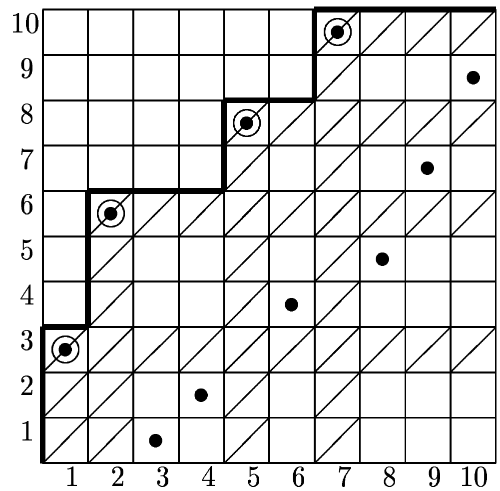
These constructions will give completed graphs (and thus some permutation ρ) in any case; however, in order to have $\psi(\rho) = p$, there must hold the following condition:

> Whenever there is a column j which does not yet contain a dot, then there is also some row i without a dot, which lies *below* the left–to–right maximum *preceding* (in the same sense as before) column j.

But this is precisely the condition, that the path p does not go below the horizontal axis.

The fact that the permutation ρ is indeed unique is best seen "by inspection", see Figures 4 (for $(3, 1, 2)$–avoiding permutations) and 5 (for $(3, 2, 1)$–avoiding permutations). Consider an arbitrary "configuration of left–to–right maxima" and observe:

FIGURE 5. Illustration of the mapping ψ from permutations to Dyck–paths. (Note: The permutation $(3,6,1,2,8,4,10,5,7,9)$ is $(3,2,1)$–avoiding.)



- If a permutation is $(3,1,2)$–avoiding, the "remaining elements" must be inserted in a "descending" way,
- if a permutation is $(3,2,1)$–avoiding, the "remaining elements" must be inserted in an "ascending" way. □

3.2. **Krattenthaler's bijections.** The bijective proof for the case of zero occurrences of length–3–patterns was very simple. While the formal description of the constructions took some pages of text, the main idea is easily obtained by mere "inspection" of Figure 4 and Figure 5. This "graphical view" will turn out to be fruitful also for the cases of one and more occurrences of patterns; however, we do not have such a simple and uniform "graphical" definition any more. We now have to distinguish between the case of $(3,2,1)$–patterns and $(3,1,2)$–patterns. The respective constructions were given by Krattenthaler [9].

Definition 11. *Consider the set of lattice paths from* $(0,0)$ *to* $(2n+s,0)$*, consisting of* $(n+s)$ *up–steps* $(1,1)$*,* n *down–steps* $(1,-1)$*, and* s *down–jumps* $(0,-1)$*, which never go below the horizontal axis. Denote this set of "generalized Dyck–paths with down–jumps" by* $D_{n,s}$*, denote the union* $\bigcup_{s=0}^{\infty} D_{n,s}$ *by* D'_n *and the union* $\bigcup_{n=0}^{\infty} D'_n$ *by* D'*.*

A maximal seqence of d *consecutive down–jumps is called a* jump of depth d.

Such paths with jumps will be the main object of the following considerations. We will refer to them simply as "lattice paths" or (even simpler) as “paths”. In order to avoid complicated verbal descriptions of such paths (or segments of paths), we introduce a “graphical notation” by denoting an up–step by /, a down–step by \, and a down–jump by |. For illustration, see Figure 6: The two Dyck–paths with jumps shown there read ////\\/\|\/\ and ////\||\///\\\, respectively, in this “graphical notation”.

For specifying the length of some sequence of consecutive steps, we introduce a notation by means of example: \· k ·\ denotes a sequence of k down–steps. A down–step of height h (in the sense of Definition 10) is denoted by \h.

Definition 12. *Define the mapping*

$$\psi_{(3,1,2)} : \mathcal{S}_n \to D'_n$$

by the following construction: Read $\rho = (\rho_1, \ldots, \rho_n) \in \mathcal{S}_n$ from left to right and define the height h_i to be the number of elements ρ_k which are smaller than ρ_i and lie to the right of i, i.e., with $k > i$ and $\rho_k < \rho_i$.

It is clear that this amounts to another unique encoding of the permutation ρ by a “height–vector” $(h_1, \ldots, h_n)$. This height–vector is translated into a path in the following way.

> *Start at $(0, 0)$.*
> *For $i = 1, \ldots, n$ do the following:*
>
> > *If the last point of the path constructed so far lies below h_i+1, then draw as many up–steps as necessary to reach height h_i+1, otherwise make as may down–jumps as necessary to reach height $h_i + 1$.*
> > *Finally, draw a down–step at height h_i.*

For the resulting lattice path p, set $\psi_{(3,1,2)}(\rho) = p$.

It is easy to see that $\psi_{(3,1,2)}$ is a well–defined injection.

Definition 13. *Define the mapping*

$$\psi_{(3,2,1)} : \mathcal{S}_n \to D'_n$$

by the following construction: Read $\rho = (\rho_1, \ldots, \rho_n) \in \mathcal{S}_n$ from left to right and define the height h_i as follows.

- IF *ρ_i is a left–to–right maximum, then h_i is the number of elements ρ_k which are smaller than and lie to the right of ρ_i, i.e., with $k > i$ and $\rho_k < \rho_i$,*
- ELSE *h_i is the number of elements ρ_k which are bigger than and lie to the right of ρ_i, i.e., with $k > i$ and $\rho_k > \rho_i$,* and *which are smaller than the preceding left–to–right maximum.*

The "height–vector" $(h_1, \dots, h_n)$ is translated into a path in exactly the same way as in Definition 12. For the resulting lattice path p, set $\psi_{(3,2,1)}(\rho) = p$.

It is easy to see that $\psi_{(3,2,1)}$ is a well–defined injection, too.

See Figure 6 for an illustration of both injections $\psi_{(3,1,2)}$ and $\psi_{(3,2,1)}$ applied to the same permutation $(4, 3, 5, 1, 2)$. Note that the entries of the permutation correspond to the down–steps of the path, as before. Again, we shall more or less identify a permutation and its corresponding path.

So clearly, our direction to proceed is as follows:

- Determine the subsets of D' which correspond to permutations containing one, two, etc. occurrences of pattern $(3, 2, 1)$ or $(3, 1, 2)$, respectively,
- identify the generating functions for such subsets.

In order to describe the relevant subsets of paths, we must have a close look at the properties of the mappings $\psi_{(3,1,2)}$ and $\psi_{(3,2,1)}$.

Observation 14. *Let τ be one of the patterns $(3, 1, 2)$ or $(3, 2, 1)$.*

(1) *By construction, a jump in $\psi_\tau(\rho)$ must be preceded and followed immediately by a down–step; i.e., patterns /| and |/ are impossible. If a jump occurs between the i–th and $(i+1)$–th down–step, we call it a* jump at position i.

(2) *For every down–step and for every jump in $\psi_\tau(\rho)$, there must exist a preceding left–to–right maximum (in the sense of Definition 10).*

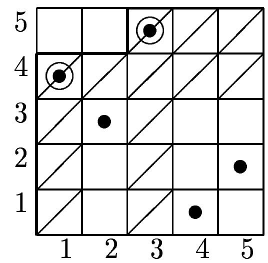
(3) *Every left–to–right maximum in permutation ρ corresponds to a "peak /\" in the lattice path $\psi_tau(\rho)$. However, there might be peaks in $\psi_\tau(\rho)$ which do not correspond to left–to–right maxima in ρ (see Figure 6 for an example).*

(4) *Every τ–occurrence $(\sigma_1, \sigma_2, \sigma_3)$ in permutaion ρ is associated with a jump in $\psi(\rho)$: σ_1 lies weakly to the left of the jump, σ_2 and σ_3 to the right.*

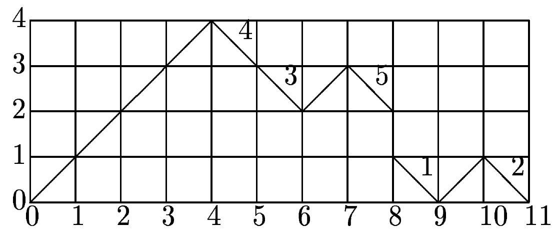
(5) *$\psi_\tau(\rho)$ is an* ordinary *Dyck–path (i.e., a Dyck–path without any down–jump) if and only if ρ is τ–avoiding: Mappings ψ_τ and ψ coincide, when restricted to τ–avoiding permutations, as a second look at the proof given in Section 3.1 shows immediately.*

(6) *To the right of an arbitrary down–step of height l in $\psi_\tau(\rho)$, there must be at least l down–steps: This simple statement implies that to the right of a jump of depth d, there must be at least d up–steps.*

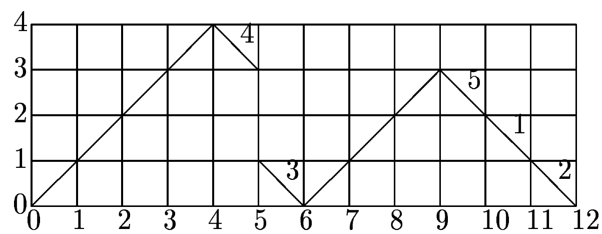
FIGURE 6. Illustration of the bijections $\psi_{(3,2,1)}$ and $\psi_{(3,2,1)}$ for the permutation $\rho = (4,3,5,1,2)$.



The image of ρ under $\psi_{(3,1,2)}$: One jump of depth 1.



The image of ρ under $\psi_{(3,2,1)}$: One jump of depth 2.



Now we investigate the relationship between jumps in $\psi_\tau(\rho)$ and the number of τ–occurrences in ρ more closely.

Lemma 15. *Let $\tau \in \mathcal{S}_3$ be one of the patterns $(3,1,2)$ or $(3,2,1)$, let ρ be an arbitrary permutation. A jump of depth d at position i in $\psi_\tau(\rho)$, which is followed immediately by l ($l > 0$ by Observation 14!) consecutive down–steps, implies $(d \cdot l)$ τ–occurrences of type $(\rho_r, \rho_{i+j}, \rho_k)$, where*

- *ρ_r is the left–to–right maximum preceding the jump,*
- *ρ_{i+j} are the entries of ρ corresponding to the consecutive down–steps immediately following the jump ($j = 1, \ldots, l$),*
- *ρ_k is one of the d entries of ρ with $k > i$ which are*
  - *bigger than ρ_{i+1} and smaller than ρ_i for $\tau = (3,1,2)$,*
  - *smaller than ρ_{i+1} for $\tau = (3,2,1)$.*

Proof. The statement sounds a bit complicated, but it is easily obtained "by inspection":

For the case $\tau = (3,1,2)$, consider Figure 7 and observe that entry 9 is the left–to–right maximum preceding the jump, that $(2,1)$ are the entries corresponding to the down–steps immediately following the jump, and that the entries $(4,3)$ are the 2 entries to the right of the jump, which are bigger than the entry following the jump and smaller than the entry preceding the jump — in some sense, they "cause" the jump of depth 2: The four occurrences of $(3,1,2)$ in Figure 7, as stated in the lemma, are $(9,2,4)$, $(9,2,3)$, $(9,1,4)$ and $(9,1,3)$.

For the case $\tau = (3,2,1)$, consider Figure 8 and observe that entry 9 is the left–to–right maximum preceding the jump, that $(6,8)$ are the entries corresponding to the down–steps immediately following the jump, and that entries $(4,5)$ are the 2 entries to the right of the jump, which are smaller than the entry following the jump — in the same sense as above, they "cause" the jump of depth 2. The four occurrences of $(3,2,1)$ in Figure 8, as stated in the lemma, are $(9,6,4)$, $(9,6,5)$, $(9,8,4)$ and $(9,8,5)$. □

Corollary 16. *For $r > 0$ and $\tau = (3,1,2)$ or $\tau = (3,2,1)$, ψ_τ takes the set $\mathcal{S}_n(\tau, r)$ into $\bigcup_{s=1}^{r} D_{n,s}$; i.e., a permutation with exactly r occurrences of τ is mapped to a Dyck–path with at most r down–jumps.*

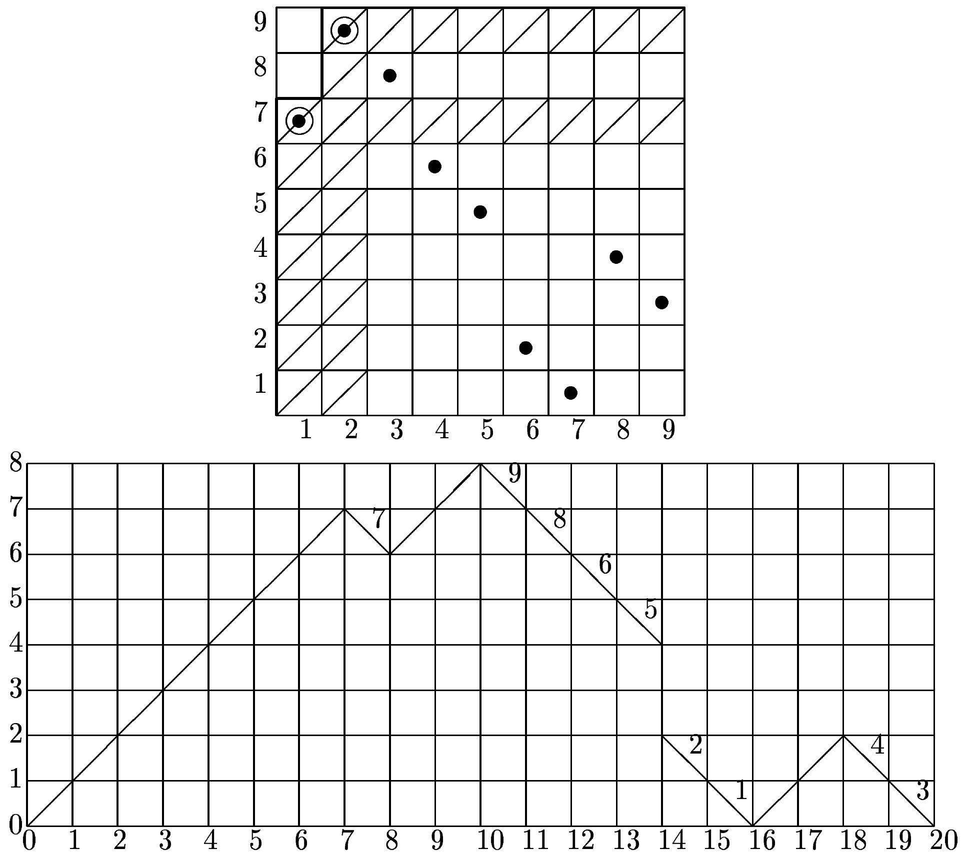
It is immediately obvious from Figures 7 and 8 that the enumeration of occurrences associated with some jump, as given in Lemma 15, is in general incomplete. Note that down–steps *immediately preceding* a jump obviously add to the number of $(3,1,2)$–occurrences, while this is not the case for $(3,2,1)$–occurrences. To the contrary, we find additional occurrences of pattern $(3,2,1)$ if there are "not enough" such down–steps.

While the general situation obviously can get quite complicated, we state the following extensions to Lemma 15.

Corollary 17. *Let $\tau = (3,1,2)$, let ρ be an arbitrary permutation. Consider a jump of depth d at position i in $\psi_\tau(\rho)$, which is followed immediately by l consecutive down–steps, and which is preceded immediately by m consecutive down–steps* including *the one corresponding to the preceding left–to–right maximum. Such a jump implies $(m \cdot d \cdot l)$ τ–occurrences of type $(\rho_{i-g+1}, \rho_{i+j}, \rho_k)$, where*

- *ρ_{i-g+1} are the consecutive down–steps immediately preceding the jump ($g = 1, \ldots, m$),*
- *ρ_{i+j} are the entries of ρ corresponding to the consecutive down–steps immediately following the jump ($j = 1, \ldots, l$),*

FIGURE 7. Illustration for Lemma 15. (Note: The permutation $(7,9,8,6,5,2,1,4,3)$ is not $(3,1,2)$–avoiding.)



- ρ_k *is one of the* d *entries of* ρ *with* $k > i$ *which are bigger than* ρ_{i+1} *and smaller than* ρ_i.

Proof. In our graphical notation, the situation is as follows:

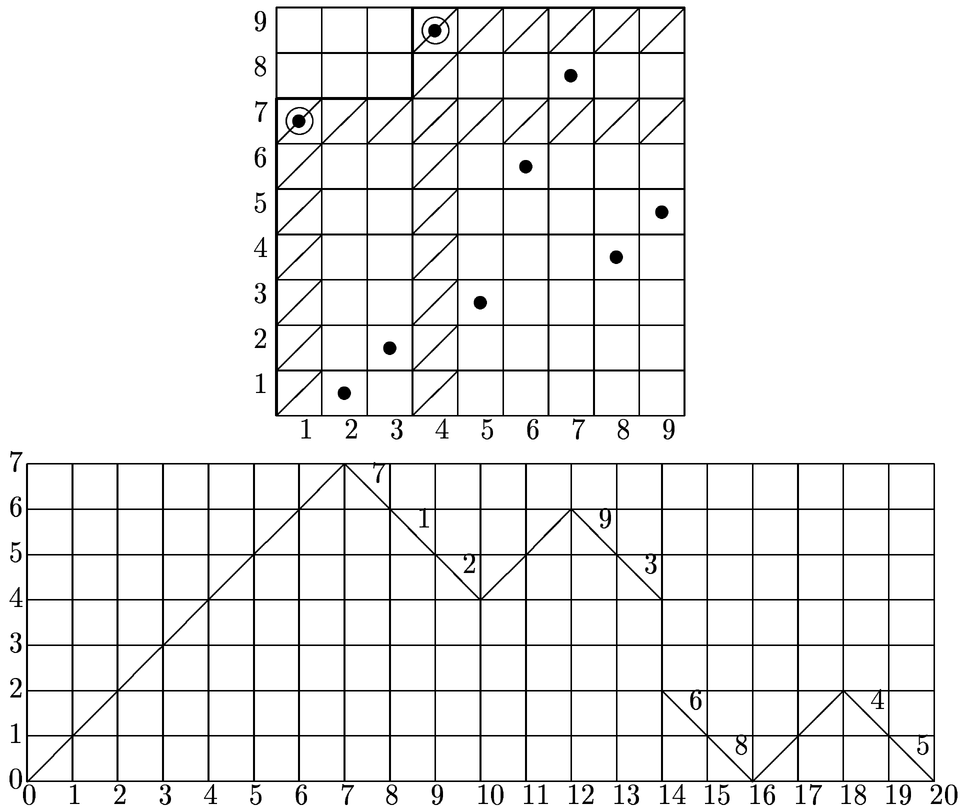
$$\diagup\diagdown\cdot\, m\,\cdot\diagdown|\cdot\, d\,\cdot|\diagdown\cdot\, l\,\cdot\diagdown\diagup.$$

The assertion is easily obtained "by inspection", see Figure 7. □

By careful inspection, we obtain yet another observation:

Corollary 18. *In the same situation as in Corollary 17, consider the left–to–right maxima* $\rho_{i_1}, \dots, \rho_{i_r}$ *to the left of the left–to–right maximum preceding the jump,* ρ_{i-m+1}. *For each of these, compute the number* s_j *of up–steps to the right, which do not correspond to peaks of left–to–right maxima and which are to the left of the jump. Let* x *be*

FIGURE 8. Illustration for Lemma 15. (Note: The permutation $(7,1,2,9,3,6,8,4,5)$ is not $(3,2,1)$–avoiding.)



the smallest index for which $s_x < m$. Then there are exactly

$$d \cdot l \cdot \sum_{\xi=x}^{r} (m - s_\xi)$$

τ–occurrences of type $(\rho_\xi, \rho_{i+j}, \rho_k)$, where

- *ρ_ξ are the left–to–right maxima as described above ($\xi = x, \ldots, r$),*
- *ρ_{i+j} are the entries of ρ corresponding to the consecutive down–steps immediately following the jump ($j = 1, \ldots, l$),*
- *ρ_k is one of the d entries of ρ with $k > i$ which are bigger than ρ_{i+1} and smaller than ρ_i.*

Lemma 19. *Let $\tau = (3,2,1)$, let ρ be an arbitrary permutation. Consider the* first *jump (counted from the left). Assume that this jump*

- *occurs at position i in $\psi_\tau(\rho)$,*
- *is of depth d,*
- *and is followed immediately by l consecutive down–steps.*

Consider the left–to–right maxima $\rho_{i_1}, \dots, \rho_{i_r}$ to the left of the jump, and their respective heights $h_{i_1}, \dots, h_{i_r}$. For each of these r left–to–right maxima compute the the number s_j of down–steps to the right, which do not correspond *to left–to–right maxima and which are to the left of the jump.*

Let x be the smallest index for which

$$h_{i_x} - d - s_x > 0.$$

Then there are exactly

$$d \cdot \sum_{g=x}^{r} \min\left(h_{i_g} - d - s_g, l\right)$$

τ–occurrences of type $\left(\rho_{i_g}, \rho_{i+j}, \rho_k\right)$, where

- *ρ_{i_g} are the left–to–right maxima as above ($g = 1, \dots, r$),*
- *ρ_{i+j} are the entries of ρ corresponding to the consecutive down–steps after the jump ($j = 1, \dots, \min\left(h_{i_g} - d - s_g, l\right)$),*
- *ρ_k are the d entries of ρ with $k > i$ which are smaller than ρ_{i+1}.*

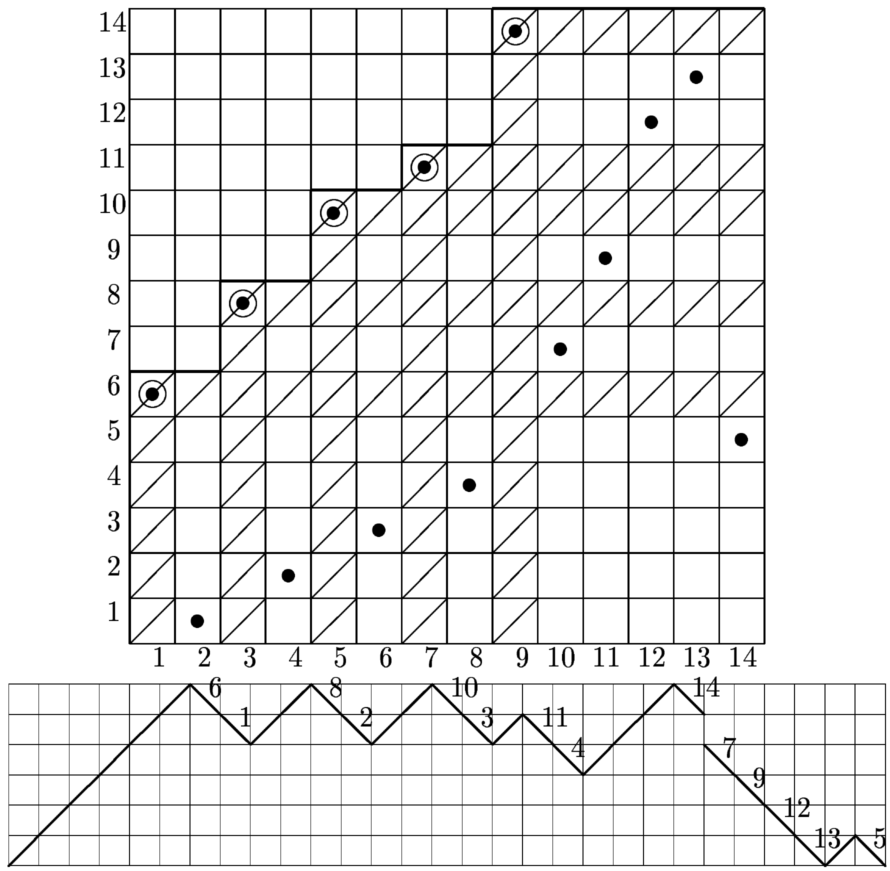
Proof. The statement sounds rather complicated, but again, it is easy to obtain "by inspection" — see Figure 9. There, we have $d = 1$ and $l = 4$. The 5 left–to–right maxima preceding the first (and only) jump are 6, 8, 10, 11 and 14; with heights 5, 5, 5, 4 and 5; the numbers s_j read 4, 3, 2, 1 and 0.

Observe that the condition for h_{i_g} and s_g simply determines whether left–to–right maximum ρ_{i_g} does "intersect the jump–configuration", which consists of $(7, 9, 12, 13, 5)$ in Figure 9. Recall that h_{i_g} is the number of entries to the right of ρ_{i_g} which are smaller than ρ_{i_g}; in order that ρ_{i_g} is involved in a τ–occurrence, the d entries which "cause" the jump must be among them, *and* there must be some entry left among them, which is not "used up" on the way to the jump. This consideration explains the sum in the lemma.

In Figure 9, the numbers $(h_{i_g} - d - s_g)$ are $0, 1, 2, 2, 4$; corresponding to no $(3, 2, 1)$–occurrence involving the first left–to–right maximum 6, 1 occurrence $(8, 7, 5)$, two occurrences $(10, 9, 5)$ and $(10, 7, 5)$, another two occurrences $(11, 9, 5)$ and $(11, 7, 5)$, and finally four occurrences $(14, 13, 5)$, $(14, 12, 5)$, $(14, 9, 5)$ and $(14, 7, 5)$. □

Now we have the prerequisites ready for the remaining proofs. In the following, we will determine the subset $P(\tau, r) \subseteq D'$, which corresponds bijectively to permutations with exactly r occurrences of τ. We will see that this set appears "naturally" as the disjoint union of a

FIGURE 9. Illustration for Lemma 19.



finite family of subsets

$$P(\tau, r) = P_1 \cup \cdots \cup P_m,$$

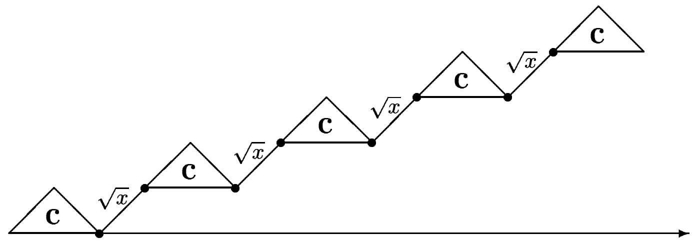
where the paths in subset P_i are characterized by a certain "pattern". Especially, the paths in P_i all have the same number s_i of down–jumps. Define the weight $w(p)$ of a path in $D_{n,s}$ to be $x^{(2n+s)/2}$ (i.e., assign weight $\sqrt{x}$ to each up–step and each down–step; and weight 1 to each of the s down–jumps), and consider the generating function

$$P_{\tau,r,i}(x) := \sum_p w(p),$$

where the summation runs over all $p \in P_i$. Then the generating function we are interested in is given as

$$F_{\tau,r}(x) = \sum_{i=1}^{m} x^{-s_i/2} P_{\tau,r,i}(x). \tag{13}$$

FIGURE 10. The generating function for lattice paths "climbing up" from height 0 to height 4, which never go below the horizontal axis, is $c^5 \left(\sqrt{x}\right)^4 = c^5 x^2$.



3.3. **The case of one occurrence of length–3–patterns.** What we want to do is "assemble the respective lattice paths from appropriate parts", and "translate" this construction into the corresponding generating function. For convenience, we introduce the notation

$$c = \frac{1}{2x} - \frac{1}{2x}\sqrt{1-4x}$$

for the generating function of ordinary Dyck–paths (without jumps).

Observe that lattice paths from $(a,0)$ to (b,l) ($b \geq a$, $l \geq 0$), which do not go below the horizontal axis, can be "composed" quite easily (see Figure 10), and that the respective generating function is

$$c^{l+1} x^{l/2}. \tag{14}$$

Now let us turn to our lattice paths with jumps: By Corollary 16, for an arbitrary permutation ρ with exactly one occurrence of $\tau = (3,1,2)$ or $\tau = (3,2,1)$, respectively, the paths $\psi_\tau(\rho)$ must contain exactly one down–jump. Moreover, by Corollaries 17 and 18 or Lemma 19, respectively, there must hold certain conditions on

- down–steps *before*,
- down–steps *after*,
- and left–to–right maxima *before*

this single jump.

Let us start with a proof of (4): Observe that for any permutation ρ with exactly one occurrence of $\tau = (3,1,2)$, the path–segment around the jump must look *exactly* like //\|\/ by Corollaries 17 and 18. In turn, it is easy to see that all lattice paths with precisely one such "down–jump path–segment" and no other jumps correspond to permutations with precisely one occurrence of $(3, 1, 2)$.

So we can immediately write down the generating function $F_{(3,1,2),\,1}(x)$ we are interested in:

$$F_{(3,1,2),\,1}(x) = \frac{1}{\sqrt{x}} \sum_{l=1}^{\infty} \left(c^{l+1} x^{l/2}\right)^2 x^{5/2} = \frac{c^4 x^3}{1 - c^2 x},$$

which coincides with (4). (Here, the summation index l is simply to be interpreted as the height of the path–segment //\|\/.) □

Now we turn to the proof of (8). The conditions of Lemma 19 imply, that in order *not* to have another left–to–right maximum "intersecting" our jump (and thus introducing at least a second occurrence of $\tau = (3, 2, 1)$), there must be at least $\max(1, l)$ down–steps after the left–to–right maximum preceding the jump, where l is the height of the "valley" preceding this left–to–right maximum — expressed in our graphical notation, it is easy to see that we must have precisely the following situation \l/ . k+2+z ./\. max(1,l)+z . \|\/ (where $z \geq 0$ is an arbitrary integer), at the end of which we are at height $(k + 1)$. Consider the generating function of the "lattice paths, starting from this path–segment up to the endpoint": For $l > 0$, it is given as

$$\sum_{l=1}^{\infty} \frac{x^{(k+5+l)/2}}{1 - x} c^{k+2} x^{(k+1)/2} = \frac{c^2 x^{(l+6)/2}}{(1 - cx)(1 - x)}.$$

(The summands are not simplified on purpose, in order to make "visible" the connection to the "path–segment".) For $l = 0$, we have

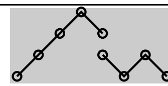
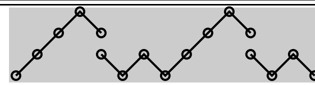
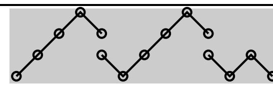
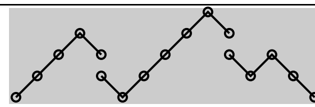
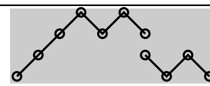
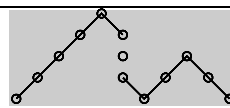
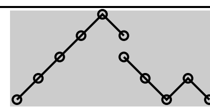
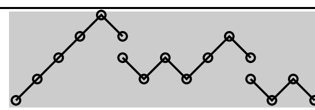
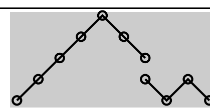
$$\frac{c^2 x^{7/2}}{(1 - cx)(1 - x)}.$$

Thus, we may write down immediately the desired generating function $F_{(3,2,1),\,1}(x)$:

$$F_{(3,2,1),\,1}(x) = \frac{1}{\sqrt{x}} \left(c \frac{c^2 x^{7/2}}{(1 - cx)(1 - x)} + \sum_{l=1}^{\infty} c^{l+2} x^{(l+1)/2} \frac{c^2 x^{(l+6)/2}}{(1 - cx)(1 - x)} \right) = \frac{c^3 x^3 \left(c^2 x - cx + 1\right)}{(1 - x)(1 - cx)^2},$$

which coincides with (8). □

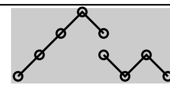
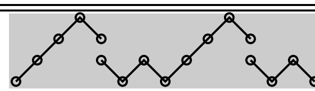
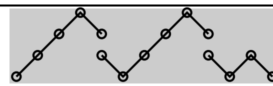
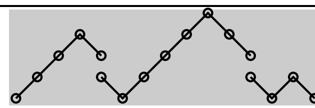
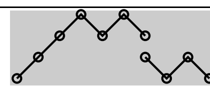
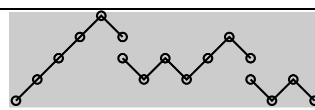
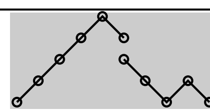
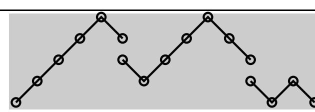
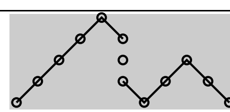
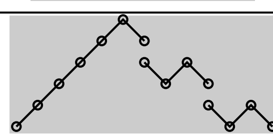
TABLE 1. 312–bases and corresponding Dyck–paths with jumps for one and two occurrences

Base ρ	$\psi_{312}(\rho)$	Reference in the text
One occurrence:		
312		
Two occurrences:		
312645		a
31524		b
316452		c
3412		d
4132		e
4213		f
423615		g
4312		h

3.4. **The case of two occurrences of length–3–patterns.** If a permutation contains exactly two occurrences of τ, then the corresponding lattice path $\psi_\rho(\tau)$ has one or two down–jumps (by Corollary 16). So we have to consider three cases for the "list of depths of jumps", $\mathcal{J}$:

- $\mathcal{J} = (1)$, i.e., there is precisely one jump of depth 1, (which is "responsible" for two τ–occurrences),
- $\mathcal{J} = (2)$, i.e., there is precisely one jump of depth 2, (which is "responsible" for two τ–occurrences),

Table 2. 321–bases and corresponding Dyck–paths with jumps for one and two occurrences

Base ρ	$\psi_{321}(\rho)$	Reference in the text
	One occurrence:	
321		
	Two occurrences:	
321654		a
32541		b
326154		c
3421		d
421653		e
4231		f
426153		g
4312		h
52143		i

- $\mathcal{J} = (1,1)$, i.e., there are precisely two jumps of depth 1 each (each of which is "responsible" for precisely one τ–occurrence).

As we shall see, the cases must be divided even further: However, we can make use of certain symmetries.

Definition 20. *For a permutation $\rho \in \mathcal{S}_n$, consider all entries that are involved in a τ–occurrence. Assume that the length of the subword σ formed by these entries is k. Then there is a permutation $\mu \in \mathcal{S}_k$ with entries "in the same relative order" as the the entries of σ: Call this permutation μ the τ–*base *of ρ.*

Clearly, the τ–base of ρ has the same number of occurrences as ρ itself. For permutations with exactly one τ–occurrence, the only possible τ–base is τ itself.

Remark 21. *This concept is closely related to the "kernel" defined by Mansour and Vainshtein: The kernel of a permutation in $\rho \in \mathcal{S}_n$ is the connected (in a certain sense) component of the τ–base containing the letter n; for the details of this powerful concept see* [11, Section 2].

Observation 22. *For all r, the τ–bases of permutations with exactly r τ–occurrences are of length $\leq 3r$: In particular, for r fixed there is only a finite set of possible τ–bases with r τ–occurrences.*

It is immediately clear that the number of τ–occurrences in some permutation ρ stays unchanged, if

- *The permutation graph $G(\rho)$ is reflected across the downwards–sloping diagonal for both $\tau = (3, 1, 2)$ and $\tau = (3, 2, 1)$,*
- *The permututation graph $G(\rho)$ is reflected across the upwards–sloping diagonal for $\tau = (3, 2, 1)$ only.*

Therefore, the generating function of permutations with some fixed τ–base is the same as the generating function of permutations with the "reflected" τ–base.

Tables 1 and 2 list all τ–bases for 1 and 2 occurrences of $(3, 1, 2)$ and $(3, 2, 1)$, respectively. The lists were compiled by a brute–force computer search.

3.4.1. *$\tau = (3, 1, 2)$, $r = 2$, $\mathcal{J} = (2)$.* By the same type of considerations as in our proof of (4), we observe that permutations with exactly one jump of depth 2 and exactly two occurrences of $\tau = (3, 1, 2)$ are in one–to–one correspondence to lattice paths, where the "path–segment" around the jump looks *exactly* like ///\‖\// . (The τ–base corresponding to these permutations is $(4, 1, 3, 2)$; see Table 1 e.) We can immediately write down the generating function of such lattice paths:

$$\sum_{l=1}^{\infty} \left(c^{l+1} x^{l/2}\right) x^{7/2} \left(c^{l+2} x^{(l+1)/2}\right) = \frac{c^5 x^5}{1 - c^2 x}. \qquad (15)$$

Here, the summation index l should be be interpreted as the height where the path–segment begins. The summands are not simplified on purpose, in order to make visible the "composition" of the paths.

3.4.2. $\tau = (3,1,2)$, $r = 2$, $\mathcal{J} = (1)$. Recall the considerations for the proof of (4): There are exactly three ways to obtain precisely two occurrences from a single jump of depth 1, namely

(1) There is a second "peak" immediately preceding the jump, i.e., we face path–segment //\/\|\/ (τ–base $(3,4,1,2)$, see Table 1 d),
(2) There is a second down–step immediately before the jump, i.e., we face path–segment //\\|\/ (τ–base $(4,3,1,2)$, see Table 1 h),
(3) There is a second down–step immediately after the jump, i.e., we face path–segment //\|\\/ (τ–base $(4,2,1,3)$, see Table 1 f).

The generating function for the permutations corresponding to the first case is $xF_{(3,1,2),1}(x)$: We simply have to "insert" another peak /\ immediately before the peak preceding the jump.

For the second case, note that the path–segment //\\|\/ can be easily transformed into the path–segment //\/\|\/ of the first case (by replacing a "peak" by a "valley"), hence the generating function is $xF_{(3,1,2),1}(x)$ again.

For the third case, note that the τ–base $(4,2,1,3)$, when reflected at the downwards–sloping diagonal, gives $(4,1,3,2)$, which corresponds to one jump of depth 2: So the generating function must be the same as in (15).

3.4.3. $\tau = (3,1,2)$, $r = 2$, $\mathcal{J} = (1,1)$. We need a simple generalization of (14): The generating function of all lattice paths which

- start at (a,k) for some $a \in \mathbb{Z}$ and $k \geq 0$,
- end at (b,l) for $l \geq k$ and $b \geq a$,
- and do not go below the horizontal axis

can be expressed in terms of c and x:

$$c_{k,l} := \sum_{h=0}^{k} x^{(l-k+2h)/2} c^{l-k+2h+1} = \frac{\left((c^2x)^{k+1} - 1\right) c^{l-k+1} x^{(l-k)/2}}{c^2x - 1}. \quad (16)$$

(To see this, interpret the summation index h as the *minimal* height reached by the respective path.)

Now it is easy to write down the generating function for the set of lattice paths which contain precisely two path–segments, connected by some arbitrary path–segment p:

//\|\/ ... some path p ... //\|\/

(the corresponding τ–bases: $(3,1,2,6,4,5)$, $(3,1,5,2,4)$, $(3,1,6,4,5,2)$ and $(4,2,3,6,1,5)$; see Table 1 a, b, c and g). Using (16), we obtain:

$$\sum_{l=1}^{\infty} c^{l+1} x^{l/2} x^{5/2} \left(\sum_{k=1}^{l} c_{k,l} x^{5/2} c^{k+1} x^{k/2} + \sum_{k=l+1}^{\infty} c_{l,k} x^{5/2} c^{k+1} x^{k/2} \right) = \frac{c^5 x^6 \left(1 + c^2 x - c^4 x^2\right)}{\left(1 - c^2 x\right)^3}. \tag{17}$$

3.4.4. *All together.* Putting together the results from sections 3.4.1, 3.4.2 and 3.4.3, and recalling (13), we get:

$$\begin{aligned} F_{(3,1,2),\,2}(x) &= 2x \frac{c^4 x^3}{1 - c^2 x} + \frac{2c^5 x^5}{x\left(1 - c^2 x\right)} + \frac{c^5 x^6 \left(1 + c^2 x - c^4 x^2\right)}{x\left(1 - c^2 x\right)^3} \\ &= \frac{c^4 x^4}{\left(1 - c^2 x\right)^3} \\ &\quad \times \left(2 + 2c + cx - 4c^2 x - 4c^3 x + c^3 x^2 + 2c^4 x^2 + 2c^5 x^2 - c^5 x^3\right), \end{aligned}$$

which coincides with (6). □

Now we turn to $\tau = (3,2,1)$.

3.4.5. $\tau = (3,2,1)$*,* $r = 2$*,* $\mathcal{J} = (2)$*.* By the same type of considerations as in our proof of (8), we observe that permutations with exactly one jump of depth 2 and exactly two occurrences of $\tau = (3,2,1)$ are in one–to–one correspondence to lattice paths, where the "path–segment" around the jump looks *exactly* like \l/ . k+2+z ./\. max(1,l)+z . \||\//, at the end of which we are at height $(k+2)$ (the τ–base corresponding to these permutations is $(4,3,1,2)$, see Table 2 h). Consider the generating function of the "lattice paths, starting from this path–segment up to the endpoint": For $l > 1$, it is given as

$$\sum_{k=0}^{\infty} \frac{x^{(k+4+l)/2}}{1 - x} c^{k+3} x^{(k+2)/2} = \frac{c^3 x^{(l+6)/2}}{(1 - cx)(1 - x)}.$$

For $l = 1$, we have

$$\frac{c^3 x^{9/2}}{(1 - cx)(1 - x)}.$$

For $l = 0$, we have

$$\frac{c^3 x^5}{(1 - cx)(1 - x)}.$$

Thus, we may write down immediately the desired generating function

$$\begin{aligned} c\frac{c^3 x^5}{(1 - cx)(1 - x)} + c^3 x^{3/2} \frac{c^3 x^{9/2}}{(1 - cx)(1 - x)} \\ + \sum_{l=2}^{\infty} c^{l+2} x^{(l+1)/2} x^{1/2} \frac{c^3 x^{(l+6)/2}}{(1 - cx)(1 - x)} \\ = \frac{c^4 x^5 \left(1 - cx + c^2 x + c^3 x - c^3 x^2\right)}{(1 - x)(1 - cx)^2}. \end{aligned} \tag{18}$$

3.4.6. $\tau = (3, 2, 1)$, $r = 2$, $\mathcal{J} = (1)$. Recall the considerations for the proof of (8): There are exactly two ways to obtain precisely two occurrences from a single jump of depth 1, namely

(1) There is a second "peak" which is "too near" before the jump (see Lemma 19; the corresponding τ–base is $(3, 4, 2, 1)$, see Table 2 d),
(2) There is a second down–step immediately after the jump, (τ–base $(4, 2, 3, 1)$, see Table 2 f).

For the first case, note that the τ–base $(3, 4, 2, 1)$, when reflected at the upwards–sloping diagonal, is mapped to $(4, 3, 1, 2)$: Hence the generating function of this case is the same as in (18).

The generating function for the permutations corresponding to the second case is $xF_{(3,2,1),\,1}(x)$: We simply have to "insert" one up–step before the jump and one down–step after the jump in the path–segment we considered in the proof of (8); i.e, we must consider

\l / . k+3+z ./\. max(1,l)+z . \|\\/

here.

3.4.7. $\tau = (3, 2, 1)$, $r = 2$, $\mathcal{J} = (1, 1)$. We want to determine the generating function for the set of lattice paths which do contain precisely the two path–segments in the following way

\l / . k+2+z ./\. max(1,l)+z . \|\/ ... some path p ... \L / . j+2+Z ./\. max(1,L)+Z . \|\/,

where p denotes some lattice path segment connecting the two path–segments.

The corresponding τ–bases are $(3, 2, 1, 6, 5, 4)$ and $(4, 2, 1, 6, 5, 3)$; see Table 2 a and e.

However, note that the "connecting path–segment" may be absent, i.e., we face the situation

\l / . k+2+z ./\. max(1,l)+z . \|\k+1 / . j+2+Z ./\. max(1,k+1)+Z . \|\/ .

$(3,2,5,4,1)$, $(3,2,6,1,5,4)$, $(4,2,6,1,5,3)$ and $(5,2,1,4,3)$ are the τ–bases corresponding to this case, see Table 2 b, c, g and i.

The same considerations as before finally lead to the generating function

$$\frac{c^3x^6\left(1-cx+c^2x\right)}{(1-x)^3\,(1-cx)^4} \times \left(2-x-2cx+c^2x+cx^2+c^3x^2+c^4x^2-c^3x^3-2c^4x^3+c^4x^4\right). \tag{19}$$

3.4.8. *All together.* So, adding the respective generating functions (19), divided by x, and 2 times (18), divided by x, and $\left(xF_{(3,2,1),1}(x)\right)$, we obtain

$$\frac{c^3x^4}{(1-x)^3\,(1-cx)^4} \times \big(1+2\,c-7\,c\,x-5\,c^2\,x+2\,c^3\,x+2\,c^4\,x+4\,c\,x^2+16\,c^2\,x^2-10\,c^4\,x^2-4\,c^5\,x^2 - c\,x^3-10\,c^2\,x^3-9\,c^3\,x^3+15\,c^4\,x^3+14\,c^5\,x^3+2\,c^6\,x^3+2\,c^2\,x^4+6\,c^3\,x^4 -7\,c^4\,x^4-16\,c^5\,x^4-5\,c^6\,x^4-c^3\,x^5+c^4\,x^5+7\,c^5\,x^5+4\,c^6\,x^5-c^5\,x^6-c^6\,x^6\big),$$

which after some simplification coincides with (10). □

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