Soliton solutions for the elastic metric on spaces of curves

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joint work with

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Abstract:
Some first order Sobolev metrics on spaces of curves admit soliton-like geodesics, i.e., geodesics whose momenta are sums of delta distributions. It turns out that these geodesics can be found within the submanifold of piecewise linear curves, which is totally geodesic for these metrics. Consequently, the geodesic equation reduces to a finite-dimensional ordinary differential equation for a dense set of initial conditions.

What are solitons?

From Wikipedia: Solitons

In mathematics and physics, a soliton is a self-reinforcing solitary wave packet that maintains its shape while it propagates at a constant velocity. Solitons are caused by a cancellation of nonlinear and dispersive effects in the medium. (The term "dispersive effects" refers to a property of certain systems where the speed of the waves varies according to frequency.) Solitons are the solutions of a widespread class of weakly nonlinear dispersive partial differential equations describing physical systems.

A single, consensus definition of a soliton is difficult to find. Drazin & Johnson (1989, p. 15) ascribe three properties to solitons:

- They are of permanent form;
- They are localized within a region;
- They can interact with other solitons, and emerge from the collision unchanged, except for a phase shift.

More formal definitions exist, but they require substantial mathematics.

Soliton solutions considered here

- Hamiltonian systems on (infinite dimensional) function spaces:

$$q_t = \partial_p H(q, p), \qquad p_t = -\partial_q H(q, p)$$

- Solitons are solutions whose momenta p are sums of delta distributions.

This is unprecise on many levels, but it captures the spirit.

Motivation

- Theoretical understanding
- Numerics

Soliton geodesics on diffeomorphism groups

Right-invariant Riemannian metric on $\mathrm{Diff}_c(M)$ generated by an inner product on $\mathfrak{X}_c(M)$:

$$G_{\mathrm{Id}}(h,k)\rangle = \int_{\mathbb{R}^n} \langle Lh, k\rangle dx, \quad L : X_c(M) \to X_c(M)'\,.$$

Geodesic equation (EPDiff) in terms of the momentum:

$$p(t) := G_\varphi(\varphi_t, \cdot) = \varphi_* p(0).$$

One can ask whether (generalized) solutions $u(t) = \varphi_t(t) \circ \varphi(t)^{-1}$ exist such that the momenta $\check{G}(u(t)) =: p(t)$ are distributions with finite support. The geodesic $\varphi(t)$ may exist only in some suitable Sobolev completion. Momentum
$\mathrm{Ad}(\varphi(t))^* p(t) = \varphi(t)^* p(t) = p(0)$ is constant, i.e.,
$p(t) = (\varphi(t)^{-1})^* p(0) = \varphi(t)_* p(0)$. i.e., the momentum is carried by the flow and stays a distributions with finite support. The infinitesimal version (take ∂_t of the last expression) is

$$p_t(t) = -\mathcal{L}_{u(t)} p(t) = -\,\mathrm{ad}_{u(t)}{}^* \, p(t).$$

Solitons for the Hunter-Saxton equation on $\mathrm{Diff}_c(\mathbb{R}) \rtimes \mathbb{R}$

Let $C_1^\infty(\mathbb{R}) := \{f \in C^\infty(\mathbb{R}) : f' \in C_c^\infty, f(-\infty) = 0\}$ and $\mathrm{Diff}_{c,1}(R) = \{\varphi = \mathrm{Id} + f : f \in C_1^\infty(\mathbb{R}) : f' > -1\}$ the corresponding regular Lie group. We use the right invariant metric

$$G_{\mathrm{Id}}^{\dot{H}^1}(X, Y) = \int_{\mathbb{R}} X'.Y'.\,dx = \int (-X'').Y.\,dx$$

Theorem. *The geodesic equation is the Hunter-Saxton equation*

$$(\varphi_t) \circ \varphi^{-1} = u \qquad u_t = -uu_x + \frac{1}{2}\int_{-\infty}^{x} (u_x(z))^2\,dz\ ,$$

and the induced geodesic distance is positive. The geodesic equation is locally well-posed.

One obtains the classical form of the Hunter-Saxton equation by differentiating: $u_{tx} = -uu_{xx} - \frac{1}{2}u_x^2$.

M.Bauer, M.Bruveris, P.Harms, P.W.Michor: Soliton solutions for the elastic metric on spaces of curves. Discrete and Continuous Dynamical Systems 38, 3 (March 2018)

On $\mathit{Diff}(S^1)/S^1$ see: J. Lenells: The Hunter-Saxton equation describes the geodesic flow on a sphere. J. Geometry and Physics, 57(10):2049-2064, 2007.

J. Lenells. The Hunter-Saxton equation: a geometric approach. SIAM J. Math. Anal., 40(1):266-277, 2008.

We define the R-map by:

$$R : \begin{cases} \mathrm{Diff}_{c,1}(\mathbb{R}) \to C_c^\infty\big(\mathbb{R}, \mathbb{R}_{>-2}\big) \subset C_c^\infty(\mathbb{R}, \mathbb{R}) \\ \varphi \mapsto 2\left((\varphi')^{1/2} - 1\right) . \end{cases}$$

The R-map is invertible with inverse

$$R^{-1} : \begin{cases} C_c^\infty\big(\mathbb{R}, \mathbb{R}_{>-2}\big) \to \mathrm{Diff}_{c,1}(\mathbb{R}) \\ \gamma \mapsto x + \dfrac{1}{4}\displaystyle\int_{-\infty}^{x} \gamma^2 + 4\gamma \, dx . \end{cases}$$

Theorem. *The pull-back of the flat L^2-metric via R is the $\dot{H}^1$-metric on* $\mathrm{Diff}_{c,1}(\mathbb{R})$*, i.e.,* $R^*\langle \cdot, \cdot \rangle_{L^2} = G^{\dot{H}^1}$*.*
Thus the space $\big(\mathrm{Diff}_{c,1}(\mathbb{R}), G^{\dot{H}^1}\big)$ *is a flat space in the sense of Riemannian geometry. It gives explicit formulas for geodesics.*

Soliton-Like Solutions of the Hunter Saxton equation

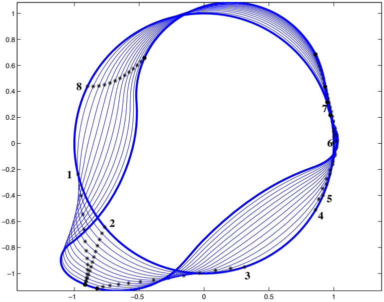
The space of N-solitons of order 0 consists of momenta of the form $p_{y,a} = \sum_{i=1}^{N} a_i \delta_{y_i}$ with $(y, a) \in \mathbb{R}^{2N}$. Consider an initial soliton $p_0 = \check{G}(u_0) = -u_0'' = \sum_{i=1}^{N} a_i \, \delta_{y_i}$ with $y_1 < y_2 < \cdots < y_N$. Let H be the Heaviside function (with $H(0) = 1/2$) and $D(x) = 0$ for $x \leq 0$ and $D(x) = x$ for $x > 0$. Then $u_0''(x) = -\sum_{i=1}^{N} a_i \delta_{y_i}(x)$, $u_0'(x) = -\sum_{i=1}^{N} a_i H(x - y_i)$, and $u_0(x) = -\sum_{i=1}^{N} a_i D(x - y_i)$. The geodesic with initial velocity u_0 is given by

$$\varphi(t, x) = x + \frac{1}{4} \int_{-\infty}^{x} t^2 (u_0'(y))^2 + 4t u_0'(y) \, dy$$

$$u(t, x) = u_0(\varphi^{-1}(t, x)) + \frac{t}{2} \int_{-\infty}^{\varphi^{-1}(t,x)} u_0'(y)^2 \, dy.$$

Solitons on $\mathrm{Diff}(S^1)/PSL(2,\mathbb{R})$

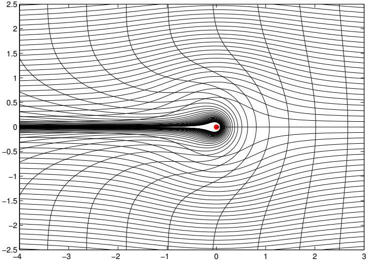
The metric is the Weil–Petersson metric (Sobolev $H^{3/2}$) on universal Teichmueller space, visualized as shapes using conformal welding. The solitons are called *Teichons* (as suggested by D. Holm).



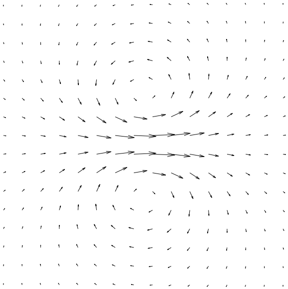
David Mumford, Eitan Sharon: 2D-Shape Analysis using Conformal Mapping, Int. J. of Computer Vision, 70, 2006, pp.55-75; preliminary version in Proc. IEEE Conf. Comp. Vision and Patt. Rec., 2004.
Kushnarev; Narayan: Approximating the Weil-Petersson metric geodesics on the universal Teichmller space by singular solutions. SIAM J. Imaging Sci. 7 (2014), no. 2, 900923.
Kushnarev: Teichons: solitonlike geodesics on universal Teichmller space. Experiment. Math. 18 (2009), no. 3, 325336.

Approximating Incompressible flow on $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$

The metric is (for $\varepsilon \to 0$ and later $\eta \to 0$)
$G^{\varepsilon}_{\mathrm{Id}}(v,v) = \int \langle v, v\rangle + \frac{1}{\varepsilon^2}\,\mathrm{div}(v).\,\mathrm{div}(v)dx = \int \langle L^\varepsilon v, v\rangle dx$, where
$L^\varepsilon = Id - \frac{1}{\varepsilon^2}\,\mathrm{grad}\,\mathrm{div}$, regularized as
$L^{\varepsilon,\eta} = (I - \frac{\eta^2}{p}\triangle)^p \circ (I - \frac{1}{\varepsilon^2}\,\mathrm{grad}\,\mathrm{div})$. The corresponding soliton solutions were called **vortons**.



The momentum moves uniformly in a straight line.



Momentum is transformed to vortex-like velocity field by kernel.

D.Mumford, P.W.Michor: On Euler's equation and 'EPDiff'. Journal of Geometric Mechanics 5, 3 (2013), 319-344.

Solitons for the Camassa-Holm equation and Landmark space as space of solitons

The Camassa-Holm equation is the geodesic equation on the group $(\mathrm{Diff}_c(\mathbb{R}), G^{H^1})$ for the dispersionfree version and on the Virasoro group $\mathrm{Diff}_c(\mathbb{R}) \rtimes \mathbb{R}$ for the version with dispersion. Both versions admit solitons, which are called **peakons**.

Landmark space with the Riemannian metric induced by LDDMM are solitons for the group $\mathrm{Diff}_c(\mathbb{R}^n)$ with the LDDMM-metric, given by a kernel.

KdV-solitons have a different origin. They do not correspond directly to momenta with finite support. Maybe the situation is similar as for the vortons mentioned above.

Holm, Darryl D.; Marsden, Jerrold E. Momentum: maps and measure-valued solutions (peakons, filaments, and sheets) for the EPDiff equation. In the book: The breadth of symplectic and Poisson geometry, 203235, Progr. Math., 232, Birkhäuser Boston, Boston, MA, 2005.

Mario Micheli, Peter W. Michor, David Mumford: Sectional curvature in terms of the cometric, with applications to the Riemannian manifolds of landmarks. SIAM J. Imaging Sci. 5, 1 (2012), 394-433.

Soliton geodesics in spaces of curves

Note to self: check time!

- Reparametrization-invariant Riemannian metric on $\mathrm{Imm}(S^1, \mathbb{R}^2)$:

$$G_c(h,k) = \int_{S^1} \langle L_c h, k \rangle \, \mathrm{d}s,$$

  where $\mathrm{d}s$ is integration with respect to arc length.
- Geodesic equation in terms of the momentum:

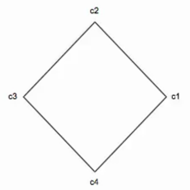
$$\begin{aligned} p &= \langle L_c c_t, \cdot \rangle \, \mathrm{d}s, \\ p_t &= \frac{1}{2} \Big(\mathrm{Adj}(\nabla L)(c_t, c_t)^\perp - 2Tc.\langle L_c c_t, \nabla c_t \rangle^\sharp - \langle L_c c_t, c_t \rangle H_c n_c \Big) \, \mathrm{d}s. \end{aligned}$$

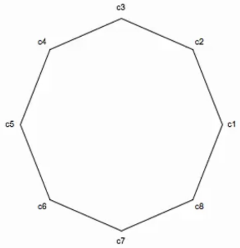
- Geodesics with sums of delta distributions as initial momenta are solitons **only for specific choices of L and c_0**.

Main result

Theorem

1. *Piecewise linear curves are a totally geodesic submanifold of the space of Lipschitz curves with the* $\dot{H}^1$ *metric.*
2. *Geodesics in this submanifold are solitons in the sense that their momenta are sums of delta distributions.*





M.Bauer, M.Bruveris, P.Harms, P.W.Michor: Soliton solutions for the elastic metric on spaces of curves. Discrete and Continuous Dynamical Systems 38, 3 (March 2018). Preprint `arxiv.org/abs/1702.04344`.

A first order metric on Lipschitz curves

Setting

- Lipschitz curves $W^{1,\infty} = W^{1,\infty}(S^1, \mathbb{R}^d)$
- Lipschitz immersions $\mathcal{I}^{1,\infty} = \mathcal{I}^{1,\infty}(S^1, \mathbb{R}^d)$
- Translations $\mathrm{Tra} \cong \mathbb{R}^d$

Agenda

- Scale- and reparametrization-invariant metric on $\mathcal{I}^{1,\infty}/\mathrm{Tra}$.
- Well-posedness of the geodesic equation.
- Piecewise linear curves are totally geodesic.

Lemma ($\dot{H}^1$ metric)

1. *The spaces* $\mathcal{I}^{1,\infty}$ *and* $\mathcal{I}^{1,\infty}/\mathrm{Tra}$ *are Banach manifolds.*
2. *The following is a smooth weak Riemannian metric on* $\mathcal{I}^{1,\infty}/\mathrm{Tra}$*:*

$$G_c(h,k) = \frac{1}{\ell_c}\int_{S^1} \langle D_s h, D_s k\rangle \,\mathrm{d}s\,, = \int_{S^1} \frac{1}{|c_\theta|}\langle h_\theta, k_\theta\rangle \,\mathrm{d}\theta \Big/ \int_{S^1} |c_\theta| \,\mathrm{d}\theta \ .$$

Proof.

1. There are explicit charts.
2. $\mathcal{I}^{1,\infty} \ni c \mapsto |c_\theta| \in L^\infty$ is smooth:
   - Quick proof: non-linear uniform boundedness thm. of convenient calculus.
   - Slow proof: definition of Fréchet derivatives.

□

Setting

- $W_0^{1,\infty} = \left\{h \in W^{1,\infty} : \int h \, \mathrm{d}\theta = 0\right\}$ and similarly for $\mathcal{I}_0^{1,\infty}$.
- $W_0^{0,\infty} = \left\{h \in W^{0,\infty} : \int h \, \mathrm{d}s = 0\right\}$; this depends on c.

Lemma (Arc-length derivative and its inverse)

For each $c \in \mathcal{I}_0^{1,\infty}$ the following diagram is commutative,

$$\begin{tikzcd}
W^{1,\infty} \arrow[r, two heads, "\pi_1"] \arrow[d, "D_s"] & W_0^{1,\infty} \arrow[r, hook, "\iota_1"] \arrow[d, shift left, "D_s"] & W^{1,\infty} \arrow[d, "D_s"] \\
W^{0,\infty} \arrow[r, two heads, "\pi_0"] & W_0^{0,\infty} \arrow[r, hook, "\iota_0"] \arrow[u, shift left, "D_s^{-1}"] & W^{0,\infty},
\end{tikzcd}$$

where π_0 is the $L^2(\mathrm{d}s)$-orthogonal projection, π_1 is the $L^2(\mathrm{d}\theta)$-orthogonal projection, and ι_0 and ι_1 are inclusions.

Theorem (Well-posedness of the geodesic equation)

1. *The geodesic equation on* $\mathcal{I}^{1,\infty}/\mathrm{Tra} \cong \mathcal{I}_0^{1,\infty}$ *exists and is given by*

$$c_{tt} = G_c(c, c_t)c_t - \frac{1}{2}G_c(c_t, c_t)c + D_s^{-1}\pi_0\left(\langle D_s c, D_s c_t\rangle D_s c_t - \frac{1}{2}|D_s c_t|^2 D_s c\right).$$

2. *The geodesic equation is locally well-posed, and the exponential map is a local diffeomorphism.*

Remark. Previously known only for $\mathcal{I}^{k,2}$, $k > 5/2$.

Proof.

1. Variational calculus; exponential law of convenient calculus.
2. The geodesic spray is a smooth vector field on $T\mathcal{I}_0^{1,\infty}$.

$\square$

Solitons on the submanifold of piecewise linear curves

Setting

- Grid $0 = \theta^1 < \ldots < \theta^{n+1} = 2\pi$ on $S^1 \cong \mathbb{R}/(2\pi\mathbb{Z})$
- Piecewise linear curves $\mathcal{P}W^{1,\infty}$ and immersions $\mathcal{P}\mathcal{I}^{1,\infty}$
- Piecewise constant left-continuous curves $\mathcal{P}L^\infty$.

Agenda

- Piecewise linear curves are totally geodesic.
- Comparison to landmark spaces.

Theorem

1. $\mathcal{PI}^{1,\infty}/\mathrm{Tra}$ *is a totally geodesic submanifold of* $\mathcal{I}^{1,\infty}/\mathrm{Tra}$.
2. *Geodesics in this submanifold are solitons.*

Proof.

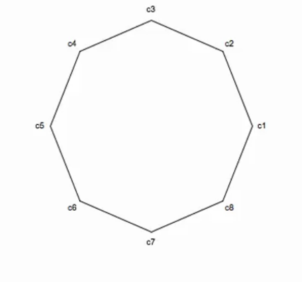
1. If c and c_t are piecewise linear, then c_{tt} is piecewise linear as well:

$$
\begin{aligned}
c_{tt} = {} & G_c(c, c_t)c_t - \frac{1}{2} G_c(c_t, c_t)c \\
& + D_s^{-1}\pi_0\left(\langle D_s c, D_s c_t\rangle D_s c_t - \frac{1}{2}|D_s c_t|^2 D_s c\right).
\end{aligned}
$$

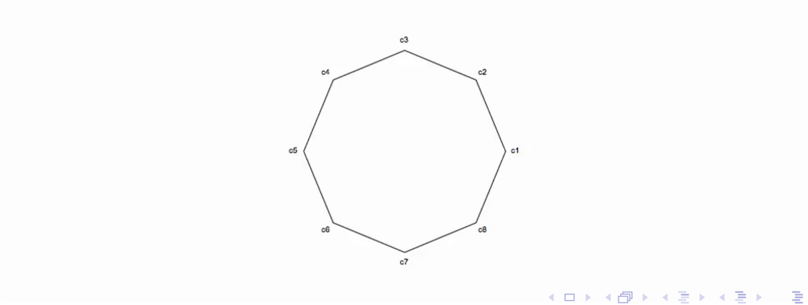
2. The velocity c_t is piecewise linear iff the momentum $G_c(c_t, \cdot)$ is a sum of delta distributions.

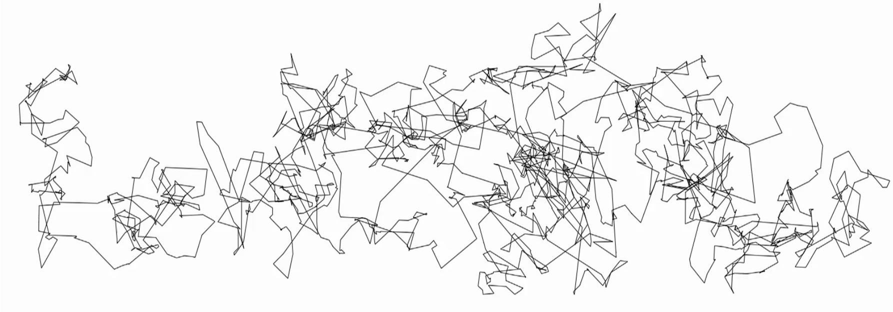
$\square$

Interactions between adjacent solitons under the $\dot{H}^1$ metric:



Interactions between all solitons under the LDDMM metric:





THANK YOU

Many thanks to the organizers for this great conference!