General Sobolev metrics on the manifold of all Riemannian metrics

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Workshop:
Shape analysis and computational anatomy
Isaac Newton Institute,
Cambridge, UK

Mon 13th Nov 2017 - Fri 17th Nov 2017

Based on collaborations with: M. Bauer, M. Bruveris, P. Harms

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For a compact manifold M^m equipped with a smooth fixed background Riemannian metric $\hat{g}$ we consider the space $\mathrm{Met}_{H^s}(M)$ of all Riemannian metrics of Sobolev class H^s for real $s < \frac{m}{2}$ with respect to $\hat{g}$. The L^2-metric on $\mathrm{Met}_{C^\infty}(M)$ was considered by DeWitt, Ebin, Freed and Groisser, Gil-Medrano and Michor, Clarke. Sobolev metrics of integer order on $\mathrm{Met}_{C^\infty}(M)$ were considered in [M.Bauer, P.Harms, and P.W. Michor: Sobolev metrics on the manifold of all Riemannian metrics. J. Differential Geom., 94(2):187-208, 2013.] In this talk we consider variants of these Sobolev metrics which include Sobolev metrics of any positive real (not integer) order $s < \frac{m}{2}$. We derive the geodesic equations and show that they are well-posed under some conditions and induce a locally diffeomorphic geodesic exponential mapping.

The diagram

$$
\begin{tikzcd}
\mathrm{Diff}(M) \arrow[rr, "\text{r-acts}"] \arrow[dr, "\text{r-acts}"] \arrow[dd, two heads, "\mathrm{Diff}(M,\mu)"] & & \mathrm{Imm}(M,N) \arrow[dl, "\text{needs } \bar g"] \arrow[dr, two heads, "\mathrm{Diff}(M)"'] & & \mathrm{Diff}_{\mathcal{A}}(N) \arrow[ll, "\text{l-acts}"', "\text{(LDDMM)}"] \arrow[dl, "\text{l-acts}"', "\text{(LDDMM)}"] \arrow[dd, "\text{r-acts}"] \\
& \mathrm{Met}(M) \arrow[dr, two heads, "\mathrm{Diff}(M)"'] & & B_i(M,N) \arrow[dl, "\text{needs } \bar g"] & \\
\mathrm{Vol}^1_+(M) & & \dfrac{\mathrm{Met}(M)}{\mathrm{Diff}(M)} & & \mathrm{Met}_{\mathcal{A}}(N)
\end{tikzcd}
$$

M compact , N possibly non-compact manifold

$\mathrm{Met}(N) = \Gamma(S^2_+ T^* N)$	space of all Riemann metrics on N
$\bar g$	one Riemann metric on N
$\mathrm{Diff}(M)$	Lie group of all diffeos on compact mf M
$\mathrm{Diff}_{\mathcal{A}}(N),\ \mathcal{A} \in \{H^\infty, \mathcal{S}, c\}$	Lie group of diffeos of decay $\mathcal{A}$ to Id_N
$\mathrm{Imm}(M, N)$	mf of all immersions $M \to N$
$B_i(M, N) = \mathrm{Imm}/\mathrm{Diff}(M)$	shape space
$\mathrm{Vol}^1_+(M) \subset \Gamma(\mathrm{vol}(M))$	space of positive smooth probability densities

Sobolev spaces of sections of vector bundles

For each $s \in \mathbb{R}$ we write $H^s(\mathbb{R}^m, \mathbb{R}^n)$ for the Sobolev space of order s of real-valued functions on $\mathbb{R}^m$, described via Fourier transform $\|f\|_{H^s} = \|\hat{f}(\xi)(1 + |\xi|^2)^{s/2}\|_{L^2}$.

Let E be a vector bundle of rank $n \in \mathbb{N}_{>0}$ over M. We choose a finite vector bundle atlas and a subordinate partition of unity in the following way: Let $(u_\alpha : U_\alpha \to u_\alpha(U_\alpha) \subseteq \mathbb{R}^m)_{\alpha \in A}$ be a finite atlas for M, let $(\varphi_\alpha)_{\alpha \in A}$ be a smooth partition of unity subordinated to $(U_\alpha)_{\alpha \in A}$, and let $\psi_\alpha : E|U_\alpha \to U_\alpha \times \mathbb{R}^n$ be vector bundle charts.

Note that we can choose open sets U_α° such that $\mathrm{supp}(\psi_\alpha) \subset U_\alpha^\circ \subset \overline{U_\alpha^\circ} \subset U_\alpha$ such that each $u_\alpha(U_\alpha^\circ)$ is an open set in $\mathbb{R}^m$ with Lipschitz boundary.

A. Behzadan and M. Holst. On certain geometric operators between Sobolev spaces of sections of tensor bundles on compact manifolds equipped with rough metrics, 2017.

Then we define for each $s \in \mathbb{R}$ and $f \in \Gamma_{C^\infty}(E)$

$$\|f\|^2_{\Gamma_{H^s}(E)} := \sum_{\alpha \in A} \| \mathrm{pr}_{\mathbb{R}^n} \circ \psi_\alpha \circ (\varphi_\alpha \cdot f) \circ u_\alpha^{-1} \|^2_{H^s(\mathbb{R}^m, \mathbb{R}^n)}.$$

Then $\| \cdot \|_{\Gamma_{H^s}(E)}$ is a norm, which comes from a scalar product, and we write $\Gamma_{H^s}(E)$ for the Hilbert completion of $\Gamma_{C^\infty}(E)$ under the norm. Then $\Gamma_{H^s}(E)$ is independent of the choice of atlas and partition of unity, up to equivalence of norms.

C. Schneider and N. Grosse. Sobolev spaces on Riemannian manifolds with bounded geometry: General coordinates and traces, 2013

H. Triebel. Theory of functions spaces II

About Met(M)

Let $\mathrm{Met}(M) = \Gamma(S^2_+ T^* M)$ be the space of all smooth Riemannian metrics on a compact manifold M.

Let $\mathrm{Met}_{H^s}(M) = \Gamma_{H^s}(S^2_+ T^* M)$ the space of all Sobolev H^s sections of the bundle of Riemannian metrics, where $s > \frac{m}{2} = \frac{\dim(M)}{2}$; by the Sobolev inequality then it makes sense to speak of positive definite metrics.

Weak Riemann metrics on $\mathrm{Met}(M)$

All of them are $\mathrm{Diff}(M)$-invariant; natural, tautological.

$$
\begin{aligned}
G_g(h,k) &= \int_M g_2^0(h,k)\,\mathrm{vol}(g) = \int \mathrm{Tr}(g^{-1}hg^{-1}k)\,\mathrm{vol}(g), \quad L^2\text{-metr.}\\
\text{or } &= \Phi(\mathrm{Vol(g)}) \int_M g_2^0(h,k)\,\mathrm{vol}(g) \qquad \text{conformal}\\
\text{or } &= \int_M \Phi(\mathrm{Scal}^g).g_2^0(h,k)\,\mathrm{vol}(g) \qquad \text{curvature modified}\\
\text{or } &= \int_M \Big(g_2^0(h,k) + g_3^0(\nabla^g h, \nabla^g k) + \dots + g_p^0((\nabla^g)^p h, (\nabla^g)^p k)\Big)\,\mathrm{vol}(g)\\
\text{or } &= \int_M g_2^0((1+\Delta^g)^p h, k)\,\mathrm{vol}(g) \qquad \text{Sobolev order } p \in \mathbb{R}_{>0}\\
\text{or } &= \int_M g_2^0\Big(f(1+\Delta^g)h, k\Big)\,\mathrm{vol}(g)
\end{aligned}
$$

where $\Phi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, $\mathrm{Vol} = \int_M \mathrm{vol}(g)$ is total volume of (M,g), Scal is scalar curvature, and g_2^0 is the induced metric on $\binom{0}{2}$-tensors. Here f is a suitable spectral function; see below.

$\Delta^g h := (\nabla^g)^{*,g} \nabla^g h = -\operatorname{Tr}^{g^{-1}}((\nabla^g)^2 h)$ is the Bochner-Laplacian. It can act on all tensor fields h, and it respects the degree of the tensor field it is acting on.

We consider Δ^g as an unbounded self-adjoint positive semidefinite operator on the Hilbert space H^0 with compact resolvent. The domain of definition of Δ^g is the space

$$H^2 = H^{2,g} := \{h \in H^0 : (1 + \Delta^g)h \in H^0\} = \{h \in H^0 : \Delta^g h \in H^0\}$$

which is again a Hilbert space with inner product

$$\int_M g_2^0((1 + \Delta^g)h, k) \operatorname{vol}(g).$$

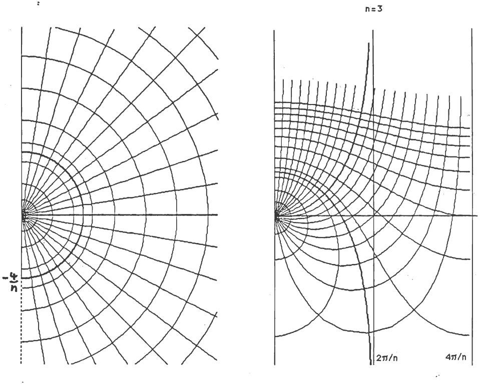
Again H^2 does not depend on the choice of g, but the inner products for different g induce different but equivalent norms on H^2. Similarly we have

$$\begin{aligned} H^{2k} = H^{2k,g} :&= \{h \in H^0 : (1 + \Delta^g)^k h \in H^0\} \\ &= \{h \in H^0 : \Delta^g h, (\Delta^g)^2, \dots (\Delta^g)^k \in H^0\} \end{aligned}$$

The L^2-metric on the space of all Riemann metrics

[DeWitt 1969]. [Ebin 1970]. Geodesics and curvature [Freed Groisser 1989]. [Gil-Medrano Michor 1991] for non-compact M. [Clarke 2009] showed that geodesic distance for the L^2-metric is positive, and he determined the metric completion of $\mathrm{Met}(M)$. The geodesic equation is completely decoupled from space, it is an ODE:

$$g_{tt} = g_t g^{-1} g_t + \tfrac{1}{4}\operatorname{Tr}(g^{-1} g_t g^{-1} g_t)\, g - \tfrac{1}{2}\operatorname{Tr}(g^{-1} g_t)\, g_t$$



$$
\begin{aligned}
A = g^{-1}a \quad & \text{for } a \in T_g\mathrm{Met}(M) \\
\exp_0(A) = \tfrac{2}{n}\log\left(\left(1+\tfrac{1}{4}\,\mathrm{Tr}(A)\right)^2 + \tfrac{n}{16}\,\mathrm{Tr}(A_0^2)\right) Id & \\
+ \frac{4}{\sqrt{n\,\mathrm{Tr}(A_0^2)}}\arctan\left(\frac{\sqrt{n\,\mathrm{Tr}(A_0^2)}}{4+\mathrm{Tr}(A)}\right) A_0. &
\end{aligned}
$$

Back to the the general metric on $\mathrm{Met}(M)$.

We describe all these metrics uniformly as

$$\begin{aligned} G_g^P(h,k) &= \int_M g_2^0(P_g h, k)\,\mathrm{vol}(g) \\ &= \int_M \mathrm{Tr}(g^{-1}.P_g(h).g^{-1}.k)\,\mathrm{vol}(g), \end{aligned}$$

where

$$P_g : \Gamma(S^2 T^* M) \to \Gamma(S^2 T^* M)$$

is a positive, symmetric, bijective pseudo-differential operator of order $2p$, $p \geq 0$, depending smoothly on the metric g, and also $\mathrm{Diff}(M)$-equivariantly:

$$\varphi^* \circ P_g = P_{\varphi^* g} \circ \varphi^*$$

The geodesic equation in this notation:

$$g_{tt} = P^{-1}\Big[(D_{(g,.)}Pg_t)^*(g_t) + \frac{1}{4}.g.\operatorname{Tr}(g^{-1}.Pg_t.g^{-1}.g_t) + \frac{1}{2}g_t.g^{-1}.Pg_t + \frac{1}{2}Pg_t.g^{-1}.g_t - (D_{(g,g_t)}P)g_t - \frac{1}{2}\operatorname{Tr}(g^{-1}.g_t).Pg_t\Big]$$

We can rewrite this equation to get it in a slightly more compact form:

$$\begin{aligned}(Pg_t)_t &= (D_{(g,g_t)}P)g_t + Pg_{tt}\\ &= (D_{(g,.)}Pg_t)^*(g_t) + \frac{1}{4}.g.\operatorname{Tr}(g^{-1}.Pg_t.g^{-1}.g_t)\\ &\quad + \frac{1}{2}g_t.g^{-1}.Pg_t + \frac{1}{2}Pg_t.g^{-1}.g_t - \frac{1}{2}\operatorname{Tr}(g^{-1}.g_t).Pg_t\end{aligned}$$

Conserved Quantities on $\mathrm{Met}(M)$.

Right action of $\mathrm{Diff}(M)$ on $\mathrm{Met}(M)$ given by

$$(g, \phi) \mapsto \phi^* g.$$

Fundamental vector field (infinitesimal action):

$$\zeta_X(g) = \mathcal{L}_X g = -2\,\mathrm{Sym}\,\nabla(g(X)).$$

If metric G^P is invariant, we have the following conserved quantities

$$\begin{aligned} \text{const} &= G^P(g_t, \zeta_X(g)) \\ &= -2 \int_M g_1^0\big(\nabla^*\,\mathrm{Sym}\,P g_t, g(X)\big)\,\mathrm{vol}(g) \\ &= -2 \int_M g\big(g^{-1}\nabla^* P g_t, X\big)\,\mathrm{vol}(g) \end{aligned}$$

Since this holds for all vector fields X,

$$\boxed{(\nabla^* P g_t)\,\mathrm{vol}(g) \in \Gamma(T^*M \otimes_M \mathrm{vol}(M)) \text{ is const. in } t.}$$

On $\mathbb{R}^n$: The pullback of the Ebin metric to $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$

We consider here the right action
$r : \mathrm{Met}_{\mathcal{A}}(\mathbb{R}^n) \times \mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n) \to \mathrm{Met}_{\mathcal{A}}(\mathbb{R}^n)$ which is given by
$r(g, \varphi) = \varphi^* g$, together with its partial mappings
$r(g, \varphi) = r^\varphi(g) = r_g(\varphi) = \mathrm{Pull}^g(\varphi)$.

Theorem. *If $n \geq 2$, the image of $\mathrm{Pull}^{\bar{g}}$, i.e., the $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$-orbit through $\bar{g}$, is the set $\mathrm{Met}^{\mathrm{flat}}_{\mathcal{A}}(\mathbb{R}^n)$ of all flat metrics in $\mathrm{Met}_{\mathcal{A}}(\mathbb{R}^n)$.*

The pullback of the Ebin metric to the diffeomorphism group is a right invariant metric G given by

$$G_{\mathrm{Id}}(X, Y) = 4 \int_{\mathbb{R}^n} \mathrm{Tr}\left((\mathrm{Sym}\, dX).(\mathrm{Sym}\, dY)\right) dx = \int_{\mathbb{R}^n} \langle X, PY \rangle dx$$

Using the inertia operator P we can write the metric as
$\int_{\mathbb{R}^n} \langle X, PY \rangle dx$, with

$$P = -2(\mathrm{grad}\,\mathrm{div} + \Delta)\,.$$

The pullback of the general metric to $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$

We consider now a weak Riemannian metric on $\mathrm{Met}_{\mathcal{A}}(\mathbb{R}^n)$ in its general form

$$G^P_g(h,k) = \int_M g^0_2(P_g h, k)\,\mathrm{vol}(g) = \int_M \mathrm{Tr}(g^{-1}.P_g(h).g^{-1}.k)\,\mathrm{vol}(g),$$

where $P_g : \Gamma(S^2T^*M) \to \Gamma(S^2T^*M)$ is as described above. *If the operator P is equivariant for the action of $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$ on $\mathrm{Met}_{\mathcal{A}}(\mathbb{R}^n)$, then the induced pullback metric $(\mathrm{Pull}^{\bar{g}})^*G^P$ on $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$ is right invariant:*

$$G_{\mathrm{Id}}(X,Y) = -4\int_{\mathbb{R}^n} \partial_j(P_{\bar{g}}\,\mathrm{Sym}\,dX)^i_j.Y^i\,dx \tag{1}$$

Thus we we get the following formula for the corresponding inertia operator $(\tilde{P}X)^i = \sum_j \partial_j(P_{\bar{g}}\,\mathrm{Sym}\,dX)^i_j$. Note that the pullback metric $(\mathrm{Pull}^{\bar{g}})^*G^P$ on $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$ is always of one order higher then the metric G^P on $\mathrm{Met}_{\mathcal{A}}(\mathbb{R}^n)$.

The Sobolev metric of order p.

The Sobolev metric G^P

$$G^P_g(h,k) = \int_{\mathbb{R}^n} \mathrm{Tr}(g^{-1}.((1+\Delta)^p h).g^{-1}.k)\,\mathrm{vol}(g)\,.$$

The pullback of the Sobolev metric G^P to the diffeomorphism group is a right invariant metric G given by

$$G_{\mathrm{Id}}(X,Y) = -2\int_{\mathbb{R}^n} \Big\langle (\mathrm{grad}\,\mathrm{div} + \Delta)(1-\Delta)^p X, Y \Big\rangle dx\,.$$

Thus the inertia operator is given by

$$\tilde{P} = -2(1-\Delta)^p(\Delta + \mathrm{grad}\,\mathrm{div}) = -2(1-\Delta)^p(\Delta + \mathrm{grad}\,\mathrm{div})\,.$$

It is a linear isomorphism $H^s(\mathbb{R}^n)^n \to H^{s-2p-2}(\mathbb{R}^n)^n$ for every s.

Theorem

Module properties of Sobolev spaces.
Let E_1, E_2 be vector bundles over M, and let $s_1, s_2, s \in \mathbb{R}$ satisfy

(i) $s_1 + s_2 \geq 0$, $\min(s_1, s_2) \geq s$, *and* $s_1 + s_2 - s > \frac{m}{2}$, *or*

(ii) $s \in \mathbb{N}$, $\min(s_1, s_2) > s$, *and* $s_1 + s_2 - s \geq \frac{m}{2}$, *or*

(iii) $-s_1 \in \mathbb{N}$ *or* $-s_2 \in \mathbb{N}$, $s_1 + s_2 > 0$, $\min(s_1, s_2) > s$, $s_1 + s_2 - s \geq \frac{m}{2}$.

Then the tensor product of smooth sections extends to a bounded bilinear mapping

$$\Gamma_{H^{s_1}}(E_1) \times \Gamma_{H^{s_2}}(E_2) \to \Gamma_{H^s}(E_1 \otimes E_2).$$

A. Behzadan and M. Holst. On certain geometric operators between Sobolev spaces of sections of tensor bundles on compact manifolds equipped with rough metrics, 2017.

The module properties are invariant under multiplication and adjoints. Indeed, letting $p(s_1, s)$ denote the set of all s_2 such that (s_1, s_2, s) satisfies condition (i), (ii), or (iii) of theTheorem above, then the following statements hold for all $r, s, t \in \mathbb{R}$:

- If $\alpha \in p(r, s)$ and $\beta \in p(s, t)$, then $\min(\alpha, \beta) \in p(r, t)$, and the tensor product of smooth sections extends to a bounded bilinear mapping

$$\Gamma_{H^\alpha}(E_1) \times \Gamma_{H^\beta}(E_2) \to \Gamma_{H^{\min(\alpha,\beta)}}(E_1 \otimes E_2).$$

- If $\beta \in p(r, s)$, then $\beta \in p(-s, -r)$.

Metrics of Sobolev order

For any $\alpha \in (\frac{m}{2}, \infty]$, we define the space of Riemannian metrics of Sobolev order α as

$$\mathrm{Met}_{H^\alpha}(M) := \Gamma_{H^\alpha}(S^2_+ T^* M).$$

This is well-defined because the condition $\alpha > \frac{m}{2}$ ensures that the tensors in $\Gamma_{H^\alpha}(S^2 T^* M)$ are continuous and that $\mathrm{Met}_{H^\alpha}(M)$ is an open subset of $\Gamma_{H^\alpha}(S^2 T^* M)$. Similarly, fiber metrics on vector bundles E are defined as elements of $\Gamma_{H^\alpha}(S^2_+ E^*)$.

Lemma *Let $g \in \mathrm{Met}_{H^\alpha}(M)$ with $\alpha > \frac{m}{2}$ and let E be a natural first order vector bundle over M. Then ∇^g extends to a bounded linear mapping*

$$\nabla^g \colon \Gamma_{H^s}(E) \to \Gamma_{H^{s-1}}(T^* M \otimes E).$$

for each $s \in (-\infty, \alpha]$.

Proof of the lemma

This follows from the module properties: The differential operator ∇^g can be written in each vector bundle chart of E as

$$\nabla^g = \sum_{i=1}^{m} a^i \partial_{x_i} + a,$$

where $a^i \in C^\infty(\mathbb{R}^m, \mathbb{R}^{n\times n})$ and $a \in H^{\alpha-1}(\mathbb{R}^m, \mathbb{R}^{n\times n})$. This can be seen in several equivalent ways. We only consider the case $E = TM$ because the general case follows by multilinear algebra.

(1) Using the Levi-Civita covariant derivative $\nabla^{\hat{g}}$ for a smooth background Riemannian metrig $\hat{g}$, we express the Levi-Civita connection of $g \in \mathrm{Met}_{H^\alpha}(M)$ as

$$\nabla^g_X = \nabla^{\hat{g}}_X + A^g(X, \quad)$$

for a suitable
$A^g \in \Gamma_{H^{\alpha-1}}(T^*M \otimes T^*M \otimes TM) = \Gamma_{H^{\alpha-1}}(T^*M \otimes L(TM, TM))$.
This tensor field A has to satisfy the following conditions (for smooth vector fields X, Y, Z):

$$(\nabla^{\hat{g}}_X g)(Y, Z) = g(A(X, Y), Z) + g(Y, A(X, Z)) \iff \nabla^g_X g = 0,$$
$$A(X, Y) = A(Y, X) \iff \nabla^g \quad \text{is torsionfree.}$$

We take the cyclic permutations of the first equation, sum them with signs $+, +, -$, and use symmetry of A to obtain

$$2g(A(X, Y), Z) = (\nabla^{\hat{g}}_X g)(Y, Z) + (\nabla^{\hat{g}}_Y g)(Z, X) - (\nabla^{\hat{g}}_Z g)(X, Y)\,;$$

this equation determines A uniquely as a $H^{\alpha-1}$-tensor field. It is easy checked that it satisfies the two requirements above.

(2) For each local chart $u : U \to \mathbb{R}^m$ which extends to a compact neighborhood of $U \subset M$, the Christoffel forms are given by the usual formula

$$\Gamma^k_{ij} = \frac{1}{2}\sum_l g^{kl}\Big(\frac{\partial g_{ij}}{\partial u^l} - \frac{\partial g_{lj}}{\partial u^i} - \frac{\partial g_{il}}{\partial u^j}\Big) \in H^{\alpha-1}(U, \mathbb{R})\,.$$

They transform as the last part in the second tangent bundle. The associated spray S^g is an $H^{\alpha-1}$-section of both $\pi_{TM} : T^2M \to TM$ and $T(\pi_M) : T^2M \to TM$. If $\alpha > \frac{\dim(M)}{2} + 1$, then the spray S^g is continuous and we have local existence (but not uniqueness) of geodesics in each chart separately, by Peano's theorem. If $\alpha > \frac{\dim(M)}{2} + 2$, we have the usual existence and uniqueness of geodesics, by Picard-Lindelöf, since then S^g is C^1 and thus Lipschitz.

(3) $\nabla^g : (X, Y) \mapsto \nabla^g_X Y$ is a bilinear bounded mapping

$$\Gamma_{H^\alpha}(TM) \times \Gamma_{H^l}(TM) \to \Gamma_{H^{l-1}}(TM) \quad \text{for } 1 \le l \le \alpha;$$

we write $\nabla^g \in L^2(\Gamma_{H^\alpha}(TM), \Gamma_{H^l}(TM); \Gamma_{H^{l-1}}(TM))$ to express this fact. Moreover, ∇^g has the expected properties

$$\begin{aligned} \nabla^g_{fX} Y &= f \nabla^g_X Y && \text{for } f \in H^\alpha(M, \mathbb{R}), \\ \nabla^g_X (fY) &= df(X)Y + f \nabla^g_X Y && \text{for } f \in H^\alpha(M, \mathbb{R}). \end{aligned}$$

Its expression in a local chart is

$$\nabla_{X^i \partial_{u^i}} Y^j \partial_{u^j} = X^i (\partial_{u^i} Y^j) \partial_{u^j} - X^i Y^j \Gamma^k_{ij} \partial_{u^k}.$$

The global implicit equation holds for $X, Z \in \Gamma_{H^\alpha}(TM)$ and $Y \in \Gamma_{H^l}(TM)$ for $\frac{m}{4} + \frac{1}{2} < l \le \alpha$:

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &\quad - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]). \end{aligned}$$

Note that $[X, Y], [Z, Y] \in \Gamma_{H^{l-1}}(TM)$, by differentiation and the module properties of Sobolev spaces.

Theorem The Bochner Laplacian

Let $g \in \mathrm{Met}_{H^\alpha}(M)$ for $\alpha \in (\frac{m}{2}, \infty]$, let E be a natural first order vector bundle over M, and let $s \in [2-\alpha, \alpha]$. Then the Bochner-Laplacian Δ^g extends to a bounded Fredholm operator of index zero

$$\Delta^g \colon \Gamma_{H^s}(E) \to \Gamma_{H^{s-2}}(E),$$

which is self-adjoint as an unbounded linear operator on $\Gamma_{H^{s-2}}(E)$ for the inner product $H^{s-2}(g)$. For functions the Laplacian extends to a bounded Fredholm operator of index zero

$$\Delta^g \colon H^s(M, \mathbb{R}) \to H^{s-2}(M, \mathbb{R}),$$

even for $s \in [2-\alpha, \alpha+1]$.

The proof involves elliptic estimates from the following papers, where one also finds similar statements for more general differential operators with Sobolev coefficients:

O. Müller. Applying the index theorem to non-smooth operators. Journal of Geometry and Physics, 116:140-145, 2017. Theorem 2.

M. Holst, G. Nagy, and G. Tsogtgerel. Rough solutions of the Einstein constraints on closed manifolds without near-CMC conditions. Communications in Mathematical Physics, 288(2):547-613, 2009. Lemma 34.

Theorem Fractional powers of the Laplacian

Let $g \in \mathrm{Met}_{H^\alpha}(M)$ for $\alpha \in (\frac{m}{2}, \infty]$, and let E be a natural first order vector bundle over M. Then the following statements hold:
(1) There exists a $L^2(g)$-orthonormal basis $(e_i)_{i\in\mathbb{N}}$ of $\Gamma_{H^0}(E)$ and a non-decreasing sequence $(\lambda_i)_{i\in\mathbb{N}}$ of positive real numbers converging to ∞ such that it holds for each $i \in \mathbb{N}$ that
$e_i \in \Gamma_{H^2}(E)$ and $(1+\Delta^g)e_i = \lambda_i e_i$.
(2) For each function $f\colon \{\lambda_1, \lambda_2, \dots\} \to \mathbb{R}$ the linear operator

$$\Gamma_{H^0}(E) \supset D(f(1+\Delta^g)) \xrightarrow{f(1+\Delta^g)} \Gamma_{H^0}(E), \quad h \mapsto \sum_{i\in\mathbb{N}} \langle h_i, e_i \rangle f(\lambda_i) e_i$$

domain $D(f(1+\Delta^g)) = \{h \in \Gamma_{H^0}(E); \sum_{i\in\mathbb{N}} \langle h_i, e_i\rangle^2 f(\lambda_i)^2 < \infty\}$ is densely defined and self-adjoint with respect to $L^2(g)$.
(3) Let $\alpha \geq 2$. If $s \in [0, \alpha]$ then the Hilbert space $D((1+\Delta^g)^s)$ with

$$\|\cdot\|_{D((1+\Delta^g)^s)} = \|(1+\Delta^g)^s(\cdot)\|_{\Gamma_{H^0}(E)}$$

is equal to $\Gamma_{H^s}(E)$ up to equivalence of norms. If $E = \mathbb{R}$ then the statement holds for $s \in [0, \alpha + 1]$.

A message from convenient analysis

Theorem Let E be a vector bundle over M. Then for each $s \in (m/2, \infty]$ the space $C^\infty(\mathbb{R}, \Gamma_{H^s}(E))$ of smooth curves in $\Gamma_{H^s}(E)$ consists of all continuous mappings $c : \mathbb{R} \times M \to E$ with $p \circ c = \mathrm{pr}_2 : \mathbb{R} \times M \to M$ such that:

- For each $x \in M$ the curve $t \mapsto c(t, x) \in E_x$ is smooth; let $(\partial_t^p c)(t, x) = \partial_t^p(c(t, x))$, and
- For each $p \in \mathbb{N}_{\geq 0}$, the curve $\partial_t^p c$ has values in $\Gamma_{H^s}(E)$ so that $\partial_t^p c : \mathbb{R} \to \Gamma_{H^s}(E)$, and $t \mapsto \|\partial_t c(t, \quad)\|_{H^s}$ is bounded, locally in t.

the proof is based on [4.1.19 and 4.1.23 of Frölicher Kriegl: Linear spaces and differentiation theory, 1988]

Corollary *Let E_1, E_2 be vector bundles over M, let $U \subset E_1$ be an open neighborhood of the image of a smooth section, let $F : U \to E_2$ be a fiber preserving smooth mapping, and let $s \in (m/2, \infty]$. Then the set $\Gamma_{H^s}(U) := \{h \in \Gamma_{H^s}(E_1) : h(M) \subset U\}$ is open in $\Gamma_{H^s}(E_1)$, and the mapping $F_* : \Gamma_{H^s}(U) \to \Gamma_{H^s}(E_2)$ given by $h \mapsto F \circ h$, is smooth. If the restriction of F to each fiber of E_1 is real analytic, then F_* is real analytic.*

Theorem The Laplacian depends smoothly on the metric

Let $\alpha \in (\frac{m}{2}, \infty]$ *and let* $E \to M$ *be a natural bundle of first order. Then* $g \mapsto \nabla^g$ *is a real analytic mapping:*

$$\nabla : \mathrm{Met}_{H^\alpha}(M) \to L^2(\Gamma_{H^\alpha}(TM), \Gamma_{H^s}(E); \Gamma_{H^{s-1}}(E)),$$
$$\nabla : \mathrm{Met}_{H^\alpha}(M) \to L(\Gamma_{H^s}(E); \Gamma_{H^{s-1}}(T^*M \otimes E)),$$

for $1 \leq s \leq \alpha$. *Consequently,* $g \mapsto \Delta^g$ *is a real analytic mapping*

$$\mathrm{Met}_{H^\alpha}(M) \to L(\Gamma_{H^s}(E), \Gamma_{H^{s-2}}(E)),$$

for $2 \leq s \leq \alpha$. *If* $E = \mathbb{R}$ *then* $g \mapsto \Delta^g$ *is a real analytic mapping*

$$\mathrm{Met}_{H^\alpha}(M) \to L(H^s(M, \mathbb{R}), H^{s-2}(M, \mathbb{R})),$$

for $2 \leq s \leq \alpha + 1$.

The operator $f(1 + \Delta^g)$

Let $g \in \mathrm{Met}_{H^\alpha}(M)$ for $\alpha \in (\frac{m}{2}, \infty]$, and let E be a natural first order vector bundle over M. Let $(e_i)_{i \in \mathbb{N}}$ be an $L^2(g)$-orthonormal basis of $\Gamma_{H^0}(E)$ of eigenvectors of $1 + \Delta^g$ with eigenvalues $(\lambda_i)_{i \in \mathbb{N}} \subset \mathbb{R}_{>0}$.
In general the eigenvalues cannot be chosen smoothly, the eigenfunctions not even continuously, as functions of g. By

[A. Kriegl, P. W. Michor, and A. Rainer. Many parameter Hölder perturbation of unbounded operators. Math. Ann., 353:519â522, 2012],

the increasingly ordered eigenvalues are Lipschitz in g. However, along any real analytic curve $t \mapsto g(t)$ in $\mathrm{Met}_{H^\alpha}(M)$ the eigenvalues and the eigenfunctions can be parameterized real analytically in t. This follows from a result due to Rellich.

The global resolvent set

$$\{(g,\lambda) \in \mathrm{Met}_{H^\alpha}(M)\times\mathbb{C} : (1+\Delta^g-\lambda) : \Gamma_{H^2}(E) \to \Gamma_{H^0}(E) \text{ invertible}\}$$

is open in $\mathrm{Met}_{H^\alpha}(M) \times \mathbb{C}$ and contains $\mathrm{Met}_{H^\alpha}(M) \times (\mathbb{C} \setminus \mathbb{R}_{>0})$. For any simple closed positively oriented C^1-curve γ in $\mathbb{C}$ which does not meet any eigenvalue of $1 + \Delta^g$ the operator

$$P(g,\gamma) = -\frac{1}{2\pi i}\int_\gamma (1+\Delta^g-\lambda)^{-1} d\lambda : \Gamma_{H^0}(E) \to \Gamma_{H^2}(E)$$

is the orthogonal projection onto the finite dimensional direct sum of all eigenspaces for those eigenvalues of $1 + \Delta^g$ which lie in the interior of γ. For fixed γ the operator $P(g,\gamma)$ is defined for all g in the open set of those g such that no eigenvalue of $1 + \Delta^g$ lies on γ. It depends smoothly, even C^ω, on those g, since inversion

$$GL(\Gamma_{H^2}(E), \Gamma_{H^0}(E)) \to L(\Gamma_{H^0}(E), \Gamma_{H^2}(E))$$

is real analytic, and since $\Gamma_{H^2}(E) \to \Gamma_{H^0}(E)$ is a compact operator.

Let $\mathbb{R}_{>}0 \subset U \xrightarrow{f} \mathbb{C}$ be a holomorphic function with $f(\mathbb{R}_{>0}) = \mathbb{R}_{>0}$ where U is an open neighborhood of $\mathbb{R}_{>0}$ in $\mathbb{C}$.

$$\Gamma_{H^0}(E) \supset D(f(1+\Delta^g)) \xrightarrow{f(1+\Delta^g)} \Gamma_{H^0}(E), \quad h \mapsto \sum_{i\in\mathbb{N}} \langle h_i, e_i \rangle f(\lambda_i) e_i$$

domain $D(f(1+\Delta^g)) = \left\{ h \in \Gamma_{H^0}(E); \sum_{i\in\mathbb{N}} \langle h_i, e_i \rangle^2 f(\lambda_i)^2 < \infty \right\}$ is densely defined and self-adjoint with respect to $L^2(g)$. The domain $D(f(1+\Delta^g))$ is a Hilbert space.

Theorem *Let $\alpha \in$ The mapping*

$$g \mapsto f(1+\Delta^g)$$
$$\mathrm{Met}_{H^\alpha}(M) \to L(D(f(1+\Delta^g)), \Gamma_{H^0}(E))$$

is smooth, and conjecturally, even real analytic.
The proof uses the message from convenient calculus. Namely, The elements $h \otimes k \in V \otimes \Gamma_{H^0}(E)$, where V is a dense subspace in $D(f(1+\Delta^g))$, separate points in $L(D(f(1+\Delta^g)), \Gamma_{H^0}(E))$ and the latter space has a basis of bounded sets which closed with respect to it.
Then we use a smooth curve $g(t) \in \mathrm{Met}_{H^\alpha}(M)$, a curve γ enclosing finitely many eigenvalues of $1+\Delta^{g(0)}$ in its interior, $h = \sum_{i=1}^N h_i e_i$,

$$-\frac{1}{2\pi i}\int_\gamma f(\lambda)\Big\langle (1+\Delta^g-\lambda)^{-1}h, k\Big\rangle d\lambda$$

and their derivatives in t as canditates.

Assumptions for Wellposedness

Assumption 1: For each $g \in \mathrm{Met}(M)$, the operator P_g is an elliptic pseudo-differential operator of order $2p$ for $p > 0$ which is positive and symmetric with respect to the $H^0(g)$-metric on $\Gamma(S^2T^*M)$, i.e.,

$$\int_M g_2^0(P_g h, k)\,\mathrm{vol}(g) = \int_M g_2^0(h, P_g k)\,\mathrm{vol}(g) \qquad \text{for } h, k \in \Gamma(S^2T^*M).$$

Assumption 2: $P : \mathrm{Met}(M) \to L(\Gamma(S^2T^*M), \Gamma(S^2T^*M))$ and

$$g \mapsto \big((h,k) \mapsto (D_{(g,h)}Ph)^*(k)\big)$$
$$\mathrm{Met}(M) \to L^2(\Gamma(S^2T^*M), \Gamma(S^2T^*M); \Gamma(S^2T^*M))$$

are smooth and extend to smooth mappings between Sobolev completions

$$\mathrm{Met}_{H^\alpha}(M) \to L(\Gamma_{H^\alpha}(S^2T^*M), \Gamma_{H^{\alpha-2p}}(S^2T^*M))$$
$$\mathrm{Met}_{H^\alpha}(M) \to L^2(\Gamma_{H^\alpha}(S^2T^*M), \Gamma_{H^\alpha}(S^2T^*M); \Gamma_{H^{\alpha-2p}}(S^2T^*M))$$

for $\alpha \in (\dim(M)/2, \infty]$.

Corollary. *If* $g \in \mathrm{Met}_{H^\alpha}(M)$ *for* $\alpha \in (\dim(M)/2, \infty]$*, then* $P_g = (1 + \Delta^g)^p$ *satisfies the assumptions for* $p \in [0, \alpha/2]$*.*
Also $f(1 + \Delta^g)$ *satisfies the assumptions for any real analytic function* $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ *satisfying (for* p *as above)*

$$C_1.\lambda_i^p \leq f(\lambda_i) \leq C_2 \lambda_i^p \text{ for all } i$$

Theorem. *Let the assumptions above hold. Then for (real)* $\alpha > \frac{\dim(M)}{2}$*, the initial value problem for the geodesic equation has unique local solutions in the Sobolev manifold* $\mathrm{Met}^{\alpha}(M)$ *of* H^{α}*-metrics. The solutions depend* C^{∞} *on* t *and on the initial conditions* $g(0,\ .\ ) \in \mathrm{Met}^{\alpha}(M)$ *and* $g_t(0,\ .\ ) \in H^{\alpha}(S^2T^*M)$*.*
If the initial conditions are smooth, then the domain of existence (in t*) is uniform in* $\alpha > \frac{\dim(M)}{2}$ *and thus this also holds in* $\mathrm{Met}(M)$*.*
Moreover, in each Sobolev completion $\mathrm{Met}^{\alpha}(M)$*, the Riemannian exponential mapping* $\exp^P$ *exists and is smooth on a neighborhood of the zero section in the tangent bundle, and* $(\pi, \exp^P)$ *is a diffeomorphism from a (smaller) neighborhood of the zero section to a neighborhood of the diagonal in* $\mathrm{Met}^{\alpha}(M) \times \mathrm{Met}^{\alpha}(M)$*. All these neighborhoods are uniform in* $\alpha > \frac{\dim(M)}{2}$ *and can be chosen* H^{α_0}*-open for some fixed* $\alpha_0 > \frac{\dim(M)}{2}$*. Thus all properties of the exponential mapping continue to hold in* $\mathrm{Met}(M)$*.*

This theorem is more general that the result in [Bauer, Harms, M. 2011], and the proof is now complete.

Ideas of proof. We consider the geodesic equation as the flow equation of a smooth (C^∞) vector field X on the open set

$$\mathrm{Met}_{H^\alpha} \times \Gamma_{H^{\alpha-2p}}(S^2T^*M) \subset \Gamma_{H^\alpha}(S^2T^*M) \times \Gamma_{H^{\alpha-2p}}(S^2T^*M).$$

as follows, using the geodesic equation:

$$\begin{aligned} g_t &= (P_g)^{-1}h =: X_1(g,h) \\ h_t &= \frac{1}{2}\big((D_{(g,.)}P_g)(P_g)^{-1}h\big)^*\big((P_g)^{-1}h\big) + \frac{1}{4}.g.\operatorname{Tr}(g^{-1}.h.g^{-1}.(P_g)^{-1}h) \\ &\qquad + \frac{1}{2}(P_g)^{-1}h.g^{-1}.h + \frac{1}{2}h.g^{-1}.(P_g)^{-1}h - \frac{1}{2}\operatorname{Tr}(g^{-1}.(P_g)^{-1}h).h \\ &=: X_2(g,h) \end{aligned}$$

For $(g,h) \in \mathrm{Met}_{H^\alpha} \times \Gamma_{H^{\alpha-2p}}$ we have $(P_g)^{-1}h \in \Gamma_{H^\alpha}$. A term by term investigation of $X_2(g,h)$, using the assumptions on the orders and the module properties of Sobolev spaces, shows that $X_2(g,h)$ is smooth in $(g,h) \in \mathrm{Met}_{H^\alpha} \times \Gamma_{H^{\alpha-2p}}$ with values in $\Gamma_{H^{\alpha-2p}}$. Likewise $X_1(g,h)$ is smooth in $(g,h) \in \mathrm{Met}^{k+2p} \times H^k$ with values in H^{k+2p}. Now use the theory of smooth ODE's on Banach spaces.

Thank you for your attention