

# DENJOY-CARLEMAN DIFFERENTIABLE PERTURBATION OF POLYNOMIALS AND UNBOUNDED OPERATORS

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ABSTRACT. Let  $t \mapsto A(t)$  for  $t \in T$  be a  $C^M$ -mapping with values unbounded operators with compact resolvents and common domain of definition which are self-adjoint or normal. Here  $C^M$  stands for  $C^\omega$  (real analytic), a quasianalytic or non-quasianalytic Denjoy-Carleman class,  $C^\infty$ , or a Hölder continuity class  $C^{0,\alpha}$ . The parameter domain  $T$  is either  $\mathbb{R}$  or  $\mathbb{R}^n$  or an infinite dimensional convenient vector space. We prove and review results on  $C^M$ -dependence on  $t$  of the eigenvalues and eigenvectors of  $A(t)$ .

**Theorem.** *Let  $t \mapsto A(t)$  for  $t \in T$  be a parameterized family of unbounded operators in a Hilbert space  $H$  with common domain of definition and with compact resolvent. If  $t \in T = \mathbb{R}$  and all  $A(t)$  are self-adjoint then the following holds:*

- (A) *If  $A(t)$  is real analytic in  $t \in \mathbb{R}$ , then the eigenvalues and the eigenvectors of  $A(t)$  may be parameterized real analytically in  $t$ .*
- (B) *If  $A(t)$  is quasianalytic of class  $C^Q$  in  $t \in \mathbb{R}$ , then the eigenvalues and the eigenvectors of  $A(t)$  may be parameterized  $C^Q$  in  $t$ .*
- (C) *If  $A(t)$  is non-quasianalytic of class  $C^L$  in  $t \in \mathbb{R}$  and if no two unequal continuously parameterized eigenvalues (e.g., ordered by size) meet of infinite order at any  $t \in \mathbb{R}$ , then the eigenvalues and the eigenvectors of  $A(t)$  can be parameterized  $C^L$  in  $t$ .*
- (D) *If  $A(t)$  is  $C^\infty$  in  $t \in \mathbb{R}$  and if no two unequal continuously parameterized eigenvalues meet of infinite order at any  $t \in \mathbb{R}$ , then the eigenvalues and the eigenvectors of  $A(t)$  can be parameterized  $C^\infty$  in  $t$ .*
- (E) *If  $A(t)$  is  $C^\infty$  in  $t \in \mathbb{R}$ , then the eigenvalues of  $A(t)$  may be parameterized twice differentiably in  $t$ .*
- (F) *If  $A(t)$  is  $C^{1,\alpha}$  in  $t \in \mathbb{R}$  for some  $\alpha > 0$ , then the eigenvalues of  $A(t)$  may be parameterized in a  $C^1$  way in  $t$ .*

*If  $t \in T = \mathbb{R}$  and all  $A(t)$  are normal then the following holds:*

- (G) *If  $A(t)$  is real analytic in  $t \in \mathbb{R}$ , then for each  $t_0 \in \mathbb{R}$  and for each eigenvalue  $\lambda$  of  $A(t_0)$  there exists  $N \in \mathbb{N}$  such that the eigenvalues near  $\lambda$  of  $A(t_0 \pm s^N)$  and their eigenvectors can be parameterized real analytically in  $s$  near  $s = 0$ .*
- (H) *If  $A(t)$  is  $C^Q$  in  $t \in \mathbb{R}$ , then for each  $t_0 \in \mathbb{R}$  and for each eigenvalue  $\lambda$  of  $A(t_0)$  there exists  $N \in \mathbb{N}$  such that the eigenvalues near  $\lambda$  of  $A(t_0 \pm s^N)$  and their eigenvectors can be parameterized  $C^Q$  in  $s$  near  $s = 0$ .*
- (I) *If  $A(t)$  is  $C^L$  in  $t \in \mathbb{R}$ , then for each  $t_0 \in \mathbb{R}$  and for each eigenvalue  $\lambda$  of  $A(t_0)$  at which no two of the unequal continuously arranged eigenvalues (which is always possible by [12, II.5.2]) meet of infinite order, there exists*

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*Date:* March 28, 2011.

*2000 Mathematics Subject Classification.* 26C10, 26E10, 47A55.

*Key words and phrases.* Perturbation theory, differentiable choice of eigenvalues and eigenvectors, Denjoy–Carleman ultradifferentiable functions.

AK was supported by FWF-Project P 23028-N13, PM by FWF-Project P 21030-N13, AR by FWF-Projects J 2771-N13 and P 22218-N13.

$N \in \mathbb{N}$  such that the eigenvalues near  $\lambda$  of  $A(t_0 \pm s^N)$  and their eigenvectors can be parameterized  $C^L$  in  $s$  near  $s = 0$ .

- (J) If  $A(t)$  is  $C^\infty$  in  $t \in \mathbb{R}$ , then for each  $t_0 \in \mathbb{R}$  and for each eigenvalue  $\lambda$  of  $A(t_0)$  at which no two of the unequal continuously arranged eigenvalues meet of infinite order, there exists  $N \in \mathbb{N}$  such that the eigenvalues near  $\lambda$  of  $A(t_0 \pm s^N)$  and their eigenvectors can be parameterized  $C^\infty$  in  $s$  near  $s = 0$ .
- (K) If  $A(t)$  is  $C^\infty$  in  $t \in \mathbb{R}$  and no two of the unequal continuously parameterized eigenvalues meet of infinite order at any  $t \in \mathbb{R}$ , then the eigenvalues and the eigenvectors of  $A(t)$  can be parameterized by absolutely continuous functions, locally in  $t$ .

If  $t \in T = \mathbb{R}^n$  and all  $A(t)$  are normal then the following holds:

- (L) If  $A(t)$  is  $C^\omega$  or  $C^Q$  in  $t \in \mathbb{R}^n$ , then for each  $t_0 \in \mathbb{R}^n$  and for each eigenvalue  $\lambda$  of  $A(t_0)$ , there exist a neighborhood  $D$  of  $\lambda$  in  $\mathbb{C}$ , a neighborhood  $W$  of  $t_0$  in  $\mathbb{R}^n$ , and a finite covering  $\{\pi_k : U_k \rightarrow W\}$  of  $W$ , where each  $\pi_k$  is a composite of finitely many mappings each of which is either a local blow-up along a  $C^\omega$  or  $C^Q$  submanifold or a local power substitution, such that the eigenvalues of  $A(\pi_k(s))$  in  $D$  and the corresponding eigenvectors can be chosen  $C^\omega$  or  $C^Q$  in  $s$ . If  $A$  is self-adjoint, then we do not need power substitutions.
- (M) If  $A(t)$  is  $C^\omega$  or  $C^Q$  in  $t \in \mathbb{R}^n$ , then the eigenvalues and their eigenvectors of  $A(t)$  can be parameterized by functions which are special functions of bounded variation (SBV), see [9] or [3], locally in  $t$ .

If  $t \in T \subseteq E$ , a  $c^\infty$ -open subset in an infinite dimensional convenient vector space then the following holds:

- (N) For  $0 < \alpha \leq 1$ , if  $A(t)$  is  $C^{0,\alpha}$  (Hölder continuous of exponent  $\alpha$ ) in  $t \in T$  and all  $A(t)$  are self-adjoint, then the eigenvalues of  $A(t)$  may be parameterized in a  $C^{0,\alpha}$  way in  $t$ .
- (O) For  $0 < \alpha \leq 1$ , if  $A(t)$  is  $C^{0,\alpha}$  in  $t \in T$  and all  $A(t)$  are normal, then we have: For each  $t_0 \in T$  and each eigenvalue  $z_0$  of  $A(t_0)$  consider a simple closed  $C^1$ -curve  $\gamma$  in the resolvent set of  $A(t_0)$  enclosing only  $z_0$  among all eigenvalues of  $A(t_0)$ . Then for  $t$  near  $t_0$  in the  $c^\infty$ -topology on  $T$ , no eigenvalue of  $A(t)$  lies on  $\gamma$ . Let  $\lambda(t) = (\lambda_1(t), \dots, \lambda_N(t))$  be the  $N$ -tuple of all eigenvalues (repeated according to their multiplicity) of  $A(t)$  inside of  $\gamma$ . Then  $t \mapsto \lambda(t)$  is  $C^{0,\alpha}$  for  $t$  near  $t_0$  with respect to the non-separating metric

$$d(\lambda, \mu) = \min_{\sigma \in S_N} \max_{1 \leq i \leq N} |\lambda_i - \mu_{\sigma(i)}|$$

on the space of  $N$ -tuples.

Part (A) is due to Rellich [22] in 1942, see also [4] and [12, VII, 3.9]. Part (D) has been proved in [2, 7.8], see also [13, 50.16], in 1997, which contains also a different proof of (A). (E) and (F) have been proved in [14] in 2003. (G) was proved in [19, 7.1]; it can be proved as (H) with some obvious changes, but it is not a special case since  $C^\omega$  does not correspond to a sequence which is an  $\mathcal{L}$ -intersection (see ‘definitions and remarks’ below and [16]). (J) and (K) were proved in [19, 7.1]. (N) was proved in [17].

The purpose of this paper is to prove the remaining parts (B), (C), (H), (I), (L), (M), and (O).

**Definitions and remarks.** Let  $M = (M_k)_{k \in \mathbb{N} = \mathbb{N}_{\geq 0}}$  be an increasing sequence ( $M_{k+1} \geq M_k$ ) of positive real numbers with  $M_0 = 1$ . Let  $U \subseteq \mathbb{R}^n$  be open. We

denote by  $C^M(U)$  the set of all  $f \in C^\infty(U)$  such that, for each compact  $K \subseteq U$ , there exist positive constants  $C$  and  $\rho$  such that

$$|\partial^\alpha f(x)| \leq C \rho^{|\alpha|} |\alpha|! M_{|\alpha|} \quad \text{for all } \alpha \in \mathbb{N}^n \text{ and } x \in K.$$

The set  $C^M(U)$  is a *Denjoy–Carleman class* of functions on  $U$ . If  $M_k = 1$ , for all  $k$ , then  $C^M(U)$  coincides with the ring  $C^\omega(U)$  of real analytic functions on  $U$ . In general,  $C^\omega(U) \subseteq C^M(U) \subseteq C^\infty(U)$ .

Here  $Q = (Q_k)_{k \in \mathbb{N}}$  is a sequence as above which is log-convex (i.e.,  $Q_k^2 \leq Q_{k-1}Q_{k+1}$  for all  $k$ ), derivation closed (i.e.,  $\sup_k (Q_{k+1}/Q_k)^{1/k} < \infty$ ), quasianalytic (i.e.,  $\sum_k (k! Q_k)^{-1/k} = \infty$ ), and which is also an  $\mathcal{L}$ -intersection. We say that  $Q$  is an  $\mathcal{L}$ -intersection if  $C^Q = \bigcap \{C^N : N \text{ non-quasianalytic, log-convex, } N \geq Q\}$ . Moreover,  $L = (L_k)_{k \in \mathbb{N}}$  is a sequence as above which is log-convex, derivation closed, and non-quasianalytic. See [16] or [15] and references therein.

That  $A(t)$  is a real analytic,  $C^M$  (where  $M$  is either  $Q$  or  $L$ ),  $C^\infty$ , or  $C^{k,\alpha}$  family of unbounded operators means the following: There is a dense subspace  $V$  of the Hilbert space  $H$  such that  $V$  is the domain of definition of each  $A(t)$ , and such that  $A(t)^* = A(t)$  in the self-adjoint case, or  $A(t)$  has closed graph and  $A(t)A(t)^* = A(t)^*A(t)$  wherever defined in the normal case. Moreover, we require that  $t \mapsto \langle A(t)u, v \rangle$  is of the respective differentiability class for each  $u \in V$  and  $v \in H$ . From now on we treat only  $C^M = C^\omega$ ,  $C^M$  for  $M = Q$ ,  $M = L$ , and  $C^M = C^{0,\alpha}$ .

This implies that  $t \mapsto A(t)u$  is of the same class  $C^M(E, H)$  (where  $E$  is either  $\mathbb{R}$  or  $\mathbb{R}^n$ ) or is in  $C^{0,\alpha}(E, H)$  (if  $E$  is a convenient vector space) for each  $u \in V$  by [13, 2.14.4, 10.3] for  $C^\omega$ , by [15, 3.1, 3.3, 3.5] for  $M = L$ , by [16, 1.10, 2.1, 2.3] for  $M = Q$ , and by [13, 2.3], [11, 2.6.2] or [10, 4.14.4] for  $C^{0,\alpha}$  because  $C^{0,\alpha}$  can be described by boundedness conditions only and for these the uniform boundedness principle is valid.

A sequence of functions  $\lambda_i$  is said to *parameterize the eigenvalues*, if for each  $z \in \mathbb{C}$  the cardinality  $|\{i : \lambda_i(t) = z\}|$  equals the multiplicity of  $z$  as eigenvalue of  $A(t)$ .

Let  $X$  be a  $C^\omega$  or  $C^Q$  manifold. A *local blow-up*  $\Phi$  over an open subset  $U$  of  $X$  means the composition  $\Phi = \iota \circ \varphi$  of a blow-up  $\varphi : U' \rightarrow U$  with center a  $C^\omega$  or  $C^Q$  submanifold and of the inclusion  $\iota : U \rightarrow X$ . A *local power substitution* is a mapping  $\Psi : V \rightarrow X$  of the form  $\Psi = \iota \circ \psi$ , where  $\iota : W \rightarrow X$  is the inclusion of a coordinate chart  $W$  of  $X$  and  $\psi : V \rightarrow W$  is given by

$$(y_1, \dots, y_q) = ((-1)^{\epsilon_1} x_1^{\gamma_1}, \dots, (-1)^{\epsilon_q} x_q^{\gamma_q}),$$

for some  $\gamma = (\gamma_1, \dots, \gamma_q) \in (\mathbb{N}_{>0})^q$  and all  $\epsilon = (\epsilon_1, \dots, \epsilon_q) \in \{0, 1\}^q$ , where  $y_1, \dots, y_q$  denote the coordinates of  $W$  (and  $q = \dim X$ ).

This paper became possible only after some of the results of [15] and [16] were proved, in particular the uniform boundedness principles. The wish to prove the results of this paper was the main motivation for us to work on [15] and [16].

**Applications.** Let  $X$  be a compact  $C^Q$  manifold and let  $t \mapsto g_t$  be a  $C^Q$ -curve of  $C^Q$  Riemannian metrics on  $X$ . Then we get the corresponding  $C^Q$  curve  $t \mapsto \Delta(g_t)$  of Laplace–Beltrami operators on  $L^2(X)$ . By theorem (B) the eigenvalues and eigenvectors can be arranged  $C^Q$  in  $t$ . By [1], the eigenfunctions are also  $C^Q$  as functions on  $X$  (at least for those  $C^Q$  which can be described by a weight function, see [7]). Question: Are the eigenvectors viewed as eigenfunctions then also in  $C^Q(X \times \mathbb{R})$ ?

Let  $\Omega$  be a bounded region in  $\mathbb{R}^n$  with  $C^Q$  boundary, and let  $H(t) = -\Delta + V(t)$  be a  $C^Q$ -curve of Schrödinger operators with varying  $C^Q$  potential and Dirichlet

boundary conditions. Then the eigenvalues and eigenvectors can be arranged  $C^Q$ . Question: Are the eigenvectors viewed as eigenfunctions then also in  $C^Q(\Omega \times \mathbb{R})$ ?

**Example.** This is an elaboration of [2, 7.4] and [14, Example]. Let  $S(2)$  be the vector space of all symmetric real  $(2 \times 2)$ -matrices. We use the  $C^L$ -curve lemma [15, 3.6] or [16, 2.5]: *There exists a converging sequence of reals  $t_n$  with the following property: Let  $A_n, B_n \in S(2)$  be any sequences which converge fast to 0, i.e., for each  $k \in \mathbb{N}$  the sequences  $n^k A_n$  and  $n^k B_n$  are bounded in  $S(2)$ . Then there exists a curve  $A \in C^L(\mathbb{R}, S(2))$  such that  $A(t_n + s) = A_n + sB_n$  for  $|s| \leq \frac{1}{n^2}$ , for all  $n$ .*

We use it for

$$A_n := \frac{1}{2n^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_n := \frac{1}{2n^2 s_n} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{where } s_n := 2^{n-n^2} \leq \frac{1}{n^2}.$$

The eigenvalues of  $A_n + tB_n$  and their derivatives are

$$\lambda_n(t) = \pm \frac{1}{2n^2} \sqrt{1 + \left(\frac{t}{s_n}\right)^2}, \quad \lambda'_n(t) = \pm \frac{2^{n^2-2nt}}{\sqrt{1 + \left(\frac{t}{s_n}\right)^2}}.$$

Then

$$\begin{aligned} \frac{\lambda'(t_n + s_n) - \lambda'(t_n)}{s_n^\alpha} &= \frac{\lambda'_n(s_n) - \lambda'_n(0)}{s_n^\alpha} = \pm \frac{2^{n^2-2ns_n}}{s_n^\alpha \sqrt{2}} \\ &= \pm \frac{2^{n(\alpha(n-1)-1)}}{\sqrt{2}} \rightarrow \infty \text{ for } \alpha > 0. \end{aligned}$$

So the condition (in (C), (D), (I), (J), and (K)) that no two unequal continuously parameterized eigenvalues meet of infinite order cannot be dropped. By [2, 2.1], we may always find a twice differentiable square root of a non-negative smooth function, so that the eigenvalues  $\lambda$  are functions which are twice differentiable but not  $C^{1,\alpha}$  for any  $\alpha > 0$ .

Note that the normed eigenvectors cannot be chosen continuously in this example (see also example [21, §2]). Namely, we have

$$A(t_n) = A_n = \frac{1}{2n^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A(t_n + s_n) = A_n + s_n B_n = \frac{1}{2n^2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

**Resolvent Lemma.** *Let  $C^M$  be any of  $C^\omega$ ,  $C^Q$ ,  $C^L$ ,  $C^\infty$ , or  $C^{0,\alpha}$ , and let  $A(t)$  be normal. If  $A$  is  $C^M$  then the resolvent  $(t, z) \mapsto (A(t) - z)^{-1} \in L(H, H)$  is  $C^M$  on its natural domain, the global resolvent set*

$$\{(t, z) \in T \times \mathbb{C} : (A(t) - z) : V \rightarrow H \text{ is invertible}\}$$

*which is open (and even connected).*

**Proof.** By definition the function  $t \mapsto \langle A(t)v, u \rangle$  is of class  $C^M$  for each  $v \in V$  and  $u \in H$ . We may conclude that the mapping  $t \mapsto A(t)v$  is of class  $C^M$  into  $H$  as follows: For  $C^M = C^\infty$  we use [13, 2.14.4]. For  $C^M = C^\omega$  we use in addition [13, 10.3]. For  $C^M = C^Q$  or  $C^M = C^L$  we use [16, 2.1] and/or [15, 3.3] where we replace  $\mathbb{R}$  by  $\mathbb{R}^n$ . For  $C^M = C^{0,\alpha}$  we use [13, 2.3], [11, 2.6.2], or [10, 4.1.14] because  $C^{0,\alpha}$  can be described by boundedness conditions only and for these the uniform boundedness principle is valid.

For each  $t$  consider the norm  $\|u\|_t^2 := \|u\|^2 + \|A(t)u\|^2$  on  $V$ . Since  $A(t)$  is closed,  $(V, \|\cdot\|_t)$  is again a Hilbert space with inner product  $\langle u, v \rangle_t := \langle u, v \rangle + \langle A(t)u, A(t)v \rangle$ .

(1) *Claim (see [2, in the proof of 7.8], [13, in the proof of 50.16], or [14, Claim 1]). All these norms  $\|\cdot\|_t$  on  $V$  are equivalent, locally uniformly in  $t$ . We then equip  $V$  with one of the equivalent Hilbert norms, say  $\|\cdot\|_0$ .*

We reduce this to  $C^{0,\alpha}$ . Namely, note first that  $A(t) : (V, \|\cdot\|_s) \rightarrow H$  is bounded since the graph of  $A(t)$  is closed in  $H \times H$ , contained in  $V \times H$  and thus also closed in  $(V, \|\cdot\|_s) \times H$ . For fixed  $u, v \in V$ , the function  $t \mapsto \langle u, v \rangle_t = \langle u, v \rangle + \langle A(t)u, A(t)v \rangle$  is  $C^{0,\alpha}$  since so is  $t \mapsto A(t)u$ . By the multilinear uniform boundedness principle ([13, 5.18] or [11, 3.7.4]) the mapping  $t \mapsto \langle \cdot, \cdot \rangle_t$  is  $C^{0,\alpha}$  into the space of bounded sesquilinear forms on  $(V, \|\cdot\|_s)$  for each fixed  $s$ . Thus the inverse image of  $\langle \cdot, \cdot \rangle_s + \frac{1}{2}(\text{unit ball})$  in  $L((V, \|\cdot\|_s) \oplus (V, \|\cdot\|_s); \mathbb{C})$  is a  $c^\infty$ -open neighborhood  $U$  of  $s$  in  $T$ . Thus  $\sqrt{1/2}\|u\|_s \leq \|u\|_t \leq \sqrt{3/2}\|u\|_s$  for all  $t \in U$ , i.e., all Hilbert norms  $\|\cdot\|_t$  are locally uniformly equivalent, and claim (1) follows.

By the linear uniform boundedness theorem we see that  $t \mapsto A(t)$  is in  $C^M(T, L(V, H))$  as follows (here it suffices to use a set of linear functionals which together recognize bounded sets instead of the whole dual): For  $C^M = C^\infty$  we use [13, 1.7 and 2.14.3]. For  $C^M = C^\omega$  we use in addition [13, 9.4]. For  $C^M = C^Q$  or  $C^M = C^L$  we use [16, 2.2 and 2.3] and/or [15, 3.5] where we replace  $\mathbb{R}$  by  $\mathbb{R}^n$ . For  $C^M = C^{0,\alpha}$  see above.

If for some  $(t, z) \in T \times \mathbb{C}$  the bounded operator  $A(t) - z : V \rightarrow H$  is invertible, then this is true locally with respect to the  $c^\infty$ -topology on the product which is the product topology by [13, 4.16], and  $(t, z) \mapsto (A(t) - z)^{-1} : H \rightarrow V$  is  $C^M$ , by the chain rule, since inversion is real analytic on the Banach space  $L(V, H)$ .  $\square$

Note that  $(A(t) - z)^{-1} : H \rightarrow H$  is a compact operator for some (equivalently any)  $(t, z)$  if and only if the inclusion  $i : V \rightarrow H$  is compact, since  $i = (A(t) - z)^{-1} \circ (A(t) - z) : V \rightarrow H \rightarrow H$ .

**Polynomial proposition.** *Let  $P$  be a curve of polynomials*

$$P(t)(x) = x^n - a_1(t)x^{n-1} + \cdots + (-1)^n a_n(t), \quad t \in \mathbb{R}.$$

- (a) *If  $P$  is hyperbolic (i.e., all roots real) and if the coefficient functions  $a_i$  are all  $C^Q$  then there exist  $C^Q$  functions  $\lambda_i$  which parameterize all roots.*
- (b) *If  $P$  is hyperbolic, the coefficient functions  $a_i$  are  $C^L$ , and no two of the unequal continuously arranged roots (e.g., ordered by size) meet of infinite order, then there exist  $C^L$  functions  $\lambda_i$  which parameterize all roots.*
- (c) *If the coefficient functions  $a_i$  are  $C^Q$ , then for each  $t_0$  there exists  $N \in \mathbb{N}$  such that the roots of  $s \mapsto P(t_0 \pm s^N)$  can be parameterized  $C^Q$  in  $s$  for  $s$  near 0.*
- (d) *If the coefficient functions  $a_i$  are  $C^L$  and no two of the unequal continuously arranged roots (by [12, II.5.2]) meet of infinite order, then for each  $t_0$  there exists  $N \in \mathbb{N}$  such that the roots of  $s \mapsto P(t_0 \pm s^N)$  can be parameterized  $C^L$  in  $s$  for  $s$  near 0.*

*All  $C^Q$  or  $C^L$  solutions differ by permutations.*

The proof of parts (a) and (b) is exactly as in [2] where the corresponding results were proven for  $C^\infty$  instead of  $C^L$ , and for  $C^\omega$  instead of  $C^Q$ . For this we need only the following properties of  $C^Q$  and  $C^L$ :

- They allow for the implicit function theorem (for [2, 3.3]).
- They contain  $C^\omega$  and are closed under composition (for [2, 3.4]).
- They are derivation closed (for [2, 3.7]).

Part (a) is also in [8, 7.6] which follows [2]. It also follows from the multidimensional version [20, 6.10] since blow-ups in dimension 1 are trivial. The proofs of parts (c) and (d) are exactly as in [19, 3.2] where the corresponding result was proven for  $C^\omega$  instead of  $C^Q$ , and for  $C^\infty$  instead of  $C^L$ , if none of the different roots meet of infinite order. For these we need the properties of  $C^Q$  and  $C^L$  listed above.

**Matrix proposition.** *Let  $A(t)$  for  $t \in T$  be a family of  $(N \times N)$ -matrices.*

- (e) If  $T = \mathbb{R} \ni t \mapsto A(t)$  is a  $C^Q$ -curve of Hermitian matrices, then the eigenvalues and the eigenvectors can be chosen  $C^Q$ .
- (f) If  $T = \mathbb{R} \ni t \mapsto A(t)$  is a  $C^L$ -curve of Hermitian matrices such that no two unequal continuously arranged eigenvalues meet of infinite order, then the eigenvalues and the eigenvectors can be chosen  $C^L$ .
- (g) If  $T = \mathbb{R} \ni t \mapsto A(t)$  is a  $C^L$ -curve of normal matrices such that no two unequal continuously arranged eigenvalues meet of infinite order, then for each  $t_0$  there exists  $N_1 \in \mathbb{N}$  such that the eigenvalues and eigenvectors of  $s \mapsto A(t_0 \pm s^{N_1})$  can be parameterized  $C^L$  in  $s$  for  $s$  near 0.
- (h) Let  $T \subseteq \mathbb{R}^n$  be open and let  $T \ni t \mapsto A(t)$  be a  $C^\omega$  or  $C^Q$ -mapping of normal matrices. Let  $K \subseteq T$  be compact. Then there exist a neighborhood  $W$  of  $K$ , and a finite covering  $\{\pi_k : U_k \rightarrow W\}$  of  $W$ , where each  $\pi_k$  is a composite of finitely many mappings each of which is either a local blow-up along a  $C^\omega$  or  $C^Q$  submanifold or a local power substitution, such that the eigenvalues and the eigenvectors of  $A(\pi_k(s))$  can be chosen  $C^\omega$  or  $C^Q$  in  $s$ . Consequently, the eigenvalues and eigenvectors of  $A(t)$  are locally special functions of bounded variation (SBV). If  $A$  is a family of Hermitian matrices, then we do not need power substitutions.

The proof of the matrix proposition in case (e) and (f) is exactly as in [2, 7.6], using the polynomial proposition and properties of  $C^Q$  and  $C^L$ . Item (g) is exactly as in [19, 6.2], using the polynomial proposition and properties of  $C^L$ . Item (h) is proved in [20, 9.1 and 9.6], see also [18].

**Proof of the theorem.** We have to prove parts (B), (C), (H), (I), (L), (M), and (O). So let  $C^M$  be any of  $C^\omega$ ,  $C^Q$ ,  $C^L$ , or  $C^{0,\alpha}$ , and let  $A(t)$  be normal. Let  $z$  be an eigenvalue of  $A(t_0)$  of multiplicity  $N$ . We choose a simple closed  $C^1$  curve  $\gamma$  in the resolvent set of  $A(t_0)$  for fixed  $t_0$  enclosing only  $z$  among all eigenvalues of  $A(t_0)$ . Since the global resolvent set is open, see the resolvent lemma, no eigenvalue of  $A(t)$  lies on  $\gamma$ , for  $t$  near  $t_0$ . By the resolvent lemma,  $A : T \rightarrow L((V, \|\cdot\|_0), H)$  is  $C^M$ , thus also

$$t \mapsto -\frac{1}{2\pi i} \int_{\gamma} (A(t) - z)^{-1} dz =: P(t, \gamma) = P(t)$$

is a  $C^M$  mapping. Each  $P(t)$  is a projection, namely onto the direct sum of all eigenspaces corresponding to eigenvalues of  $A(t)$  in the interior of  $\gamma$ , with finite rank. Thus the rank must be constant: It is easy to see that the (finite) rank cannot fall locally, and it cannot increase, since the distance in  $L(H, H)$  of  $P(t)$  to the subset of operators of rank  $\leq N = \text{rank}(P(t_0))$  is continuous in  $t$  and is either 0 or 1.

So for  $t$  in a neighborhood  $U$  of  $t_0$  there are equally many eigenvalues in the interior of  $\gamma$ , and we may call them  $\lambda_i(t)$  for  $1 \leq i \leq N$  (repeated with multiplicity).

Now we consider the family of  $N$ -dimensional complex vector spaces  $t \mapsto P(t)H \subseteq H$ , for  $t \in U$ . They form a  $C^M$  Hermitian vector subbundle over  $U$  of  $U \times H \rightarrow U$ : For given  $t$ , choose  $v_1, \dots, v_N \in H$  such that the  $P(t)v_i$  are linearly independent and thus span  $P(t)H$ . This remains true locally in  $t$ . Now we use the Gram Schmidt orthonormalization procedure (which is  $C^\omega$ ) for the  $P(t)v_i$  to obtain a local orthonormal  $C^M$  frame of the bundle.

Now  $A(t)$  maps  $P(t)H$  to itself; in a  $C^M$  local frame it is given by a normal  $(N \times N)$ -matrix parameterized  $C^M$  by  $t \in U$ .

Now all local assertions of the theorem follow:

- (B) Use the matrix proposition, part (e).
- (C) Use the matrix proposition, part (f).

- (H) Use the matrix proposition, part (h), and note that in dimension 1 blowups are trivial.
- (I) Use the matrix proposition, part (g).
- (L,M) Use the matrix proposition, part (h), for  $\mathbb{R}^n$ .
- (O) We use the following

**Result** ([6], [5, VII.4.1]) *Let  $A, B$  be normal  $(N \times N)$ -matrices and let  $\lambda_i(A)$  and  $\lambda_i(B)$  for  $i = 1, \dots, N$  denote the respective eigenvalues. Then*

$$\min_{\sigma \in \mathcal{S}_N} \max_j |\lambda_j(A) - \lambda_{\sigma(j)}(B)| \leq C \|A - B\|$$

for a universal constant  $C$  with  $1 < C < 3$ . Here  $\| \cdot \|$  is the operator norm.

Finally, it remains to extend the local choices to global ones for the cases (B) and (C) only. There  $t \mapsto A(t)$  is  $C^Q$  or  $C^L$ , respectively, which imply both  $C^\infty$ , and no two different eigenvalues meet of infinite order. So we may apply [2, 7.8] (in fact we need only the end of the proof) to conclude that the eigenvalues can be chosen  $C^\infty$  on  $T = \mathbb{R}$ , uniquely up to a global permutation. By the local result above they are then  $C^Q$  or  $C^L$ . The same proof then gives us, for each eigenvalue  $\lambda_i : T \rightarrow \mathbb{R}$  with generic multiplicity  $N$ , a unique  $N$ -dimensional smooth vector subbundle of  $\mathbb{R} \times H$  whose fiber over  $t$  consists of eigenvectors for the eigenvalue  $\lambda_i(t)$ . In fact this vector bundle is  $C^Q$  or  $C^L$  by the local result above, namely the matrix proposition, part (e) or (f), respectively.  $\square$

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