DUALITY OF COMPACTOLOGICAL AND LOCALLY COMPACT GROUPS

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We develop a duality for compactological groups, based on a concrete realization of the dual category of the category of compactological spaces in terms of a mixed topology on $\ell^\infty(S)$. The dual of a compactological group appears as a Hopf-algebra with mixed topology.

We are able to treat the following notions in terms of the dual: characterization of locally compact groups, Bohr compactification, almost periodic functions, Pontryagin duality, and some connections with representation theory.

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§1. Preliminaries and Notation

1.1. Compactological spaces: The model for a compactological space is a topological Hausdorff space together with the collection of all its compact subsets, disregarding its original topology. So a compactological space S is a set S together with a collection $\mathcal{K}(S)$ of subsets of S, each $K \in \mathcal{K}(S)$ bearing a compact (Hausdorff) topology τ_K such that

(1) $\mathcal{K}(S)$ is closed under formation of finite unions and taking closed subsets.

(2) for each $K \subset L$; $K, L \in \mathcal{K}(S)$ the inclusion $K \to L$ is continuous.

The category CPTOL of compactological spaces has as morphisms maps $f: S \to T$ such that for each $K \in \mathcal{K}(S)$ there is $L \in \mathcal{K}(T)$ with $f(K) \subset L$ and $f|K:K \to L$ is $\tau_K - \tau_L$ - continuous.

By $C^\infty(S)$ we mean the vector space of all bounded complex valued functions on S whose restrictions to each $K \in \mathcal{K}(S)$ are τ_K-continuous. A compactological space is said to be regular, if $C^\infty(S)$ separates points on S. We denote the full subcategory of regular compactological spaces by RCPTOL. We note that compactological spaces may be regarded as formal inductive limits of systems of compact spaces. For more information see BUCHWALTER.

1.2. The category MIXC*: Objects are triples $(E, \|.\|, \tau)$ where E is a commutative involutive algebra with unit over C, $\|.\|$ is a norm on E and τ is a locally convex topology on E such that:

(1) $B_{\|.\|} = \{x \in E: \|x\| \leqslant 1\}$ is bounded and complete for τ

(2) τ may be defined by a family of seminorms P on E such that $p(xy) \leqslant p(x)p(y)$, $p(1) = 1$ and $p(x^*x) = p(x)^2$ holds for all $x, y \in E$ and $p \in P$ and $\|x\| = \sup\{p(x), p \in P\}$ for all $x \in E$.

Morphisms are multiplicative linear maps, respecting involution and unit, contractive for the norm and continuous for the locally convex topology.

MIXC* may be regarded as the category of formal projective limits of systems of commutative C*-algebras with unit. For more information see COOPER 1975.

1.3. The category MIXTOP: Objects are triples $(E, \|.\|, \tau)$ where E is a vector space (over C), $\|.\|$ is a norm on E and τ is a locally convex topology on E such that $B_{\|.\|} = \{x \in E, \|x\| \leqslant 1\}$ is τ-bounded. Morphisms are linear maps, contractive for $\|.\|$ and continuous for τ. $(E, \|.\|, \tau)$ is said to be complete, if $B_{\|.\|}$ is τ-complete. Then $(E, \|.\|)$ is a Banach space. The complete objects in MIXTOP are exactly the formal projective limits of systems of Banach spaces.

1.4. The functor C^∞:

Let S be a compactological space. Let $C^\infty(S)$ be the space of all bounded complex valued functions on S whose restrictions to all $K \in \mathcal{K}(S)$ are τ_K-continuous. Consider the following structures on $C^\infty(S)$: $\| \ \|$ – the supremum norm

τ – the topology of uniform convergence on members of $\mathcal{K}(S)$.

Then $(C^\infty(S), \|.\|, \tau)$ is an object of $MIXC^*$. If $\varphi: S \to T$ is a CPTOL-morphism, then $C^\infty(\varphi): C^\infty(T) \to C^\infty(S)$, given by $x \to x \circ \varphi$, is a $MIXC^*$-morphism.

We have constructed a contravariant functor C^∞: CPTOL $\to MIXC^*$

1.5. The functor M_γ:

Let $(E, \|.\|, \tau)$ be an object of $MIXC^*$.

Denote by $M_\gamma(E)$ the set of all $MIXC^*$-morphisms $E \to C$. We equip it with the following compactology: members of $\mathcal{K}(M_\gamma(E))$ are the weak*-closed subsets of $M_\gamma(E)$, whose restriction to $B_{\|.\|}$ is τ-equicontinuous, and they bear the restriction of the weak*-topology. If $\varphi: E \to F$ is a $MIXC^*$-morphism, then $M_\gamma(\varphi): M_\gamma(F) \to M_\gamma(E)$, given by $f \to f \circ \varphi$, is a CPTOL-morphism. We have constructed a contravariant functor $MIXC^* \to$ RCPTOL.

1.6. Proposition: (COOPER 1975): The categories MIXC^* and RCPTROL are quasi-dual to each other under the functors C^∞ and M_γ.
The maps $\delta: s \to (x \to x(s))$ gives a natural isomorphism $S \to M_\gamma C^\infty(S)$ for each $S \in$ RCPTOL. We call it the Dirac transformation.
The map $\hat{}: x \to (f \to f(x))$ give a natural isomorphism $E \to C^\infty M_\gamma(E)$ for each $E \in \mathrm{MIXC}^*$. We call it the Gelfand-Naimark transformation.
These two maps produce the quasi-duality.

1.7. The tensor product in MIXC^* and MIXTOP:

The category RCPTOL has products (the obvious ones), so MIXC^* as the (quasi)-dual category has coproducts. The γ-tensorproduct, which we will now describe, is an explicit construction of the coproduct in MIXC^*.

Yet $(E, \|\cdot\|_E, \tau_E)$ and $(F, \|\cdot\|_F, \tau_F)$ be two objects of MIXC^*. We consider the following structures on $E \otimes F$, the vector-space tensor product of E and F: $\|\cdot\|^{\hat{\hat{}}}$-the inductive tensor product of the norms $\|\cdot\|_E$, $\|\cdot\|_F$ (i.e. that induced by the operator norm via the embedding $E \otimes F \to L(E',F)$.

$\tau = \tau_E \hat{\hat{\otimes}} \tau_F$ -the inductive tensor product of the locally convex topologies τ_E, τ_F.

Let B denote the closure of $\{u \in E \otimes F, \|u\|^{\hat{\hat{}}} \leq 1\}$ in the completion of $(E \otimes F, \tau_E \hat{\hat{\otimes}} \tau_F)$ and let $E \hat{\hat{\otimes}}_\gamma F$ denote the subspace $\bigcup_{n>0} nB$ of this completion, and let $\|\cdot\|^{\hat{\hat{}}}$ be the Minkowski functional of B. $(E \hat{\hat{\otimes}}_\gamma F, \| \|^{\hat{\hat{}}}, \tau_E \hat{\hat{\otimes}} \tau_F)$

is again an object of MIXC*. The same construction works for MIXTOP. A result to be found in COOPER 1975 asserts that $E \hat{\otimes}_\gamma F \cong C^\infty(M_\gamma(E) \times M_\gamma(F))$ in MIXC*, so the γ-tensor product is the coproduct in MIXC*.

1.8. The strict topology:

If $(E, \|.\|, \tau)$ is an element of MIXC*, let $\gamma = \gamma[\|.\|, \tau]$ be the finest locally convex topology on E which agrees with τ on $B_{\|.\|}$; γ is a complete topology.

We note that $\gamma[\|.\|^{\wedge}, \tau_E \hat{\otimes} \tau_F] = \gamma[\|.\|_E, \tau_E] \hat{\otimes} \gamma[\|.\|_F, \tau_F]$ on $E \hat{\otimes}_\gamma F$.

For further details see COOPER 1975.

The same definition holds of course for objects of MIXTOP.

1.9. It is well known, that locally compact topological spaces are k-spaces, i.e. their topology is uniquely determined by their natural compactology. In this spirit we can regard the category of locally compact topological spaces (we call it LOCCOMP) as a full subcategory of RCPTOL and we will speak of locally compact compactological spaces.

1.10. Let $(E, \|.\|, \tau)$ be an object of MIXC* and let P be a defining family of C^*-seminorms on E (i.e. a family P satisfying 1.2. (2)). If $p \in P$ we denote by I_p the ideal $\{x \in E: p(x) = 0\}$ and by A_p its annihilator in E, i.e. $A_p = \{y \in E: y\, I_p = 0\}$.

E is said to be perfect (APOSTOL 1971) if the sum $\sum_{p \in P} A_p$ is γ-dense (1.8) in E.

If S is a regular compactological space and if $K \in \mathcal{K}(S)$ and p_K the associated C^*- seminorm, then $I_{p_K} = \{x \in C^\infty(S): x|K = 0\}$ and $A_{p_K} = \{y \in C^\infty(S): y(s) = 0 \text{ for } s \notin K\}$ So $\sum_{K \in \mathcal{K}(s)} A_{p_K}$ is the subspace $C_c(S)$ of functions in $C^\infty(S)$ with compact support (i.e. $x \in C_c(S)$ iff there is $K \in \mathcal{K}(S)$ with $x(s) = 0$ for $s \notin K$).

1.11. Proposition (COOPER 1975) Let S be a regular compactological space. S is locally compact if and only if $C^\infty(S)$ is perfect.

§2. Compactological groups and duality

2.1. Definition. A compactological group is a group in the category CPTOL of compactological spaces, i.e. it is a quadruple (S,m,e,i) where S is a compactological space and $m : S \times S \to S$, $e : I \to S$, $i : S \to S$ are CPTOL-morphisms so that the following diagrams commute:

$$\begin{tikzcd}
S \times S \times S \arrow[r, "id_S \times m"] \arrow[d, "m \times id_S"'] & S \times S \arrow[d, "m"] \\
S \times S \arrow[r, "m"] & S
\end{tikzcd}$$

$$\begin{tikzcd}
I \times S \arrow[r, "e \times id_S"] \arrow[dr] & S \times S \arrow[d, "m"] & S \times I \arrow[l, "id_S \times e"'] \arrow[dl] \\
& S &
\end{tikzcd}$$

$$\begin{tikzcd}
S \arrow[r, "d"] \arrow[d] & S \times S \arrow[r, "id_S \times i"] & S \times S \arrow[d, "m"] & S \times S \arrow[l, "i \times id_S"'] & S \arrow[l, "d"'] \arrow[d] \\
I \arrow[rr, "e"] & & S & & I \arrow[ll, "e"']
\end{tikzcd}$$

(I is the final object of CPTOL, i.e. the one-point set, d is the diagonal map). The compactological groups form a category which we denote by GCPTOL (the morphisms are those CPTOL-morphisms which respect the maps (m,e,i). A compactological group is said to be <u>regular</u> if its underlying compactological space is regular, i.e. if $C^\infty(S)$ separates S. GRCPTOL denotes the full subcategory of regular compactological groups.

2.2. Problem: Do there exist compactological groups which are not regular?

2.3. Definition: CMIXC* denotes the category of cogroups in MIXC*. Thus an object of CMIXC* is a quadruple (E,c,η,a) where E is an object of MIXC*, and $c: E \to E \hat{\otimes}_\gamma E$, $\eta: E \to C$, $a: E \to E$ are MIXC*-morphisms so that the following diagrams commute:

$$
\begin{tikzcd}
E \arrow[r, "c"] \arrow[d, "c"'] & E \hat{\otimes}_\gamma E \arrow[d, "id_E \otimes c"] \\
E \hat{\otimes}_\gamma E \arrow[r, "c \otimes id_E"] & E \hat{\otimes}_\gamma E \hat{\otimes}_\gamma E
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & E \arrow[dl] \arrow[d, "c"] \arrow[dr] & \\
C \hat{\otimes}_\gamma E & E \hat{\otimes}_\gamma E \arrow[l, "\eta \otimes id_E"'] \arrow[r, "id_E \otimes \eta"] & E \hat{\otimes}_\gamma C
\end{tikzcd}
$$

$$
\begin{tikzcd}
C \arrow[d] & E \arrow[l, "\eta"'] \arrow[r, "\eta"] \arrow[d, "c"] & C \arrow[d] \\
E & E \hat{\otimes}_\gamma E \arrow[l, "(a,id_E)"'] \arrow[r, "(id_E,a)"] & E
\end{tikzcd}
$$

((a,id_E) denotes the canonical morphism from $E \hat{\otimes}_\gamma E$, the coproduct in MIXC* (cf 1.6) into E, defined by the maps a and id_E; C is the initial object of MIXC*).

2.4. If (S,m,e,i) is a compactological group, then clearly $(C^\infty(S),C^\infty(m),C^\infty(e),C^\infty(i))$ is a cogroup in $MIXC^*$. Clearly C^∞ lifts to a functor $GCPTOL \to CMIXC^*$.

If on the other hand (E,c,η,a) is a cogroup in $MIXC^*$, then again $(M_\gamma(E),M_\gamma(c),M_\gamma(a))$ is a regular compactological group. M_γ lifts to a functor $CMIXC^* \to GRCPTOL$.

Proposition: The functors C^∞ and M_γ induce a duality between $CMIXC^*$ and $GRCPTOL$.

2.5. A compactological group (S,m,e,i) is locally compact if and only if its dual $C^\infty(S)$ is perfect. (cf 1.11).

§3. The Bohr Compactification

3.1. Let (E,c,η,a) be a cogroup in MIXC*. If we consider the C^*-algebra $(E,\|.\|)$, then $M_\gamma(E,\|.\|)$ is a compact topological space, in fact, the Stone-Čech compactification of $M_\gamma(E)$. It is, however, in general not a topological group, since $(E,\|.\|)$ is not a cogroup in the category C^* of commutative C^*-algebras with unit, since $E \hat{\hat{\otimes}} E \neq E \hat{\otimes}_\gamma E$, where $E \hat{\hat{\otimes}} E$, is the C^*-algebra tensor product or the inductive tensor product of Banach spaces.

Let $\tilde{c} = (id_E \otimes a) \circ c$.

3.2. Lemma: There is a largest C^*-subalgebra $\tilde{E}$ of E with the property that $\tilde{c}(\tilde{E}) \subset \tilde{E} \hat{\hat{\otimes}} \tilde{E}$. $\tilde{E}$, with the induced norm and cogroup structure is a cogroup in C^*. The assignment $E \to \tilde{E}$ is functorial.

Proof: For each ordinal α, we define a subalgebra E_α of E inductively by

$$E_o := E$$

$$E_\alpha := \tilde{c}^{-1}(E_\beta \hat{\hat{\otimes}} E_\beta) \quad (\alpha = \beta + 1)$$

$$E_\alpha := \bigcap \{E_\beta, \beta < \alpha\} \quad (\alpha \text{ is a limit ordinal}).$$

Then the family $\{E_\alpha\}$ is eventually stationary and we denote its limit by $\tilde{E}$. Then $\tilde{E}$ is C^*-subalgebra of E with the desired properties.

3.3. The functor $E \to \tilde{E}$ is a "forgetful functor" from CMIXC* into CC*. We denote it by U. We can now define a functor $B := M_\gamma \circ U \circ C^\infty$ from GCPTOL into GCOMP, the category of compact groups. If S is a compactological group, we call $B(S)$ the Bohr-compactification of S. There is a natural morphism $j_S: S \to B(S)$, $j_S = M_\gamma(\lambda)$, where $\lambda: \widetilde{C^\infty(S)} \to C^\infty(S)$ is the embedding. j_S has dense image, since λ is injective.

3.4. Proposition: B(S) has the following universal property: every GCPTOL-morphism from S into a compact group factorises over j_S.

Proof: If $\emptyset: S \to T$ is a GCPTOL-morphism, where T is a compact group, then $C^\infty(\emptyset): C(T) \to C^\infty(S)$ is a CMIXC*-morphism. Since $UC(T) = C(T)$ and U is a functor, acting on morphisms by restricting them, we conclude that $\widetilde{C^\infty(\emptyset)} = C^\infty(\emptyset)$ maps C(T) into $\widetilde{C^\infty(S)}$, i.e. factors over $\lambda: \widetilde{C^\infty(S)} \to C^\infty(S)$.

So $\emptyset = \tilde{\emptyset} \circ j_S$, $\tilde{\emptyset} = M_\gamma \widetilde{C^\infty(\emptyset)}$.

3.5. If S is a compactological group and $x \in C^\infty(S)$, then we define $(L_a x)(s) = x(as)$, $(R_a x)(s) = x(sa)$ for a, $s \in S$. $x \in C^\infty(S)$ is said to be (left) almost periodic if $\{L_a x, a \in S\}$ is relatively norm-compact in $C^\infty(S)$. We denote by AP(S) the set of left almost periodic functions on S.

With induced norm AP(S) is C^*-subalgebra of $C^\infty(S)$.

3.6. Lemma: $\widetilde{C^\infty(S)} \subset AP(S)$.

Proof: If $x \in \widetilde{C^\infty(S)}$, there is an $\tilde{x} \in C(B(S))$ so that $x = \tilde{x} \circ j_S$ by definition of B(S). Then $L_a x = (L_{j_S(a)} \tilde{x}) \circ j_S$ and the result follows from the fact that $\tilde{x}$ is almost periodic on B(S).

3.7. Conjecture: $\widetilde{C^\infty(S)} = AP(S)$

§4. The Algebra $M_t(S)$ and representations

4.1. Let (E,c,η,a) be a cogroup in $MIXC^*$. Equip E with the strict topology $\gamma[\|.\|,\tau]$ (1.8) and let E_γ be its dual. Define a multiplication on E_γ' in the following way: if $f,g \in E_\gamma'$, then $f * g$ be given by $x \to f \otimes g(c(x))$.

Proposition: If E is a cogroup in $MIXC^*$, then E_γ' is Banach algebra with identity. It is commutative if E is cocommutative.

4.2. Let S be a compactological space.

A premeasure on S is a member of the projective limit of the system $\{\tilde{f}_{K_1 K_2}: M(K_1) \to M(K_2): K_2 \subset K_1, K_1, K_2 \in \mathcal{K}(S)\}$ where $M(K)$ denotes the space of all Radon-measures on K.

If $\mu = \{\mu_K\}_{K\in\mathcal{K}}$ is a premeasure on S and $|\mu_K|^*$ denotes the outer measure on K defined by $|\mu_K|$ then we define, for a set $C \subset S$ $|\mu|^*(C) = \sup\{|\mu_K|^*(C \cap K): K \in \mathcal{K}(S)\}$. A premeasure μ on S is said to be tight if for each $\varepsilon > 0$ there is a $K \in \mathcal{K}$ so that $|\mu|^*(S\setminus K) < \varepsilon$. Equivalent is the existence of an increasing sequence K_n in $\mathcal{K}(S)$ with $|\mu|^*(S\setminus K_n) \to 0$.

We denote by $M_t(S)$ the space of all tight (pre) measures on S.

If $x \in \mathcal{C}^\infty(S)$ and $\mu \in M_t(S)$, then the limit $\lim_{n\to\infty} \int x \mid K_n \, d\mu_{K_n}$ exists and is independent of the particular choice of the sequence K_n. We write $\int x \, d\mu$ for this limit.

A premeasure $\mu = \{\mu_K\}$ is said to have compact support if there is a $K \in \mathcal{K}(S)$ so that $|\mu|^*(S \setminus K) = 0$. The space $M_o(S)$ of premeasures with compact support is identifiable with $\bigcup_{K \in \mathcal{K}} M(K)$.

Proposition: (COOPER) If S is a regular compactology, then the dual of $[C^\infty(S), \tau_{\mathcal{K}(S)}]$ is naturally isomorphic to $M_o(S)$ under the bilinear form $(x,\mu) \to \int x \, d\mu$; and the dual of $[C^\infty(S), \gamma[\|.\|, \tau]]$ is naturally isomorphic to $M_t(S)$ under the bilinear form

$$(x,\mu) \to \int x \, d\mu .$$

So $C^\infty(S)'_\gamma = M_t(S)$.

4.3. If S is a compactological group, then we can give an explicit description of the multiplication * (3.1) in $M_t(S)$:

If $x \in C^\infty(S)$, μ, $\nu \in M_t(S)$, then we have

$$\begin{aligned} \int x \, d\,(\mu * \nu) &= (\mu \otimes \nu)\,(c(x)) \\ &= \int x(s.t)\, d(\mu \otimes \nu)(s.t) \\ &= \iint x(s.t)\, d\mu(s) d\nu(t), \end{aligned}$$

i.e. we have the ordinary convolution.

4.4. If S is a regular compactological space and E a complete object of MIXTOP, then define $C^\infty(S;E)$ as the space of all $\|.\|$-bounded maps $f: S \to E$, equipped with the pointwise linear structure and the following mixed structure:

$\|.\|$ - the norm $\|f\| = \sup \{ \|f(s)\|_E, s \in S \}$.

τ - the topology of uniform τ_E-convergence on members of $\mathcal{K}(S)$.

Proposition: If S is a regular compactological space, then the embedding $\delta : S \to M_t(S)$ has the following universal property: for every complete object E of MIXTOP and every $f \in C^\infty(S;E)$ there is a unique $T \in L(M_t(S);E)$ which extends f via δ; here $L(M_t(S);E) \cong C^\infty(S)\hat{\hat{\otimes}}\gamma E$ is the space of linear maps which are continuous in a rather complicated structure on $M_t(S)$ – for simplicity's sake we take this equality for definition. So we have $C^\infty(S;E) \cong C^\infty(S)\, \hat{\hat{\otimes}}\, \gamma\, E \cong L(M_t(S);E)$.

4.5. If S is a compactological group and E a complete object in MIXTOP, then $C^\infty(S,E)$ has a natural map c_E: $C^\infty(S;E) \to C^\infty(S\times S;E)$, given by $c_E : x \mapsto ((s,t) \mapsto x(st))$;

i.e. $c_E = c \otimes id_E$: $C^\infty(S)\, \hat{\hat{\otimes}}_\gamma\, E \;\to\; C^\infty(S)\, \hat{\hat{\otimes}}_\gamma\, C^\infty(S)\, \hat{\hat{\otimes}}_\gamma\, E$.

Let $(E,\|.\|,\tau)$ be a complete algebra with unit e in MIXTOP, i.e. there is an associative multiplication $m: E \times E \to E$ so that $(E,\|.\|)$ is a Banach algebra and $m|_{B_{\|.\|} \times B_{\|.\|}}$ is $\tilde{\tau} \times \tilde{\tau}$ -continuous.

Then $C^\infty(S;E)$ has a natural "multiplication".

m_E: $C^\infty(S;E) \times C^\infty(S;E) \to C^\infty(S \times S;E)$, given by

m_E: $(x,y) \mapsto ((s,t) \mapsto m(x(s),y(t)))$.

We say that an element $x \in C^\infty(S;E)$ is primitive, if $c_E(x) = m_E(x,x)$.

4.6. Proposition: Under the identification $C^\infty(S;E) \cong L(M_t(S),E)$ the primitive elements correspond to the Banach algebra morphisms. $x \in C^\infty(S;E)$ induces a unit preserving operator if and only if $x(\varepsilon) = e_E$.

Proof: Suppose that $x \in C^\infty(S;E)$ is primitive. The image T_x of x in $L(M_t(S);E)$ is defined by $\mu \mapsto \int_S x\, d\mu$.

Then $T_x\ (\mu * \nu) = \int_S x\, d(\mu * \nu) =$

$$\int_{S\times S} c_E(x)\, d\,(\mu \otimes \nu) = \int_{S\times S} x \otimes x\, d(\mu \otimes \nu)$$

$$= \int_S x\, d\mu \int_S y d\nu = T_x(\mu)\, T_x(\nu).$$

On the other hand, if $T \in L(M_t(S);E)$, then $T = T_x$ where $x = T \circ \delta$.

Then $c_E(x)(s,t) = x(st) = T(\delta_{st})$

$$= T(\delta_s * \delta_t) = m(T(\delta_s), T(\delta_t))$$

$$= m(x(s), x(t)) = m_E(x,x)(s,t).$$

4.7. Corollary: Let S be a regular compactological group, X a Banach space. Then there is a one-one correspondence between

(i) the set of strongly continuous representations of S in X.

(ii) the unit preserving Banach algebra morphisms in $L(M_t(S);E)$

(iii) the primitive elements x of $C^\infty(S;E)$ with $x(e) = e_E$.

(E denotes the object $(L(X,X), \|.\|, \tau_s)$ of MIXTOP - τ_s is the strong operator topology).

§5. Pontryagin duality

5.1. Let (E,c,η,a) be an object of CMIXC*. $x \in E$ is called <u>strongly primitive</u> if

(i) $c(x) = x \otimes x$

(ii) $\eta(x) = 1$

(iii) $a(x) = x^{-1}$ (in E).

We denote by $P(E)$ the set of strongly primitive elements of E.

5.2. <u>Proposition</u>. $P(E)$, with the topology and the multiplicative structure induced from $(E,\gamma(\|.\|_E,\tau_E))$, is a topological group. It is contained in $\{x \in E: \|x\| = 1\}$.

<u>Proof</u>: $P(E)$ is closed under multiplication:

Let $x, y \in P(E)$, then

$c(xy) = c(x)c(y) = (x \otimes x)(y \otimes y) = xy \otimes xy$.

$\eta(xy) = \eta(x)\,\eta(y) = 1.1 = 1$

$a(xy) = a(x)a(y) = x^{-1}\,y^{-1} = (xy)^{-1}$

(E is commutative: the C^*-algebra part).

The constant function 1 is a unit for $P(E)$.

If $x \in P(E)$, then $a(x) = x^{-1}$ is an inverse for x. Since multiplication is $\gamma[\|.\|_E, \tau_E)$ continuous, $(P(E),\gamma[\|.\|_E,\tau_E])$ is a topological group. Since $\|x\|\,\|x\| = \|x \otimes x\| = \|c(x)\| \leq \|x$ and $1 = \|1\| = \|\eta(x)\| \leq \|x\|$ we conclude that $\|x\| = 1$ for all $x \in P(E)$.

5.3. Definition: Let S be a commutative compactological group. A character on S is a GCPTOL-morphism from S into the circle group T. The set $\hat{S}$ of all characters form a group. $\hat{S}$, with the topology of uniform convergence on members of $\mathcal{K}(S)$, is a topological group, and it is complete in this uniformity.

5.4. Proposition: Let S be a commutative, compactological group. The $\hat{S} = P(C^\infty(S))$ (a topological group).

Proof: If $x \in P(C^\infty(S))$, then $\| x \| = 1 = \| x^{-1} \|$, so x takes its values in T.

$c(x)(s,t) = x(st)$ and

$(x \otimes x)(s,t) = x(s)x(t)$ show that the strong primitivity of x is equivalent to its being a character.

Since $\gamma [\|.\|, \tau] = \tau$ on $B_{\|.\|}$ we see that even the topologies coincide.

5.5. Corollary: Let S be a commutative, regular compactological group. The $\hat{S}$ separates S if and only if the vector space generated by $P(C^\infty(S))$ is γ-dense in $C^\infty(S)$.

REFERENCES

C. APOSTOL — B^*-algebras and their representations, Jour. Lond. Math. Soc. (2) 3 (1971) 30-38.

H. BUCHWALTER — Topologies, bornologies et compactologies, Lyon

J. B. COOPER — The mixed topology and applications.

H. HEYER — Dualität lokalkompakter Gruppen, Springer Lecture Notes Nr. 150 (Berlin, 1970).

E. HEWITT, K.A. ROSS — The Tannaka-Krein duality theorems, Jahresber. D.M.V. 71 (1969) 61-83.

Abstract harmonic analysis II (Berlin, 1970).

K. H. HOFMANN — The duality of compact semigroups and C^*-bigebras, Springer Lecture Notes Nr. 129 (Berlin 1970).

P. MICHOR — Duality in Groups, unpublished note - 1972.

J.W. NEGREPONTIS — (J.W. Pelletier) Duality in analysis from the point of view of triples, Jour. of Algebra 19 (1971) 228-253.

N. NOBLE — k-groups and duality, Trans. Amer. Math. Soc. 151 (1970) 551-561.

D. W. ROEDER — Category theory applied to Pantryagin duality, Pac. J. Math. 52 (1974) 519-527.

S. SANKARAN, S.A. SELESNICK — Some remakrs on C^*-bigebras and duality, Semigroups Forum 3 (1971) 108-129.

S. A. SELESNICK — Watts cohomology for a class of Banach algebras and the duality of compact abelian groups, Math. Z. 130 (1973) 313-323.

M. TAKESAKI — Duality and von Neumann algebras (in "Lectures on OPerator Algebras" - Springer Lecture Notes Nr. 247 - Berlin, 1972).