Overview on Geometries of Diffeomorphism Groups

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- An extremely short introduction to convenient calculus in infinite dimensions. (I resisted the temptation to present this. See http://www.mat.univie.ac.at/˜michor/Pisa-2014.pdf)
- A zoo of diffeomorphism groups on $\mathbb{R}^n$
- A diagram of actions of diffeomorphism groups
- Riemannian geometries on diffeomorphism groups and spaces of immersions and shape spaces.
- Transforming a Riemannian metric on $\mathrm{Diff}(\mathbb{R})$ to a flat space, or solving the Hunter-Saxton equation.
- Sobolev Metrics on Diffeomorphism Groups, and the Derived Geometry of Shape Spaces.

A Zoo of diffeomorphism groups on $\mathbb{R}^n$

For suitable convenient vector space $\mathcal{A}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$ let $\operatorname{Diff}_{\mathcal{A}}(\mathbb{R}^n)$ be the group of all diffeomorphisms of $\mathbb{R}^n$ of the form $\operatorname{Id} + f$ for $f \in \mathcal{A}(\mathbb{R}^n)^n$ with $\det(\mathbb{I}_n + df(x)) \geq \varepsilon > 0$.

Theorem. *The sets of diffeomorphisms* $\operatorname{Diff}_c(\mathbb{R}^n)$, $\operatorname{Diff}_{\mathcal{S}}(\mathbb{R}^n)$, $\operatorname{Diff}_{H^\infty}(\mathbb{R}^n)$, *and* $\operatorname{Diff}_{\mathcal{B}}(\mathbb{R}^n)$ *are all smooth regular Lie groups. We have the following smooth injective group homomorphisms*

$$\operatorname{Diff}_c(\mathbb{R}^n) \longrightarrow \operatorname{Diff}_{\mathcal{S}}(\mathbb{R}^n) \longrightarrow \operatorname{Diff}_{H^\infty}(\mathbb{R}^n) \longrightarrow \operatorname{Diff}_{\mathcal{B}}(\mathbb{R}^n)\,.$$

Each group is a normal subgroup in any other in which it is contained, in particular in $\operatorname{Diff}_{\mathcal{B}}(\mathbb{R}^n)$. *Similarly for suitable Denjoy-Carleman spaces of ultradifferentiable functions both of Roumieu and Beurling type:*

$$\operatorname{Diff}_{\mathcal{D}^{[M]}}(\mathbb{R}^n) \rightarrowtail \operatorname{Diff}_{\mathcal{S}^{[M]}_{[L]}}(\mathbb{R}^n) \rightarrowtail \operatorname{Diff}_{W^{[M],p}}(\mathbb{R}^n) \rightarrowtail \operatorname{Diff}_{W^{[M],q}}(\mathbb{R}^n) \rightarrowtail \operatorname{Diff}_{\mathcal{B}^{[M]}}(\mathbb{R}^n).$$

Here we require that the $M = (M_k)$ *is log-convex and has moderate growth, and that also* $C_b^{(M)} \supseteq C^\omega$ *in the Beurling case.*

[M,Mumford,2013], partly [B.Walter,2012]; for Denjoy-Carleman ultradifferentiable diffeomorphisms [Kriegl, M, Rainer 2014].

A diagram of actions of diffeomorphism groups.

$$
\begin{tikzcd}
\mathrm{Diff}(M) \arrow[rr, "\text{r-acts}"] \arrow[dr, "\text{r-acts}"] \arrow[dd, "\mathrm{Diff}(M,\mu)"] & & \mathrm{Imm}(M,N) \arrow[dl, "\text{needs } \bar{g}"] \arrow[dr, "\mathrm{Diff}(M)"'] & & \mathrm{Diff}_{\mathcal{A}}(N) \arrow[ll, "\text{l-acts}"', "\text{(LDDMM)}"] \arrow[dl, "\text{l-acts}"', "\text{(LDDMM)}"] \arrow[dd, "\text{r-acts}"] \\
& \mathrm{Met}(M) \arrow[dr, "\mathrm{Diff}(M)"'] & & B_i(M,N) \arrow[dl, "\text{needs } \bar{g}"] & \\
\mathrm{Vol}^1_+(M) & & \dfrac{\mathrm{Met}(M)}{\mathrm{Diff}(M)} & & \mathrm{Met}_{\mathcal{A}}(N)
\end{tikzcd}
$$

M compact , N pssibly non-compact manifold

$\mathrm{Met}(N) = \Gamma(S^2_+ T^* N)$	space of all Riemann metrics on N
$\bar{g}$	one Riemann metric on N
$\mathrm{Diff}(M)$	Lie group of all diffeos on compact mf M
$\mathrm{Diff}_{\mathcal{A}}(N),\ \mathcal{A} \in \{H^\infty, \mathcal{S}, c\}$	Lie group of diffeos of decay $\mathcal{A}$ to Id_N
$\mathrm{Imm}(M, N)$	mf of all immersions $M \to N$
$B_i(M, N) = \mathrm{Imm}/\mathrm{Diff}(M)$	shape space
$\mathrm{Vol}^1_+(M) \subset \Gamma(\mathrm{vol}(M))$	space of positive smooth probability densities

$$
\begin{tikzcd}
\mathrm{Diff}(S^1) \arrow[rr, "\text{r-acts}"] \arrow[d, "\text{r-acts}"] \arrow[dr, "\text{r-acts}"] & & \mathrm{Imm}(S^1,\mathbb{R}^2) \arrow[dl, "\text{needs } \bar{g}"] \arrow[dr, "\mathrm{Diff}(S^1)"] & & \mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^2) \arrow[ll, "\text{l-acts}", "\text{(LDDMM)}"'] \arrow[dl, "\text{l-acts}", "\text{(LDDMM)}"'] \arrow[dd, "\text{r-acts}"] \\
\mathrm{Vol}_+(S^1) \arrow[d] & \mathrm{Met}(S^1) \arrow[dr, "\mathrm{Diff}(S^1)"] & & B_i(S^1,\mathbb{R}^2) \arrow[dl, "\text{needs } \bar{g}"] & \\
\dfrac{\mathrm{Vol}_+(S^1)}{\mathrm{Diff}(S^1)} \arrow[r, "\int f d\theta", "="'] & \mathbb{R}_{>0} & \dfrac{\mathrm{Met}(S^1)}{\mathrm{Diff}(S^1)} \arrow[l, "=", "\int \sqrt{g} d\theta"'] & & \mathrm{Met}(\mathbb{R}^2)
\end{tikzcd}
$$

$\mathrm{Diff}(S^1)$	Lie group of all diffeos on compact mf S^1
$\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^2)$, $\mathcal{A} \in \{\mathcal{B}, H^\infty, \mathcal{S}, c\}$	Lie group of diffeos of decay $\mathcal{A}$ to $\mathrm{Id}_{\mathbb{R}^2}$
$\mathrm{Imm}(S^1,\mathbb{R}^2)$	mf of all immersions $S^1 \to \mathbb{R}^2$
$B_i(S^1,\mathbb{R}^2) = \mathrm{Imm}/\mathrm{Diff}(S^1)$	shape space
$\mathrm{Vol}_+(S^1) = \left\{ f\, d\theta : f \in C^\infty(S^1,\mathbb{R}_{>0}) \right\}$	space of positive smooth probability densities
$\mathrm{Met}(S^1) = \left\{ g\, d\theta^2 : g \in C^\infty(S^1,\mathbb{R}_{>0}) \right\}$	space of metrics on S^1

L^2 metric

$$\mathrm{Diff}(S^1): \qquad G^0_\varphi(X\circ\varphi, Y\circ\varphi) = G^0_{\mathrm{Id}}(X,Y) = \int_{S^1} X(\theta)Y(\theta)d\theta$$

$$\mathrm{Imm}(S^1,\mathbb{R}^2): \qquad G^0_c(h,k) = \int_{S^1}\langle h(\theta), k(\theta)\rangle ds$$

$$\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^2): \qquad G^0_\varphi(X\circ\varphi, Y\circ\varphi) = G^0_{\mathrm{Id}}(X,Y) = \int_{\mathbb{R}^2} X(x)Y(x)dx$$

Problem: The induced geodesic distance vanishes.

$$\begin{tikzcd}
\mathrm{Diff}(S^1) \arrow[rr, "\text{r-acts}"] \arrow[dr, "\text{r-acts}"'] & & \mathrm{Imm}(S^1,\mathbb{R}^2) \arrow[dl, "\text{needs } \bar{g}"] \arrow[dr, "\mathrm{Diff}(S^1)"'] & & \mathrm{Diff}_c(\mathbb{R}^2) \arrow[ll, "\text{l-acts}"', "\text{(LDDMM)}"] \arrow[dl, "\text{l-acts}"', "\text{(LDDMM)}"] \\
& \mathrm{Met}(S^1) & & B_i(S^1,\mathbb{R}^2) &
\end{tikzcd}$$

Movies about vanishing: $\mathrm{Diff}(S^1)$ $\quad B_i(S^1,\mathbb{R}^2)$

[MichorMumford2005a,2005b], [BauerBruverisHarmsMichor2011,2012]

Metrics on the space of immersions of the form:

$$\mathrm{Diff}_{\mathcal{A}}(N): \qquad G^P_\varphi(X\circ\varphi, Y\circ\varphi) = \int_N \bar{g}(PX, Y)\,\mathrm{vol}(\bar{g})$$

$$\mathrm{Imm}(M,N): \qquad G^P_f(h,k) = \int_M \bar{g}(P^f h, k)\,\mathrm{vol}(f^*\bar{g})$$

where $\bar{g}$ is some fixed metric on N, $g = f^*\bar{g}$ is the induced metric on M, $h,k \in \Gamma(f^*TN)$ are tangent vectors at f to $\mathrm{Imm}(M,N)$, and P^f is a positive, selfadjoint, bijective (scalar) pseudo differential operator of order $2p$ depending smoothly on f. Good example: $P^f = 1 + A(\Delta^g)^p$, where Δ^g is the Bochner-Laplacian on M induced by the metric $g = f^*\bar{g}$. Also P has to be $\mathrm{Diff}(M)$-invariant: $\varphi^* \circ P_f = P_{f\circ\varphi}\circ\varphi^*$.

Advantages of Sobolev type metrics:

1. Positive geodesic distance
2. Geodesic equations are well posed
3. Spaces are geodesically complete for $p > \frac{\dim(M)}{2} + 1$.

   [Bruveris, M, Mumford, 2013] for plane curves. A remark in [Ebin, Marsden, 1970], and [Bruveris, Meyer, 2014] for diffeomorphism groups.

Problems:

1. Analytic solutions to the geodesic equation?
2. Curvature of shape space with respect to these metrics?
3. Numerics are in general computational expensive

$$
\begin{tikzcd}
\begin{array}{c}\text{wellp.:}\\ \text{Space:}\\ \text{geod. dist.:}\end{array} & \begin{array}{c}p\geq 1/2\\ \mathrm{Diff}(S^1)\\ +:p>\frac{1}{2},-:p\leq\frac{1}{2}\end{array} \arrow[rr, "\text{r-acts}"] \arrow[dr, "\text{r-acts}"] & & \begin{array}{c}p\geq 1\\ \mathrm{Imm}(S^1,\mathbb{R}^2)\\ -:p=0,+:p\geq 1\end{array} \arrow[dl, "\text{needs } \bar{g}"'] \arrow[dr, "\mathrm{Diff}(S^1)"'] & & \begin{array}{c}p\geq 1\\ \mathrm{Diff}_c(\mathbb{R}^2)\\ -:p<\frac{1}{2},+:p\geq 1\end{array} \arrow[ll, "\text{l-acts}", "\text{(LDDMM)}"'] \arrow[dl, "\text{l-acts}"', "\text{(LDDMM)}"] \\
\begin{array}{c}\text{wellp.:}\\ \text{Space:}\\ \text{geod. dist.:}\end{array} & & \begin{array}{c}p\geq 0\\ \mathrm{Met}(S^1)\\ +:p\geq 0\end{array} & & \begin{array}{c}p\geq 1\\ B_i(S^1,\mathbb{R}^2)\\ -:p=0,+:p\geq 1\end{array} &
\end{tikzcd}
$$

Sobolev type metrics

Advantages of Sobolev type metrics:

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   [Bruveris, M, Mumford, 2013] for plane curves. A remark in [Ebin, Marsden, 1970], and [Bruveris, Viallard, 2014] for diffeomorphism groups.

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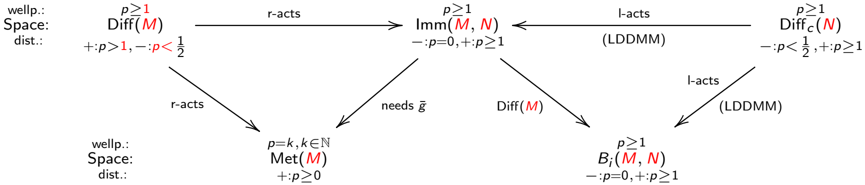

Right invariant Riemannian geometries on Diffeomorphism groups.

For $M = N$ the space $\mathrm{Emb}(M, M)$ equals the *diffeomorphism group of* M. An operator $P \in \Gamma\big(L(T\,\mathrm{Emb}; T\,\mathrm{Emb})\big)$ that is invariant under reparametrizations induces a right-invariant Riemannian metric on this space. Thus one gets the geodesic equation for right-invariant Sobolev metrics on diffeomorphism groups and well-posedness of this equation. The geodesic equation on $\mathrm{Diff}(M)$ in terms of the momentum p is given by

$$\begin{cases} \qquad p = Pf_t \otimes \mathrm{vol}(g), \\ \nabla_{\partial_t} p = -Tf.\bar{g}(Pf_t, \nabla f_t)^\sharp \otimes \mathrm{vol}(g). \end{cases}$$

Note that this equation is not right-trivialized, in contrast to the equation given in [Arnold 1966]. The special case of theorem now reads as follows:

Theorem. [Bauer, Harms, M, 2011] *Let* $p \geq 1$ *and* $k > \frac{\dim(M)}{2} + 1$ *and let* P *satisfy suitable assumptions.*
The initial value problem for the geodesic equation has unique local solutions in the Sobolev manifold Diff^{k+2p} *of* H^{k+2p}*-diffeomorphisms. The solutions depend smoothly on* t *and on the initial conditions* $f(0,\ .\ )$ *and* $f_t(0,\ .\ )$. *The domain of existence (in* t*) is uniform in* k *and thus this also holds in* $\mathrm{Diff}(M)$.
Moreover, in each Sobolev completion Diff^{k+2p}*, the Riemannian exponential mapping* $\exp^P$ *exists and is smooth on a neighborhood of the zero section in the tangent bundle, and* $(\pi, \exp^P)$ *is a diffeomorphism from a (smaller) neigbourhood of the zero section to a neighborhood of the diagonal in* $\mathrm{Diff}^{k+2p} \times \mathrm{Diff}^{k+2p}$. *All these neighborhoods are uniform in* $k > \dim(M)/2 + 1$ *and can be chosen* H^{k_0+2p}*-open, for* $k_0 > \dim(M)/2 + 1$. *Thus both properties of the exponential mapping continue to hold in* $\mathrm{Diff}(M)$.

Geodesics of a Right-Invariant Metric on a Lie Group

Let $\gamma = \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be a positive-definite bounded (weak) inner product. Then

$$\gamma_x(\xi, \eta) = \gamma\big(T(\mu^{x^{-1}}) \cdot \xi,\ T(\mu^{x^{-1}}) \cdot \eta\big) = \gamma\big(\kappa(\xi),\ \kappa(\eta)\big)$$

is a right-invariant (weak) Riemannian metric on G. Denote by $\check{\gamma} : \mathfrak{g} \to \mathfrak{g}^*$ the mapping induced by γ, and by $\langle \alpha, X \rangle_{\mathfrak{g}}$ the duality evaluation between $\alpha \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$.
Let $g : [a, b] \to G$ be a smooth curve. The velocity field of g, viewed in the right trivializations, coincides with the right logarithmic derivative

$$\delta^r(g) = T(\mu^{g^{-1}}) \cdot \partial_t g = \kappa(\partial_t g) = (g^*\kappa)(\partial_t).$$

The energy of the curve $g(t)$ is given by

$$E(g) = \frac{1}{2} \int_a^b \gamma_g(g', g')dt = \frac{1}{2} \int_a^b \gamma\big((g^*\kappa)(\partial_t), (g^*\kappa)(\partial_t)\big)\, dt.$$

Thus the curve $g(0,t)$ is critical for the energy if and only if

$$\check{\gamma}(\partial_t(g^*\kappa)(\partial_t)) + (\mathrm{ad}_{(g^*\kappa)(\partial_t)})^*\check{\gamma}((g^*\kappa)(\partial_t)) = 0.$$

In terms of the right logarithmic derivative $u : [a,b] \to \mathfrak{g}$ of $g : [a,b] \to G$, given by $u(t) := g^*\kappa(\partial_t) = T_{g(t)}(\mu^{g(t)^{-1}}) \cdot g'(t)$, the geodesic equation has the expression

$$\boxed{\partial_t u = -\,\check{\gamma}^{-1}\,\mathrm{ad}(u)^*\,\check{\gamma}(u)} \tag{1}$$

Thus the geodesic equation exists in general if and only if $\mathrm{ad}(X)^*\check{\gamma}(X)$ is in the image of $\check{\gamma} : \mathfrak{g} \to \mathfrak{g}^*$, i.e.

$$\mathrm{ad}(X)^*\check{\gamma}(X) \in \check{\gamma}(\mathfrak{g}) \tag{2}$$

for every $X \in \mathfrak{X}$. Condition (2) then leads to the existence of the Christoffel symbols. [Arnold 1966] has the more restrictive condition $\mathrm{ad}(X)^*\check{\gamma}(Y) \in \check{\gamma} \in \mathfrak{g}$. The geodesic equation for the *momentum* $p := \gamma(u)$:

$$p_t = -\,\mathrm{ad}(\check{\gamma}^{-1}(p))^*p.$$

Soon we shall encounter situations where only the more general condition is satisfied, but where the usual transpose $\mathrm{ad}^\top(X)$ of $\mathrm{ad}(X)$,

$$\mathrm{ad}^\top(X) := \check{\gamma}^{-1} \circ \mathrm{ad}^*_X \circ \check{\gamma}$$

does not exist for all X.

Groups related to $\mathrm{Diff}_c(\mathbb{R})$

The reflexive nuclear (LF) space $C_c^\infty(\mathbb{R})$ of smooth functions with compact support leads to the well-known regular Lie group $\mathrm{Diff}_c(\mathbb{R})$.
Define $C_{c,2}^\infty(\mathbb{R}) = \{f : f' \in C_c^\infty(\mathbb{R})\}$ to be the space of antiderivatives of smooth functions with compact support. It is a reflexive nuclear (LF) space. We also define the space $C_{c,1}^\infty(\mathbb{R}) = \left\{f \in C_{c,2}^\infty(\mathbb{R}) \;:\; f(-\infty) = 0\right\}$ of antiderivatives of the form $x \mapsto \int_{-\infty}^x g\, dy$ with $g \in C_c^\infty(\mathbb{R})$.
$\mathrm{Diff}_{c,2}(\mathbb{R}) = \left\{\varphi = \mathrm{Id} + f : f \in C_{c,2}^\infty(\mathbb{R}), f' > -1\right\}$ *is the corresponding group.*
Define the two functionals $\mathrm{Shift}_\ell, \mathrm{Shift}_r : \mathrm{Diff}_{c,2}(\mathbb{R}) \to \mathbb{R}$ by

$$\mathrm{Shift}_\ell(\varphi) = \mathrm{ev}_{-\infty}(f) = \lim_{x\to-\infty} f(x), \quad \mathrm{Shift}_r(\varphi) = \mathrm{ev}_\infty(f) = \lim_{x\to\infty} f(x)$$

for $\varphi(x) = x + f(x)$.

Then the short exact sequence of smooth homomorphisms of Lie groups

$$\mathrm{Diff}_c(\mathbb{R}) \rightarrowtail \mathrm{Diff}_{c,2}(\mathbb{R}) \xrightarrow{(\mathrm{Shift}_\ell, \mathrm{Shift}_r)} (\mathbb{R}^2, +)$$

describes a semidirect product, where a smooth homomorphic section $s : \mathbb{R}^2 \to \mathrm{Diff}_{c,2}(\mathbb{R})$ is given by the composition of flows $s(a, b) = \mathrm{Fl}_a^{X_\ell} \circ \mathrm{Fl}_b^{X_r}$ for the vectorfields $X_\ell = f_\ell \partial_x$, $X_r = f_r \partial_x$ with $[X_\ell, X_r] = 0$ where $f_\ell, f_r \in C^\infty(\mathbb{R}, [0, 1])$ satisfy

$$f_\ell(x) = \begin{cases} 1 & \text{for } x \leq -1 \\ 0 & \text{for } x \geq 0, \end{cases} \qquad f_r(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ 1 & \text{for } x \geq 1. \end{cases} \tag{3}$$

The normal subgroup $\mathrm{Diff}_{c,1}(\mathbb{R}) = \ker(\mathrm{Shift}_\ell) = \{\varphi = \mathrm{Id} + f : f \in C^\infty_{c,1}(\mathbb{R}), f' > -1\}$ of diffeomorphisms which have no shift at $-\infty$ will play an important role later on.

Some diffeomorphism groups on $\mathbb{R}$

We have the following smooth injective group homomorphisms:

$$
\begin{tikzcd}
\mathrm{Diff}_c(\mathbb{R}) \arrow[r] \arrow[d] & \mathrm{Diff}_{\mathcal{S}}(\mathbb{R}) \arrow[r] \arrow[d] & \mathrm{Diff}_{W^{\infty,1}}(\mathbb{R}) \arrow[d] & \\
\mathrm{Diff}_{c,1}(\mathbb{R}) \arrow[r] \arrow[d] & \mathrm{Diff}_{\mathcal{S}_1}(\mathbb{R}) \arrow[r] \arrow[d] & \mathrm{Diff}_{W_1^{\infty,1}}(\mathbb{R}) \arrow[d] & \\
\mathrm{Diff}_{c,2}(\mathbb{R}) \arrow[r] & \mathrm{Diff}_{\mathcal{S}_2}(\mathbb{R}) \arrow[r] & \mathrm{Diff}_{W_2^{\infty,1}}(\mathbb{R}) \arrow[r] & \mathrm{Diff}_{\mathcal{B}}(\mathbb{R})
\end{tikzcd}
$$

Each group is a normal subgroup in any other in which it is contained, in particular in $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R})$.
For $\mathcal{S}$ and $W^{\infty,1}$ this works the same as for C_c^∞. For $H^\infty = W^{\infty,2}$ it is surprisingly more subtle.

Solving the Hunter-Saxton equation: The setting

We will denote by $\mathcal{A}(\mathbb{R})$ any of the spaces $C_c^\infty(\mathbb{R})$, $\mathcal{S}(\mathbb{R})$ or $W^{\infty,1}(\mathbb{R})$ and by $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ the corresponding groups $\mathrm{Diff}_c(\mathbb{R})$, $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R})$ or $\mathrm{Diff}_{W^{\infty,1}}(\mathbb{R})$.
Similarly $\mathcal{A}_1(\mathbb{R})$ will denote any of the spaces $C_{c,1}^\infty(\mathbb{R})$, $\mathcal{S}_1(\mathbb{R})$ or $W_1^{\infty,1}(\mathbb{R})$ and $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ the corresponding groups $\mathrm{Diff}_{c,1}(\mathbb{R})$, $\mathrm{Diff}_{\mathcal{S}_1}(\mathbb{R})$ or $\mathrm{Diff}_{W_1^{\infty,1}}(\mathbb{R})$.
The $\dot{H}^1$-metric. For $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ and $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ the homogeneous H^1-metric is given by

$$G_\varphi(X \circ \varphi, Y \circ \varphi) = G_{\mathrm{Id}}(X, Y) = \int_{\mathbb{R}} X'(x) Y'(x)\, dx \ ,$$

where X, Y are elements of the Lie algebra $\mathcal{A}(\mathbb{R})$ or $\mathcal{A}_1(\mathbb{R})$. We shall also use the notation

$$\langle \cdot, \cdot \rangle_{\dot{H}^1} := G(\cdot, \cdot)\ .$$

Theorem

On $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ *the geodesic equation is the Hunter-Saxton equation*

$$\boxed{(\varphi_t)\circ\varphi^{-1} = u \qquad u_t = -uu_x + \frac{1}{2}\int_{-\infty}^{x}(u_x(z))^2\,dz\ ,}$$

and the induced geodesic distance is positive.
On the other hand the geodesic equation does not exist on the subgroups $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$*, since the adjoint* $\mathrm{ad}(X)^*\check{G}_{\mathrm{Id}}(X)$ *does not lie in* $\check{G}_{\mathrm{Id}}(\mathcal{A}(\mathbb{R}))$ *for all* $X\in\mathcal{A}(\mathbb{R})$*.*
One obtains the classical form of the Hunter-Saxton equation by differentiating:

$$u_{tx} = -uu_{xx} - \frac{1}{2}u_x^2\ ,$$

Note that $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ is a natural example of a non-robust Riemannian manifold.

Proof

Note that $\check{G}_{\mathrm{Id}} : \mathcal{A}_1(\mathbb{R}) \to \mathcal{A}_1(\mathbb{R})^*$ is given by $\check{G}_{\mathrm{Id}}(X) = -X''$ if we use the L^2-pairing $X \mapsto (Y \mapsto \int XYdx)$ to embed functions into the space of distributions. We now compute the adjoint of $\mathrm{ad}(X)$:

$$\begin{aligned}\left\langle \mathrm{ad}(X)^* \check{G}_{\mathrm{Id}}(Y), Z \right\rangle &= \check{G}_{\mathrm{Id}}(Y, \mathrm{ad}(X)Z) = G_{\mathrm{Id}}(Y, -[X, Z]) \\ &= \int_{\mathbb{R}} Y'(x)\big(X'(x)Z(x) - X(x)Z'(x)\big)'\, dx \\ &= \int_{\mathbb{R}} Z(x)\big(X''(x)Y'(x) - (X(x)Y'(x))''\big)\, dx \,.\end{aligned}$$

Therefore the adjoint as an element of $\mathcal{A}_1^*$ is given by

$$\mathrm{ad}(X)^* \check{G}_{\mathrm{Id}}(Y) = X''Y' - (XY')'' \,.$$

For $X = Y$ we can rewrite this as

$$\mathrm{ad}(X)^* \check{G}_{\mathrm{Id}}(X) = \tfrac{1}{2}\big((X'^2)' - (X^2)'''\big) = \frac{1}{2}\Big(\int_{-\infty}^{x} X'(y)^2\,dy - (X^2)'\Big)''$$
$$= \frac{1}{2}\check{G}_{\mathrm{Id}}\Big(-\int_{-\infty}^{x} X'(y)^2\,dy + (X^2)'\Big)\,.$$

If $X \in \mathcal{A}_1(\mathbb{R})$ then the function $-\frac{1}{2}\int_{-\infty}^{x} X'(y)^2\,dy + \frac{1}{2}(X^2)'$ is again an element of $\mathcal{A}_1(\mathbb{R})$. This follows immediately from the definition of $\mathcal{A}_1(\mathbb{R})$. Therefore the geodesic equation exists on $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ and is as given.
However if $X \in \mathcal{A}(\mathbb{R})$, a neccessary condition for $\int_{-\infty}^{x}(X'(y))^2 dy \in \mathcal{A}(\mathbb{R})$ would be $\int_{-\infty}^{\infty} X'(y)^2 dy = 0$, which would imply $X' = 0$. Thus the geodesic equation does not exist on $\mathcal{A}(\mathbb{R})$.
The positivity of geodesic distance will follow from the explicit formula for geodesic distance below. QED.

Theorem.

[BBM2014] [A version for *Diff* (S^1) is by J.Lenells 2007,08,11]
We define the R-map by:

$$R : \begin{cases} \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R}) \to \mathcal{A}\big(\mathbb{R}, \mathbb{R}_{>-2}\big) \subset \mathcal{A}(\mathbb{R}, \mathbb{R}) \\ \varphi \mapsto 2\left((\varphi')^{1/2} - 1\right) . \end{cases}$$

The R-map is invertible with inverse

$$R^{-1} : \begin{cases} \mathcal{A}\big(\mathbb{R}, \mathbb{R}_{>-2}\big) \to \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R}) \\ \gamma \mapsto x + \dfrac{1}{4}\displaystyle\int_{-\infty}^{x} \gamma^2 + 4\gamma \, dx . \end{cases}$$

The pull-back of the flat L^2-metric via R is the $\dot{H}^1$-metric on $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$*, i.e.,*

$$R^* \langle \cdot, \cdot \rangle_{L^2} = \langle \cdot, \cdot \rangle_{\dot{H}^1} .$$

Thus the space $\left(\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R}), \dot{H}^1\right)$ *is a flat space in the sense of Riemannian geometry.*
Here $\langle \cdot, \cdot \rangle_{L^2}$ denotes the L^2-inner product on $\mathcal{A}(\mathbb{R})$ with constant volume dx.

Proof

To compute the pullback of the L^2-metric via the R-map we first need to calculate its tangent mapping. For this let $h = X \circ \varphi \in T_\varphi \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ and let $t \mapsto \psi(t)$ be a smooth curve in $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ with $\psi(0) = \mathrm{Id}$ and $\partial_t|_0 \psi(t) = X$. We have:

$$\begin{aligned}
T_\varphi R.h &= \partial_t|_0 R(\psi(t)\circ\varphi) = \partial_t|_0 2\Big(((\psi(t)\circ\varphi)_x)^{1/2} - 1\Big) \\
&= \partial_t|_0 2((\psi(t)_x \circ \varphi)\,\varphi_x)^{1/2} \\
&= 2(\varphi_x)^{1/2}\partial_t|_0((\psi(t)_x)^{1/2}\circ\varphi) = (\varphi_x)^{1/2}\Big(\frac{\psi_{tx}(0)}{(\psi(0)_x)^{-1/2}}\circ\varphi\Big) \\
&= (\varphi_x)^{1/2}(X'\circ\varphi) = (\varphi')^{1/2}(X'\circ\varphi)\,.
\end{aligned}$$

Using this formula we have for $h = X_1 \circ \varphi, k = X_2 \circ \varphi$:

$$R^*\langle h, k\rangle_{L^2} = \langle T_\varphi R.h, T_\varphi R.k\rangle_{L^2} = \int_{\mathbb{R}} X_1'(x)X_2'(x)\,dx = \langle h, k\rangle_{\dot{H}^1} \; QED$$

Corollary

Given $\varphi_0, \varphi_1 \in \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ *the geodesic* $\varphi(t,x)$ *connecting them is given by*

$$\varphi(t,x) = R^{-1}\Big((1-t)R(\varphi_0) + tR(\varphi_1)\Big)(x)$$

and their geodesic distance is

$$d(\varphi_0, \varphi_1)^2 = 4\int_{\mathbb{R}} \left((\varphi_1')^{1/2} - (\varphi_0')^{1/2}\right)^2 dx\ .$$

But this construction shows much more: For $\mathcal{S}_1$, C_1^∞, and even for many kinds of Denjoy-Carleman ultradifferentiable model spaces (not explained here). This shows that Sobolev space methods for treating nonlinear PDEs is not the only method.

Corollary: *The metric space* $\left(\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R}), \dot{H}^1\right)$ *is path-connected and geodesically convex but not geodesically complete. In particular, for every* $\varphi_0 \in \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ *and* $h \in T_{\varphi_0} \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$*,* $h \neq 0$ *there exists a time* $T \in \mathbb{R}$ *such that* $\varphi(t, \cdot)$ *is a geodesic for* $|t| < |T|$ *starting at* φ_0 *with* $\varphi_t(0) = h$*, but* $\varphi_x(T, x) = 0$ *for some* $x \in \mathbb{R}$*.*

Theorem: *The square root representation on the diffeomorphism group* $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ *is a bijective mapping, given by:*

$$R : \begin{cases} \mathrm{Diff}_{\mathcal{A}}(\mathbb{R}) \to \left(\mathrm{Im}(R), \|\cdot\|_{L^2}\right) \subset \left(\mathcal{A}\left(\mathbb{R}, \mathbb{R}_{>-2}\right), \|\cdot\|_{L^2}\right) \\ \qquad\qquad \varphi \mapsto 2\left((\varphi')^{1/2} - 1\right) . \end{cases}$$

The pull-back of the restriction of the flat L^2*-metric to* $\mathrm{Im}(R)$ *via* R *is again the homogeneous Sobolev metric of order one. The image of the* R*-map is the splitting submanifold of* $\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$ *given by:*

$$\mathrm{Im}(R) = \left\{\gamma \in \mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2}) : F(\gamma) := \int_{\mathbb{R}} \gamma(\gamma + 4)\, dx = 0\right\} .$$

On the space $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ the geodesic equation does not exist. Still:
Corollary: *The geodesic distance* $d^{\mathcal{A}}$ *on* $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ *coincides with the restriction of* $d^{\mathcal{A}_1}$ *to* $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$*, i.e., for* $\varphi_0, \varphi_1 \in \mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ *we have*

$$d^{\mathcal{A}}(\varphi_0, \varphi_1) = d^{\mathcal{A}_1}(\varphi_0, \varphi_1) \,.$$

Continuing Geodesics Beyond the Group, or How Solutions of the Hunter–Saxton Equation Blow Up

Consider a straight line $\gamma(t) = \gamma_0 + t\gamma_1$ in $\mathcal{A}(\mathbb{R}, \mathbb{R})$. Then $\gamma(t) \in \mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$ precisely for t in an open interval (t_0, t_1) which is finite at least on one side, say, at $t_1 < \infty$. Note that

$$\varphi(t)(x) := R^{-1}(\gamma(t))(x) = x + \frac{1}{4}\int_{-\infty}^{x} \gamma^2(t)(u) + 4\gamma(t)(u)\,du$$

makes sense for all t, that $\varphi(t) : \mathbb{R} \to \mathbb{R}$ is smooth and that $\varphi(t)'(x) \geq 0$ for all x and t; thus, $\varphi(t)$ is monotone non-decreasing. Moreover, $\varphi(t)$ is proper and surjective since $\gamma(t)$ vanishes at $-\infty$ and ∞. Let

$$\mathrm{Mon}_{\mathcal{A}_1}(\mathbb{R}) := \left\{ \mathrm{Id} + f : f \in \mathcal{A}_1(\mathbb{R}, \mathbb{R}), f' \geq -1 \right\}$$

be the monoid (under composition) of all such functions.

For $\gamma \in \mathcal{A}(\mathbb{R}, \mathbb{R})$ let $x(\gamma) := \min\{x \in \mathbb{R} \cup \{\infty\} : \gamma(x) = -2\}$. Then for the line $\gamma(t)$ from above we see that $x(\gamma(t)) < \infty$ for all $t > t_1$. Thus, if the 'geodesic' $\varphi(t)$ leaves the diffeomorphism group at t_1, it never comes back but stays inside $\mathrm{Mon}_{\mathcal{A}_1}(\mathbb{R}) \setminus \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ for the rest of its life. In this sense, $\mathrm{Mon}_{\mathcal{A}_1}(\mathbb{R})$ is a *geodesic completion* of $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$, and $\mathrm{Mon}_{\mathcal{A}_1}(\mathbb{R}) \setminus \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ is the *boundary*.

What happens to the corresponding solution $u(t,x) = \varphi_t(t, \varphi(t)^{-1}(x))$ of the HS equation? In certain points it has infinite derivative, it may be multivalued, or its graph can contain whole vertical intervals. If we replace an element $\varphi \in \mathrm{Mon}_{\mathcal{A}_1}(\mathbb{R})$ by its graph $\{(x, \varphi(x)) : x \in \mathbb{R}\} \subset \mathbb{R}$ we get a smooth 'monotone' submanifold, a smooth monotone relation. The inverse φ^{-1} is then also a smooth monotone relation. Then $t \mapsto \{(x, u(t,x)) : x \in \mathbb{R}\}$ is a (smooth) curve of relations. Checking that it satisfies the HS equation is an exercise left for the interested reader. What we have described here is the *flow completion* of the HS equation in the spirit of [KhesinMichor2004].

Soliton-Like Solutions of the Hunter Saxton equation

For a right-invariant metric G on a diffeomorphism group one can ask whether (generalized) solutions $u(t) = \varphi_t(t) \circ \varphi(t)^{-1}$ exist such that the momenta $\check{G}(u(t)) =: p(t)$ are distributions with finite support. Here the geodesic $\varphi(t)$ may exist only in some suitable Sobolev completion of the diffeomorphism group. By the general theory, the momentum $\mathrm{Ad}(\varphi(t))^* p(t) = \varphi(t)^* p(t) = p(0)$ is constant. In other words,

$$p(t) = (\varphi(t)^{-1})^* p(0) = \varphi(t)_* p(0),$$

i.e., the momentum is carried forward by the flow and remains in the space of distributions with finite support. The infinitesimal version (take ∂_t of the last expression) is

$$p_t(t) = -\mathcal{L}_{u(t)} p(t) = -\operatorname{ad}_{u(t)}{}^* p(t).$$

The space of N-solitons of order 0 consists of momenta of the form $p_{y,a} = \sum_{i=1}^N a_i \delta_{y_i}$ with $(y, a) \in \mathbb{R}^{2N}$. Consider an initial soliton $p_0 = \check{G}(u_0) = -u_0'' = \sum_{i=1}^N a_i\, \delta_{y_i}$ with $y_1 < y_2 < \cdots < y_N$. Let H be the Heaviside function

$$H(x) = \begin{cases} 0, & x < 0, \\ \frac{1}{2}, & x = 0, \\ 1, & x > 0, \end{cases}$$

and $D(x) = 0$ for $x \leq 0$ and $D(x) = x$ for $x > 0$. We will see later why the choice $H(0) = \frac{1}{2}$ is the most natural one; note that the behavior is called the Gibbs phenomenon. With these functions we can write

$$u_0''(x) = -\sum_{i=1}^N a_i \delta_{y_i}(x)$$

$$u_0'(x) = -\sum_{i=1}^N a_i H(x - y_i)$$

$$u_0(x) = -\sum_{i=1}^N a_i D(x - y_i).$$

We will assume henceforth that $\sum_{i=1}^{N} a_i = 0$. Then $u_0(x)$ is constant for $x > y_N$ and thus $u_0 \in H_1^1(\mathbb{R})$; with a slight abuse of notation we assume that $H_1^1(\mathbb{R})$ is defined similarly to $H_1^\infty(\mathbb{R})$. Defining $S_i = \sum_{j=1}^{i} a_j$ we can write

$$u_0'(x) = -\sum_{i=1}^{N} S_i \left(H(x - y_i) - H(x - y_{i+1})\right).$$

This formula will be useful because
$\mathrm{supp}(H(. - y_i) - H(. - y_{i+1})) = [y_i, y_{i+1}]$.
The evolution of the geodesic $u(t)$ with initial value $u(0) = u_0$ can be described by a system of ordinary differential equations (ODEs) for the variables (y, a).
Theorem *The map $(y, a) \mapsto \sum_{i=1}^{N} a_i \delta_{y_i}$ is a Poisson map between the canonical symplectic structure on $\mathbb{R}^{2N}$ and the Lie–Poisson structure on the dual $T_{\mathrm{Id}}^* \mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ of the Lie algebra.*

In particular, this means that the ODEs for (y, a) are Hamilton's equations for the pullback Hamiltonian

$$E(y,a) = \frac{1}{2} G_{\mathrm{Id}}(u_{(y,a)}, u_{(y,a)}),$$

with $u_{(y,a)} = \check{G}^{-1}(\sum_{i=1}^{N} a_i \delta_{y_i}) = -\sum_{i=1}^{N} a_i D(. - y_i)$. We can obtain the more explicit expression

$$\begin{aligned} E(y,a) &= \frac{1}{2}\int_{\mathbb{R}} \left(u_{(y,a)}(x)'\right)^2 \, dx = \frac{1}{2}\int_{\mathbb{R}} \left(\sum_{i=1}^{N} S_i \mathbb{1}_{[y_i, y_{i+1}]}\right)^2 \, dx \\ &= \frac{1}{2}\sum_{i=1}^{N} S_i^2 (y_{i+1} - y_i). \end{aligned}$$

Hamilton's equations $\dot{y}_i = \partial E / \partial a_i$, $\dot{a}_i = -\partial E / \partial y_i$ are in this case

$$\begin{aligned} \dot{y}_i(t) &= \sum_{j=i}^{N-1} S_i(t)(y_{i+1}(t) - y_i(t)), \\ \dot{a}_i(t) &= \frac{1}{2}\left(S_i(t)^2 - S_{i-1}(t)^2\right). \end{aligned}$$

Using the R-map we can find explicit solutions for these equations as follows. Let us write $a_i(0) = a_i$ and $y_i(0) = y_i$. The geodesic with initial velocity u_0 is given by

$$\varphi(t,x) = x + \frac{1}{4}\int_{-\infty}^{x} t^2(u_0'(y))^2 + 4tu_0'(y)\,dy$$
$$u(t,x) = u_0(\varphi^{-1}(t,x)) + \frac{t}{2}\int_{-\infty}^{\varphi^{-1}(t,x)} u_0'(y)^2\,dy.$$

First note that

$$\varphi'(t,x) = \left(1 + \frac{t}{2}u_0'(x)\right)^2$$
$$u'(t,z) = \frac{u_0'\left(\varphi^{-1}(t,z)\right)}{1 + \frac{t}{2}u_0'\left(\varphi^{-1}(t,z)\right)}.$$

Using the identity $H(\varphi^{-1}(t,z) - y_i) = H(z - \varphi(t,y_i))$ we obtain

$$u_0'\left(\varphi^{-1}(t,z)\right) = -\sum_{i=1}^{N} a_i H\left(z - \varphi(t,y_i)\right),$$

and thus

$$\left(u_0'\left(\varphi^{-1}(t,z)\right)\right)' = -\sum_{i=1}^{N} a_i \delta_{\varphi(t,y_i)}(z).$$

Combining these we obtain

$$\begin{aligned} u''(t,z) &= \frac{1}{\left(1 + \frac{t}{2} u_0'\left(\varphi^{-1}(t,z)\right)\right)^2} \left(-\sum_{i=1}^{N} a_i \delta_{\varphi(t,y_i)}(z)\right) \\ &= \sum_{i=1}^{N} \frac{-a_i}{\left(1 + \frac{t}{2} u_0'(y_i)\right)^2} \delta_{\varphi(t,y_i)}(z). \end{aligned}$$

From here we can read off the solution of Hamilton's equations

$$\begin{aligned} y_i(t) &= \varphi(t,y_i) \\ a_i(t) &= -a_i \left(1 + \tfrac{t}{2} u_0'(y_i)\right)^{-2}. \end{aligned}$$

When trying to evaluate $u_0'(y_i)$,

$$u_0'(y_i) = a_i H(0) - S_i,$$

we see that u_0' is discontinuous at y_i and it is here that we seem to have the freedom to choose the value $H(0)$. However, it turns out that we observe the Gibbs phenomenon, i.e., only the choice $H(0) = \frac{1}{2}$ leads to solutions of Hamilton's equations. Also, the regularized theory of multiplications of distributions (Colombeau, Kunzinger et.al.) leads to this choice. Thus we obtain

$$y_i(t) = y_i + \sum_{j=1}^{i-1} \left(\frac{t^2}{4} S_j^2 - tS_j \right) (y_{j+1} - y_j)$$

$$a_i(t) = \frac{-a_i}{\left(1 + \frac{t}{2}\left(\frac{a_i}{2} - S_i\right)\right)^2} = -\left(\frac{S_i}{1 - \frac{t}{2} S_i} - \frac{S_{i-1}}{1 - \frac{t}{2} S_{i-1}} \right).$$

It can be checked by direct computation that these functions indeed solve Hamilton's equations.

Sobolev Metrics on Diffeomorphism Groups, and the Derived Geometry of Shape Spaces

Based on:

[Mario Micheli, Peter W. Michor, David Mumford: Sectional curvature in terms of the cometric, with applications to the Riemannian manifolds of landmarks. SIAM J. Imaging Sci. 5, 1 (2012), 394-433.]

and

[Mario Micheli, Peter W. Michor, David Mumford: Sobolev Metrics on Diffeomorphism Groups and the Derived Geometry of Spaces of Submanifolds. Izvestiya: Mathematics 77:3 (2013), 541-570.]

Riem. Metric on $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$

$$G^L_\varphi(X\circ\varphi, Y\circ\varphi) = G^L_{\mathrm{Id}}(X,Y) = \int_{\mathbb{R}^n}\langle LX, Y\rangle\,dx$$
$$= \int_{\mathbb{R}^n\times\mathbb{R}^n} L(x,y)X(x)Y(y)\,dx\,dy, \quad \text{where}$$
$$L = (1-A\Delta)^l, \quad L(x,y) = \frac{1}{(2\pi)^n}\int_{\xi\in\mathbb{R}^n} e^{i\langle\xi,x-y\rangle}(1+A|\xi|^2)^l d\xi$$
$$L^{-1}\alpha = K\alpha = \int_{\mathbb{R}^n} K(x-y)\alpha(y)\,dy \quad \text{where}$$
$$K(x-y) = K_l(x-y) = \frac{1}{(2\pi)^n}\int_{\xi\in\mathbb{R}^n}\frac{e^{i\langle\xi,x-y\rangle}}{(1+A|\xi|^2)^l}d\xi$$

where K_l is a classical Bessel function of differentiability class C^{2l}.

Landmark space as homogeneus space of solitons

A *landmark* $q = (q_1, \dots, q_N)$ is an N-tuple of distinct points in $\mathbb{R}^n$; so $\mathrm{Land}^N \subset (\mathbb{R}^n)^N$ is open. Let $q^0 = (q^0_1, \dots, q^0_N)$ be a fixed standard template landmark. Then we have the the surjective mapping

$$\begin{aligned} &\mathrm{ev}_{q^0} : \mathrm{Diff}(\mathbb{R}^n) \to \mathrm{Land}^N, \\ &\varphi \mapsto \mathrm{ev}_{q^0}(\varphi) = \varphi(q^0) = (\varphi(q^0_1), \dots, \varphi(q^0_N)). \end{aligned}$$

The fiber of ev_{q^0} over a landmark $q = \varphi_0(q^0)$ is

$$\begin{aligned} &\{\varphi \in \mathrm{Diff}(\mathbb{R}^n) : \varphi(q^0) = q\} \\ &= \varphi_0 \circ \{\varphi \in \mathrm{Diff}(\mathbb{R}^n) : \varphi(q^0) = q^0\} \\ &= \{\varphi \in \mathrm{Diff}(\mathbb{R}^n) : \varphi(q) = q\} \circ \varphi_0; \end{aligned}$$

The tangent space to the fiber is

$$\{X \circ \varphi_0 : X \in \mathfrak{X}_{\mathcal{S}}(\mathbb{R}^n), X(q_i) = 0 \text{ for all } i\}.$$

A tangent vector $Y \circ \varphi_0 \in T_{\varphi_0} \operatorname{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ is $G^L_{\varphi_0}$-perpendicular to the fiber over q if

$$\int_{\mathbb{R}^n} \langle LY, X \rangle \, dx = 0 \quad \forall X \text{ with } X(q) = 0.$$

If we require Y to be smooth then $Y = 0$. So we assume that $LY = \sum_i P_i.\delta_{q_i}$, a distributional vector field with support in q. Here $P_i \in T_{q_i}\mathbb{R}^n$. But then

$$Y(x) = L^{-1}\Big(\sum_i P_i.\delta_{q_i} \Big) = \int_{\mathbb{R}^n} K(x-y) \sum_i P_i.\delta_{q_i}(y) \, dy$$

$$= \sum_i K(x - q_i).P_i$$

$$T_{\varphi_0}(\mathrm{ev}_{q^0}).(Y \circ \varphi_0) = Y(q_k)_k = \sum_i (K(q_k - q_i).P_i)_k$$

Now let us consider a tangent vector $P = (P_k) \in T_q \operatorname{Land}^N$. Its horizontal lift with footpoint φ_0 is $P^{\mathrm{hor}} \circ \varphi_0$ where the vector field P^{hor} on $\mathbb{R}^n$ is given as follows: Let $K^{-1}(q)_{ki}$ be the inverse of the $(N \times N)$-matrix $K(q)_{ij} = K(q_i - q_j)$. Then

$$P^{\mathrm{hor}}(x) = \sum_{i,j} K(x - q_i) K^{-1}(q)_{ij} P_j$$

$$L(P^{\mathrm{hor}}(x)) = \sum_{i,j} \delta(x - q_i) K^{-1}(q)_{ij} P_j$$

Note that P^{hor} is a vector field of class H^{2l-1}.

The Riemannian metric on Land^N induced by the g^L-metric on $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ is

$$\begin{aligned}
g^L_q(P,Q) &= G^L_{\varphi_0}(P^{\mathrm{hor}}, Q^{\mathrm{hor}}) \\
&= \int_{\mathbb{R}^n} \langle L(P^{\mathrm{hor}}), Q^{\mathrm{hor}}\rangle \, dx \\
&= \int_{\mathbb{R}^n} \Big\langle \sum_{i,j} \delta(x-q_i) K^{-1}(q)_{ij} P_j, \\
&\qquad\qquad \sum_{k,l} K(x-q_k) K^{-1}(q)_{kl} Q_l \Big\rangle \, dx \\
&= \sum_{i,j,k,l} K^{-1}(q)_{ij} K(q_i - q_k) K^{-1}(q)_{kl} \langle P_j, Q_l\rangle \\
g^L_q(P,Q) &= \sum_{k,l} K^{-1}(q)_{kl} \langle P_k, Q_l\rangle. \qquad (1)
\end{aligned}$$

The geodesic equation in vector form is:

$$\boxed{\begin{aligned}&\ddot{q}_n = \\ &-\frac{1}{2}\sum_{k,i,j,l} K^{-1}(q)_{ki}\,\operatorname{grad} K(q_i - q_j)(K(q)_{in} - K(q)_{jn}) \\ &\qquad\qquad K^{-1}(q)_{jl}\langle \dot{q}_k, \dot{q}_l\rangle \\ &+\sum_{k,i} K^{-1}(q)_{ki}\Big\langle \operatorname{grad} K(q_i - q_n), \dot{q}_i - \dot{q}_n\Big\rangle \dot{q}_k\end{aligned}}$$

The geodesic equation on $T^*\mathrm{Land}^N(\mathbb{R}^n)$

.

The cotangent bundle
$T^*\mathrm{Land}^N(\mathbb{R}^n) = \mathrm{Land}^N(\mathbb{R}^n) \times ((\mathbb{R}^n)^N)^* \ni (q, \alpha)$. We shall treat $\mathbb{R}^n$ like scalars; $\langle \quad , \quad \rangle$ is always the standard inner product on $\mathbb{R}^n$.
The metric looks like

$$(g^L)^{-1}_q(\alpha, \beta) = \sum_{i,j} K(q)_{ij} \langle \alpha_i, \beta_j \rangle,$$
$$K(q)_{ij} = K(q_i - q_j).$$

The energy function

$$E(q,\alpha) = \tfrac{1}{2}(g^L)_q^{-1}(\alpha,\alpha) = \tfrac{1}{2}\sum_{i,j} K(q)_{ij}\langle \alpha_i, \alpha_j\rangle$$

and its Hamiltonian vector field (using $\mathbb{R}^n$-valued derivatives to save notation)

$$H_E(q,\alpha) = \sum_{i,k=1}^{N}\Big(K(q_k - q_i)\alpha_i \frac{\partial}{\partial q_k} \\ + \operatorname{grad} K(q_i - q_k)\langle \alpha_i, \alpha_k\rangle \frac{\partial}{\partial \alpha_k}\Big).$$

So the geodesic equation is the flow of this vector field:

$$\boxed{\begin{aligned} \dot q_k &= \sum_i K(q_i - q_k)\alpha_i \\ \dot\alpha_k &= -\sum_i \operatorname{grad} K(q_i - q_k)\langle \alpha_i, \alpha_k\rangle \end{aligned}}$$

A covariant formula for curvature and its relations to O'Neill's curvature formulas.

Mario Micheli in his 2008 thesis derived the the coordinate version of the following formula for the sectional curvature expression, which is valid for **closed** 1-forms α, β on a Riemannian manifold (M, g), where we view $g : TM \to T^*M$ and so g^{-1} is the dual inner product on T^*M. Here $\alpha^\sharp = g^{-1}(\alpha)$.

$$\boxed{\begin{aligned}
&g\big(R(\alpha^\sharp, \beta^\sharp)\alpha^\sharp, \beta^\sharp\big) = \\
&-\tfrac{1}{2}\alpha^\sharp\alpha^\sharp(\|\beta\|^2_{g^{-1}}) - \tfrac{1}{2}\beta^\sharp\beta^\sharp(\|\alpha\|^2_{g^{-1}}) + \tfrac{1}{2}(\alpha^\sharp\beta^\sharp + \beta^\sharp\alpha^\sharp)g^{-1}(\alpha, \beta) \\
&\qquad\qquad\big(\text{last line } = -\alpha^\sharp\beta([\alpha^\sharp, \beta^\sharp]) + \beta^\sharp\alpha([\alpha^\sharp, \beta^\sharp]])\big) \\
&-\tfrac{1}{4}\|d(g^{-1}(\alpha, \beta))\|^2_{g^{-1}} + \tfrac{1}{4}g^{-1}\big(d(\|\alpha\|^2_{g^{-1}}), d(\|\beta\|^2_{g^{-1}})\big) \\
&+\tfrac{3}{4}\big\|[\alpha^\sharp, \beta^\sharp]\big\|^2_g
\end{aligned}}$$

Mario's formula in coordinates

Assume that $\alpha = \alpha_i dx^i, \beta = \beta_i dx^i$ where the coefficients α_i, β_i are *constants*, hence α, β are closed.
Then $\alpha^\sharp = g^{ij}\alpha_i\partial_j, \beta^\sharp = g^{ij}\beta_i\partial_j$ and we have:

$$\boxed{\begin{aligned} &4g\big(R(\alpha^\sharp,\beta^\sharp)\beta^\sharp,\alpha^\sharp\big)\\ &= (\alpha_i\beta_k - \alpha_k\beta_i)\cdot(\alpha_j\beta_l - \alpha_l\beta_j)\cdot\\ &\cdot\Big(2g^{is}(g^{jt}g^{kl}_{,t})_{,s} - \tfrac{1}{2}g^{ij}_{,s}g^{st}g^{kl}_{,t} - 3g^{is}g^{kp}_{,s}g_{pq}g^{jt}g^{lq}_{,t}\Big)\end{aligned}}$$

Covariant curvature and O'Neill's formula, finite dim.

Let $p : (E, g_E) \to (B, g_B)$ be a Riemannian submersion:
For $b \in B$ and $x \in E_b := p^{-1}(b)$ the g_E-orthogonal splitting

$$T_x E = T_x(E_{p(x)}) \oplus T_x(E_{p(x)})^{\perp, g_E} =: T_x(E_{p(x)}) \oplus \mathrm{Hor}_x(p).$$
$$T_x p : (\mathrm{Hor}_x(p), g_E) \to (T_b B, g_B)$$

is an isometry. A vector field $X \in \mathfrak{X}(E)$ is decomposed as $X = X^{\mathrm{hor}} + X^{\mathrm{ver}}$ into horizontal and vertical parts. Each vector field $\xi \in \mathfrak{X}(B)$ can be uniquely lifted to a smooth horizontal field $\xi^{\mathrm{hor}} \in \Gamma(\mathrm{Hor}(p)) \subset \mathfrak{X}(E)$.

Semilocal version of Mario's formula, force, and stress

Let (M, g) be a robust Riemannian manifold, $x \in M$, $\alpha, \beta \in g_x(T_x M)$. Assume we are given local smooth vector fields X_α and X_β such that:

1. $X_\alpha(x) = \alpha^\sharp(x), \quad X_\beta(x) = \beta^\sharp(x)$,
2. Then $\alpha^\sharp - X_\alpha$ is zero at x hence has a well defined derivative $D_x(\alpha^\sharp - X_\alpha)$ lying in $\mathrm{Hom}(T_x M, T_x M)$. For a vector field Y we have $D_x(\alpha^\sharp - X_\alpha).Y_x = [Y, \alpha^\sharp - X_\alpha](x) = \mathcal{L}_Y(\alpha^\sharp - X_\alpha)|_x$. The same holds for β.
3. $\mathcal{L}_{X_\alpha}(\alpha) = \mathcal{L}_{X_\alpha}(\beta) = \mathcal{L}_{X_\beta}(\alpha) = \mathcal{L}_{X_\beta}(\beta) = 0$,
4. $[X_\alpha, X_\beta] = 0$.

Locally constant 1-forms and vector fields will do. We then define:

$$\mathcal{F}(\alpha, \beta) := \tfrac{1}{2} d(g^{-1}(\alpha, \beta)), \qquad \text{a 1-form on } M \text{ called the } \textit{force},$$

$$\begin{aligned}\mathcal{D}(\alpha, \beta)(x) &:= D_x(\beta^\sharp - X_\beta).\alpha^\sharp(x) \\ &= d(\beta^\sharp - X_\beta).\alpha^\sharp(x), \quad \in T_x M \text{ called the } \textit{stress}.\end{aligned}$$

$$\implies \mathcal{D}(\alpha, \beta)(x) - \mathcal{D}(\beta, \alpha)(x) = [\alpha^\sharp, \beta^\sharp](x)$$

Then in the notation above:

$$\boxed{\begin{aligned}
&g\big(R(\alpha^\sharp,\beta^\sharp)\beta^\sharp,\alpha^\sharp\big)(x) = R_{11}+R_{12}+R_2+R_3\\
&R_{11} = \tfrac{1}{2}\Big(\mathcal{L}^2_{X_\alpha}(g^{-1})(\beta,\beta) - 2\mathcal{L}_{X_\alpha}\mathcal{L}_{X_\beta}(g^{-1})(\alpha,\beta)\\
&\qquad\qquad + \mathcal{L}^2_{X_\beta}(g^{-1})(\alpha,\alpha)\Big)(x)\\
&R_{12} = \langle\mathcal{F}(\alpha,\alpha),\mathcal{D}(\beta,\beta)\rangle + \langle\mathcal{F}(\beta,\beta),\mathcal{D}(\alpha,\alpha)\rangle\\
&\qquad\qquad - \langle\mathcal{F}(\alpha,\beta),\mathcal{D}(\alpha,\beta)+\mathcal{D}(\beta,\alpha)\rangle\\
&R_2 = \Big(\|\mathcal{F}(\alpha,\beta)\|^2_{g^{-1}} - \big\langle\mathcal{F}(\alpha,\alpha)),\mathcal{F}(\beta,\beta)\big\rangle_{g^{-1}}\Big)(x)\\
&R_3 = -\tfrac{3}{4}\|\mathcal{D}(\alpha,\beta)-\mathcal{D}(\beta,\alpha)\|^2_{g_x}
\end{aligned}}$$

Stress and Force on Landmark space

$$\alpha_k^\sharp = \sum_i K(q_k - q_i)\alpha_i, \quad \alpha^\sharp = \sum_{i,k} K(q_k - q_i)\langle \alpha_i, \tfrac{\partial}{\partial q^k}\rangle$$

$$\mathcal{D}(\alpha,\beta) := \sum_{i,j} dK(q_i - q_j)(\alpha_i^\sharp - \alpha_j^\sharp)\Big\langle \beta_j, \frac{\partial}{\partial q_i}\Big\rangle, \quad \text{the } \textit{stress.}$$

$$\mathcal{D}(\alpha,\beta) - \mathcal{D}(\beta,\alpha) = (D_{\alpha^\sharp}\beta^\sharp) - D_{\beta^\sharp}\alpha^\sharp = [\alpha^\sharp, \beta^\sharp], \quad \textit{Lie bracket.}$$

$$\mathcal{F}_i(\alpha,\beta) = \frac{1}{2}\sum_k \operatorname{grad} K(q_i - q_k)(\langle \alpha_i, \beta_k\rangle + \langle \beta_i, \alpha_k\rangle)$$

$$\mathcal{F}(\alpha,\beta) := \sum_i \langle \mathcal{F}_i(\alpha,\beta), dq_i\rangle = \frac{1}{2}\, d\, g^{-1}(\alpha,\beta) \quad \text{the } \textit{force}.$$

The geodesic equation on $T^* \operatorname{Land}^N(\mathbb{R}^n)$ then becomes

$$\boxed{\begin{aligned} \dot{q} &= \alpha^\sharp \\ \dot{\alpha} &= -\mathcal{F}(\alpha,\alpha) \end{aligned}}$$

Curvature via the cotangent bundle

From the semilocal version of Mario's formula for the sectional curvature expression for constant 1-forms α, β on landmark space, where $\alpha_k^\sharp = \sum_i K(q_k - q_i)\alpha_i$, we get directly:

$$\boxed{\begin{aligned}
&g^L\big(R(\alpha^\sharp,\beta^\sharp)\alpha^\sharp,\beta^\sharp\big) = \\
&= \big\langle \mathcal{D}(\alpha,\beta)+\mathcal{D}(\beta,\alpha),\mathcal{F}(\alpha,\beta)\big\rangle \\
&\quad - \big\langle \mathcal{D}(\alpha,\alpha),\mathcal{F}(\beta,\beta)\big\rangle - \big\langle \mathcal{D}(\beta,\beta),\mathcal{F}(\alpha,\alpha)\big\rangle \\
&\quad - \tfrac{1}{2}\sum_{i,j}\Big(d^2K(q_i-q_j)(\beta_i^\sharp-\beta_j^\sharp,\beta_i^\sharp-\beta_j^\sharp)\langle\alpha_i,\alpha_j\rangle \\
&\qquad\qquad - 2d^2K(q_i-q_j)(\beta_i^\sharp-\beta_j^\sharp,\alpha_i^\sharp-\alpha_j^\sharp)\langle\beta_i,\alpha_j\rangle \\
&\qquad\qquad + d^2K(q_i-q_j)(\alpha_i^\sharp-\alpha_j^\sharp,\alpha_i^\sharp-\alpha_j^\sharp)\langle\beta_i,\beta_j\rangle\Big) \\
&\quad - \|\mathcal{F}(\alpha,\beta)\|^2_{g^{-1}} + g^{-1}\big(\mathcal{F}(\alpha,\alpha),\mathcal{F}(\beta,\beta)\big). \\
&\quad + \tfrac{3}{4}\|[\alpha^\sharp,\beta^\sharp]\|_g^2
\end{aligned}}$$

Bundle of embeddings over the differentiable Chow variety.

Let M be a compact connected manifold with $\dim(M) < \dim(N)$. The smooth manifold $\mathrm{Emb}(M, N)$ of all embeddings $M \to N$ is the total space of a smooth principal bundle with structure group $\mathrm{Diff}(M)$ acting freely by composition from the right hand side.

The quotient manifold $B(M, N)$ can be viewed as the space of all submanifolds of N of diffeomorphism type M; we call it the *differentiable Chow manifold* or the *non-linear Grassmannian*.

$B(M, N)$ is a smooth manifold with charts centered at $F \in B(M, N)$ diffeomorphic to open subsets of the Frechet space of sections of the normal bundle $TF^{\perp,g} \subset TN|_F$.

Let $\ell : \mathrm{Diff}_{\mathcal{S}}(N) \times B(M, N) \to B(M, N)$ be the smooth left action. In the following we will consider just one open $\mathrm{Diff}_{\mathcal{S}}(N)$-orbit $\ell(\mathrm{Diff}_{\mathcal{S}}(N), F_0)$ in $B(M, N)$.

The induced Riemannian cometric on $T^*B(M,N)$

We follow the procedure used for $\mathrm{Diff}_{\mathcal{S}}(N)$. For any $F \subset N$, we decompose $\mathcal{H}$ into:

$$\mathcal{H}_F^{\text{vert}} = j_2^{-1}\big(\{X \in \Gamma_{C_b^2}(TN) : X(x) \in T_xF, \text{ for all } x \in F\}\big)$$
$$\mathcal{H}_F^{\text{hor}} = \text{perpendicular complement of } \mathcal{H}_F^{\text{vert}}$$

It is then easy to check that we get the diagram:

$$\begin{tikzcd}
\Gamma_{\mathcal{S}}(TN) \arrow[r, hook, "j_1"] \arrow[d, two heads, "\text{res}"] & \mathcal{H} \arrow[r, hook, "j_2"] \arrow[d, two heads] & \Gamma_{C_b^2}(TN) \arrow[d, two heads, "\text{res}"] \\
\Gamma_{\mathcal{S}}(\text{Nor}(F)) \arrow[r, hook, "j_1^f"] & \mathcal{H}_F^{\text{hor}} \arrow[r, hook, "j_2^f"] & \Gamma_{C_b^2}(\text{Nor}(F)).
\end{tikzcd}$$

Here $\text{Nor}(F) = TN|_F / TF$.

As this is an orthogonal decomposition, L and K take $\mathcal{H}_F^{\mathrm{vert}}$ and $\mathcal{H}_F^{\mathrm{hor}}$ into their own duals and, as before we get:

$$\begin{tikzcd}
\Gamma_{\mathcal{S}}(\mathrm{Nor}(F)) \arrow[d, hook, "j_1"] & & \Gamma_{\mathcal{S}'}(\mathrm{Nor}^*(F)) \\
\mathcal{H}_F^{\mathrm{hor}} \arrow[rr, shift left, "L_F"] \arrow[d, hook, "j_2"] & & (\mathcal{H}_F^{\mathrm{hor}})' \arrow[ll, shift left, "K_F"] \arrow[u, hook, "j_1'"'] \\
\Gamma_{C_b^2}(\mathrm{Nor}(F)) & & \Gamma_{M^2}(\mathrm{Nor}^*(F)) \arrow[u, hook, "j_2'"']
\end{tikzcd}$$

K_F is just the restriction of K to this subspace of $\mathcal{H}'$ and is given by the kernel:

$$K_F(x_1, x_2) := \text{image of } K(x_1, x_2) \in \mathrm{Nor}_{x_1}(F) \otimes \mathrm{Nor}_{x_2}(F)), \quad x_1, x_2 \in F.$$

This is a C^2 section over $F \times F$ of $\mathrm{pr}_1^* \mathrm{Nor}(F) \otimes \mathrm{pr}_2^* \mathrm{Nor}(F)$.

We can identify $\mathcal{H}_F^{\text{hor}}$ as the closure of the image under K_F of measure valued 1-forms supported by F and with values in $\text{Nor}^*(F)$. A dense set of elements in $\mathcal{H}_F^{\text{hor}}$ is given by either taking the 1-forms with finite support or taking smooth 1-forms. In the smooth case, fix a volume form κ on M and a smooth covector $\xi \in \Gamma_{\mathcal{S}}(\text{Nor}^*(F))$. Then $\xi.\kappa$ defines a horizontal vector field h like this:

$$h(x_1) = \int_{x_2 \in F} \big| K_F(x_1, x_2) \big| \xi(x_2).\kappa(x_2) \big\rangle$$

The horizontal lift h^{hor} of any $h \in T_F B(M, N)$ is then:

$$h^{\text{hor}}(y_1) = K(L_F h)(y_1) = \int_{x_2 \in F} \big| K(y_1, x_2) \big| L_F h(x_2) \big\rangle, \quad y_1 \in N.$$

Note that all elements of the cotangent space $\alpha \in \Gamma_{\mathcal{S}'}(\text{Nor}^*(F))$ can be pushed up to N by $(j_F)_*$, where $j_F : F \hookrightarrow N$ is the inclusion, and this identifies $(j_F)_* \alpha$ with a 1-co-current on N.

Finally the induced homogeneous weak Riemannian metric on $B(M,N)$ is given like this:

$$\begin{aligned}\langle h,k\rangle_F &= \int_N (h^{\text{hor}}(y_1), L(k^{\text{hor}})(y_1)) = \int_{y_1\in N} (K(L_F h))(y_1), (L_F k)(y_1)) \\ &= \int_{(y_1,y_2)\in N\times N} (K(y_1,y_2), (L_F h)(y_1)\otimes (L_F k)(y_2)) \\ &= \int_{(x_1,x_1)\in F\times F} \Big\langle L_F h(x_1)\Big| K_F(x_1,x_2)\Big| L_F h(x_2)\Big\rangle\end{aligned}$$

With this metric, the projection from $\text{Diff}_{\mathcal{S}}(N)$ to $B(M,N)$ is a submersion.

The inverse co-metric on the smooth cotangent bundle $\bigsqcup_{F \in B(M,N)} \Gamma(\mathrm{Nor}^*(F) \otimes \mathrm{vol}(F)) \subset T^*B(M,N)$ is much simpler and easier to handle:

$$\boxed{\langle \alpha, \beta \rangle_F = \iint_{F \times F} \big\langle \alpha(x_1) \big| K_F(x_1, x_2) \big| \beta(x_1) \big\rangle.}$$

It is simply the restriction to the co-metric on the Hilbert sub-bundle of $T^* \mathrm{Diff}_{\mathcal{S}}(N)$ defined by $\mathcal{H}'$ to the Hilbert sub-bundle of subspace $T^*B(M,N)$ defined by $\mathcal{H}'_F$.

Because they are related by a submersion, the geodesics on $B(M,N)$ are the horizontal geodesics on $\mathrm{Diff}_{\mathcal{S}}(N)$. We have two variables: a family $\{F_t\}$ of submanifolds in $B(M,N)$ and a time varying momentum $\alpha(t,\cdot) \in \mathrm{Nor}^*(F_t) \otimes \mathrm{vol}(F_t)$ which lifts to the horizontal 1-co-current $(j_{F_t})_*(\alpha(t,\cdot)$ on N. Then the horizontal geodesic on $\mathrm{Diff}_{\mathcal{S}}(N)$ is given by the same equations as before:

$$\boxed{\begin{aligned} \partial_t(F_t) &= \mathrm{res}_{\mathrm{Nor}(F_t)}(u(t,\cdot)) \\ u(t,x) &= \int_{(F_t)_y} \big|K(x,y)\big|\alpha(t,y)\big\rangle \in \mathfrak{X}_{\mathcal{S}}(N) \\ \partial_t\left((j_{F_t})_*(\alpha(t,\cdot))\right) &= -\mathcal{L}_{u(t,\cdot)}((j_{F_t})_*(\alpha(t,\cdot)). \end{aligned}}$$

This is a complete description for geodesics on $B(M,N)$ but it is not very clear how to compute the Lie derivative of $(j_{F_t})_*(\alpha(t,\cdot)$. One can unwind this Lie derivative via a torsion-free connection, but we turn to a different approach which will be essential for working out the curvature of $B(M,N)$.

Auxiliary tensors on $B(M, N)$

For $X \in \mathfrak{X}_{\mathcal{S}}(N)$ let B_X be the infinitesimal action on $B(M, N)$ given by $B_X(F) = T_{\mathrm{Id}}(\ell^F)X$ with its flow $\mathrm{Fl}_t^{B_X}(F) = \mathrm{Fl}_t^X(F)$. We have $[B_X, B_Y] = B_{[X,Y]}$.
$\{B_X(F) : X \in \mathfrak{X}_{\mathcal{S}}(N)\}$ equals the tangent space $T_F B(M, N)$.

Note that $B(M, N)$ is naturally submanifold of the vector space of m-currents on N:

$$B(M, N) \hookrightarrow \Gamma_{\mathcal{S}'}(\Lambda^m T^* N), \quad \text{via } F \mapsto \left(\omega \mapsto \int_F \omega\right).$$

Any $\alpha \in \Omega^m(N)$ is a linear coordinate on $\Gamma_{\mathcal{S}'}(TN)$ and this restricts to the function $B_\alpha \in C^\infty(B(M, N), \mathbb{R})$ given by $B_\alpha(F) = \int_F \alpha$.
If $\alpha = d\beta$ for $\beta \in \Omega^{m-1}(N)$ then

$$B_\alpha(F) = B_{d\beta}(F) = \int_F j_F^* d\beta = \int_F d j_F^* \beta = 0$$

by Stokes' theorem.

For $\alpha \in \Omega^m(N)$ and $X \in \mathfrak{X}_{\mathcal{S}}(N)$ we can evaluate the vector field B_X on the function B_α:

$$\begin{aligned} B_X(B_\alpha)(F) &= dB_\alpha(B_X)(F) = \partial_t|_0 B_\alpha(Fl_t^X(F)) \\ &= \int_F j_F^* \mathcal{L}_X \alpha = B_{\mathcal{L}_X(\alpha)}(F) \\ \text{as well as } &= \int_F j_F^*(i_X d\alpha + d i_X \alpha) = \int_F j_F^* i_X d\alpha = B_{i_X(d\alpha)}(F) \end{aligned}$$

If $X \in \mathfrak{X}_{\mathcal{S}}(N)$ is tangent to F along F then $B_X(B_\alpha)(F) = \int_F \mathcal{L}_{X|_F} j_F^* \alpha = 0$.

More generally, a pm-form α on N^k defines a function $B_\alpha^{(p)}$ on $B(M,N)$ by $B_\alpha^{(p)}(F) = \int_{F^p} \alpha$.

For $\alpha \in \Omega^{m+k}(N)$ we denote by B_α the k-form in $\Omega^k(B(M,N))$ given by the skew-symmetric multi-linear form:

$$(B_\alpha)_F(B_{X_1}(F), \ldots, B_{X_k}(F)) = \int_F {j_F}^*(i_{X_1 \wedge \cdots \wedge X_k} \alpha).$$

This is well defined: If one of the X_i is tangential to F at a point $x \in F$ then ${j_F}^*$ pulls back the resulting m-form to 0 at x.

Note that any smooth cotangent vector a to $F \in B(M,N)$ is equal to $B_\alpha(F)$ for some closed $(m+1)$-form α. Smooth cotangent vectors at F are elements of $\Gamma_{\mathcal{S}}(F, \mathrm{Nor}^*(F) \otimes \Lambda^m T^*(F))$.

Likewise, a $2m+k$ form $\alpha \in \Omega^{2m+k}(N^2)$ defines a k-form on $B(M,N)$ as follows: First, for $X \in \mathfrak{X}_{\mathcal{S}}(N)$ let $X^{(2)} \in \mathfrak{X}(N^2)$ be given by

$$X^{(2)}_{(n_1,n_2)} := (X_{n_1} \times 0_{n_2}) + (0_{n_1} \times X_{n_2})$$

Then we put

$$(B^{(2)}_\alpha)_F(B_{X_1}(F),\dots,B_{X_k}(F)) = \int_{F^2} {j_{F^2}}^*(i_{X_1^{(2)}\wedge\dots\wedge X_k^{(2)}}\alpha).$$

This is just B applied to the submanifold $F^2 \subset N^2$ and to the special vector fields $X^{(2)}$.
Using this for $p=2$, we find that for any two m-forms α, β on N, the inner product of B_α and B_β is given by:

$$g_B^{-1}(B_\alpha, B_\beta) = B^{(2)}_{\langle\alpha|K|\beta\rangle}.$$

We have

$$\begin{aligned} i_{B_X} B_\alpha &= B_{i_X\alpha} \\ dB_\alpha &= B_{d\alpha} \quad \text{for any } \alpha \in \Omega^{m+k}(N) \\ \mathcal{L}_{B_X} B_\alpha &= B_{\mathcal{L}_X\alpha} \end{aligned}$$

Force and Stress

Moving to curvature, fix F. Then we claim that for any two smooth co-vectors a, b at F, we can construct not only two closed $(m+1)$-forms α, β on N as above but also two commuting vector fields X_α, X_β on N in a neighborhood of F such that:

1. $B_\alpha(F) = a$ and $B_\beta(F) = b$,
2. $B_{X_\alpha}(F) = a^\sharp$ and $B_{X_\beta}(F) = b^\sharp$
3. $\mathcal{L}_{X_\alpha}(\alpha) = \mathcal{L}_{X_\alpha}(\beta) = \mathcal{L}_{X_\beta}(\alpha) = \mathcal{L}_{X_\beta}(\beta) = 0$
4. $[X_\alpha, X_\beta] = 0$

The force is

$$2\mathcal{F}(\alpha, \beta) = d(\langle B_\alpha, B_\beta \rangle) = d\left(B^{(2)}_{\langle \alpha | K | \beta \rangle}\right) = B^{(2)}_{d(\langle \alpha | K | \beta \rangle)}.$$

The stress $\mathcal{D} = \mathcal{D}_N$ on N can be computed as:

$$\mathcal{D}(\alpha, \beta, F)(x) = (\text{restr. to Nor}(F))\left(-\int_{y \in F} \left| \mathcal{L}_{X_\alpha^{(2)}}(x,y) K(x,y) \right| \beta(y) \right\rangle\right)$$

The curvature

Finally, the semilocal Mario formula and some computations lead to:

$$\boxed{\begin{aligned}
&\langle R_{B(M,N)}(B^\sharp_\alpha, B^\sharp_\beta)B^\sharp_\beta, B^\sharp_\alpha\rangle(F) = R_{11} + R_{12} + R_2 + R_3\\
&\qquad R_{11} = \tfrac{1}{2}\iint_{F\times F}\Big(\big\langle\beta\big|\mathcal{L}_{X^{(2)}_\alpha}\mathcal{L}_{X^{(2)}_\alpha}K\big|\beta\big\rangle + \big\langle\alpha\big|\mathcal{L}_{X^{(2)}_\beta}\mathcal{L}_{X^{(2)}_\beta}K\big|\alpha\big\rangle\\
&\qquad\qquad\qquad - 2\big\langle\alpha\big|\mathcal{L}_{X^{(2)}_\alpha}\mathcal{L}_{X^{(2)}_\beta}K\big|\beta\big\rangle\Big)\\
&\qquad R_{12} = \int_F\Big(\langle\mathcal{D}(\alpha,\alpha,F), \mathcal{F}(\beta,\beta,F)\rangle + \langle\mathcal{D}(\beta,\beta,F), \mathcal{F}(\alpha,\alpha,F)\rangle\\
&\qquad\qquad\qquad - \langle\mathcal{D}(\alpha,\beta,F) + \mathcal{D}(\beta,\alpha,F), \mathcal{F}(\alpha,\beta,F)\rangle\Big)\\
&\qquad R_2 = \|\mathcal{F}(\alpha,\beta,F)\|^2_{K_F} - \big\langle\mathcal{F}(\alpha,\alpha,F)), \mathcal{F}(\beta,\beta,F)\big\rangle_{K_F}\\
&\qquad R_3 = -\tfrac{3}{4}\|\mathcal{D}(\alpha,\beta,F) - \mathcal{D}(\beta,\alpha,F)\|^2_{L_F}
\end{aligned}}$$

Thank you for listening