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Manifolds of differentiable maps

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1. Let X, Y be smooth finite dimensional manifolds, let $C^\infty(X,Y)$ be the set of smooth mappings from X to Y; for any non negative integer n let $J^n(X,Y)$ denote the fibre bundle of n-jets of smooth maps from X to Y, equipped with the canonical manifold structure which makes $j^n f : X \longrightarrow J^n(X,Y)$ into a smooth section for each $f \in C^\infty(X,Y)$, where $j^n f(x)$ is the n-jet of f at $x \in X$.

Usually $C^\infty(X,Y)$ is equipped with the so called Whitney-C^∞-topology: a basis of open sets is given by all sets of the form $M(U) = \{f \in C^\infty(X,Y) : j^n f(X) \subseteq U\}$, where U is any open set in $J^n(X,Y)$ and $n \in \mathbb{N}$. See [3] and [6] for accounts of this topology. We may describe it intuitively by the following words: if you go to infinity on X you may control better and better partial derivatives up to a fixed order.

2. Anyone familiar with functional analysis may have heard the following words: if you go to infinity (on X) you may control better and better more and more partial derivatives. This describes the inductive limit topology on $\mathfrak{D}(\mathbb{R}) = \varinjlim \mathfrak{D}(K)$, where $\mathfrak{D}(\mathbb{R})$ is the space of all smooth functions with compact support on $\mathbb{R}$ and $\mathfrak{D}(K)$ is the subspace of those functions which have support contained in some fixed compact K of X, equipped with the topology of uniform convergence in all partial derivatives.

The topology induced by the Whitney-C^∞-topology on $\mathfrak{D}(\mathbb{R})$

could be described by the formula $\mathfrak{D}(\mathbb{R}) = \varinjlim_{r\to\infty} (\varinjlim_{K} \mathfrak{D}^r(K))$, where $\mathfrak{D}^r(K)$ is the space of all C^r-functions on $\mathbb{R}$ with support contained in K. This discussion shows (I hope) that the Whitney-C^∞-topology is not the most natural topology on $C^\infty(X,Y)$.

3. We now give an intrinsic description in terms of jets of the topology on $C^\infty(X,Y)$ referred to in 2. We call it the $\mathfrak{D}$-topology. A detailed account of it can be found in [7]. There are three equivalent descriptions of the $\mathfrak{D}$-topology on $C^\infty(X,Y)$:

(a) Fix a sequence $K = (K_n)$ of compact sets in X such that $K_0 = \emptyset$, $K_{n-1} \subseteq K_n^{\,o}$, $X = \bigcup K_n$. Then the system of sets of the form $M(m,U) = \{ f \in C^\infty(X,Y) : j^{m_n} f(X - K_n^{\,o}) \subseteq U_n$ for all $n\}$ is a base of open sets for the $\mathfrak{D}$-topology on $C^\infty(X,Y)$, where $m = (m_n)$ runs through all sequences of non negative integers and $U = (U_n)$ with U_n open in $J^{m_n}(X,Y)$. The $\mathfrak{D}$-topology is independent of the choice of the sequence (K_n).

(b) Fix a sequence (d_n) of metrics d_n on $J^n(X,Y)$, compatible with the manifold topologies. Then the system of sets of the form $V_\varphi(f) = \{ g \in C^\infty(X,Y) : \varphi_n(x)\, d_n(j^n f(x), j^n g(x)) < 1$ for all x in X and for all $n \}$ is a neighbourhood base for $f \in C^\infty(X,Y)$ in the $\mathfrak{D}$-topology, consisting of open sets, where $\varphi = (\varphi_n)$ runs through all sequences of continuous strictly positive functions on X with $(\operatorname{supp} \varphi_n)$ locally finite. The $\mathfrak{D}$-topology is independent of the choice of the metrics d_n.

(c) The system of sets of the form

$$M(L,U) = \{ f \in C^\infty(X,Y) : j^n f(X-L_n{}^o) \subseteq U_n \text{ for all } n \}$$

is a base of open sets for the $\mathfrak{D}$-topology on $C^\infty(X,Y)$, where $L = (L_n)$ runs trough all sequences of compact sets $L_n \subseteq X$ such that $(X-L_n{}^o)$ is locally finite and $U = (U_n)$ runs through all sequences of open sets $U_n \subseteq J^n(X,Y)$.

4. The $\mathfrak{D}$-topology on $C^\infty(X,Y)$ is finer than the Whitney-C^∞-topology. It is exactly the topology $\mathfrak{C}^\infty$ of MORLET in [2], who proves that $C^\infty(X,Y)$ is a Baire space in this topology. It was mistoken to be the Whitney-topology by LESLIE [5]. We now list some properties of the $\mathfrak{D}$-topology:

(a) A sequence (f_n) in $C^\infty(X,Y)$ converges to f if and only if there exists a compact set $K \subseteq X$ such that all but finitely many of the f_n's equal f off K and $j^k f_n \longrightarrow j^k f$ "uniformly" on K for all k. So convergence of sequences is the same for the Whitney-topology and for the $\mathfrak{D}$-topology. See [7].

(b) If T is a connected mètrizable compact topological space and $f: T \longrightarrow C^\infty(X,Y)$ is any continuous mapping (for the $\mathfrak{D}$-topology), then there is a compact set $K \subseteq X$ such that $t \longmapsto f(t)(x)$ is constant on T for $x \in X-K$.

Proof: Any $t \in T$ has a neighbourhood V_t in T such that the stated property holds on V_t: if not one may find a sequence $t_n \longrightarrow t$ in T such that the sequence $f(t_n)$ does not satisfy the condition in (a). Now use that T is compact and connected.

(c) For each $k \geq 0$ the map $j^k: C^\infty(X,Y) \longrightarrow C^\infty(X, J^k(X,Y))$ is continuous for the $\mathfrak{D}$-topology. See [7].

(d) If X, Y, Z are smooth finite dimensional manifolds then composition $C^\infty(Y,Z) \times C^\infty_{prop}(X,Y) \longrightarrow C^\infty(X,Z)$, given by $(f,g) \longmapsto f \circ g$, is continuous in the $\mathfrak{D}$-topology, where $C^\infty_{prop}(X,Y)$ is the space of all smooth proper maps $f: X \longrightarrow Y$, i.e. $f^{-1}(K)$ is compact if K is compact. See [7].

5. Theorem: Let X, Y be smooth manifolds. Then $C^\infty(X,Y)$ is a Baire space with the $\mathfrak{D}$-topology.

This was proved by MORLET [2]. We give here a quite different proof using the explicit description of the $\mathfrak{D}$-topology.

Proof: Let $U_1, U_2, \ldots$ be a countable sequence of $\mathfrak{D}$-open-dense subsets of $C^\infty(X,Y)$. We have to show that $\bigcap_n U_n$ is again $\mathfrak{D}$-dense. Choose metrics d_n on $J^n(X,Y)$, $n = 0, 1, \ldots$, compatible with the topologies, such that each $J^n(X,Y)$ becomes a complete metric space with d_n. Let be $f_o \in C^\infty(X,Y)$ and $V_\varphi(f_o)$ be any neighbourhood of f_o as in 3(b). It suffices to show that $V_\varphi(f_o) \cap \bigcap_i U_i \neq \emptyset$. Let

Let $\frac{1}{2}\varphi := (\frac{1}{2}\varphi_n)$, then $f_o \in V_{\frac{1}{2}\varphi}(f_o) \subseteq \overline{V_{\frac{1}{2}\varphi}(f_o)} \subseteq V_\varphi(f_o)$, where $\overline{V_{\frac{1}{2}\varphi}(f_o)} = \{ g \in C^\infty(X,Y) : \frac{1}{2}\varphi_n(x)\, d_n(j^n f(x), j^n g(x)) \leq 1$ for all $x \in X$ and for all $n \geq 0 \}$. It clearly suffices to show that $\overline{V_{\frac{1}{2}\varphi}(f_o)} \cap \bigcap_i U_i \neq \emptyset$.

To do this we choose inductively a sequence of functions (f_i) in $C^\infty(X,Y)$; a sequence $(\varphi^{(i)})$ of families as in 3(b) such that the following holds:

(Ai) $f_i \in V_{\frac{1}{2}\varphi}(f_o) \cap \bigcap_{j=1}^{i-1} V_{\varphi^{(j)}}(f_j) \cap U_i$

(Bi) $\overline{V_{\varphi^{(i)}}(f_i)} \subseteq U_i$

(Ci)$(i > 1)$ $d_s(j^s f_i(x), j^s f_{i-1}(x)) < 1/2^i$, $0 \leq s \leq i$.

Choose $f_1 \in V_{\frac{1}{2}\varphi}(f_o) \cap U_1$ which is possible, since U_1 is dense.

So (A1) holds. U_1 is open and $f_1 \in U_1$ so we can find a family $\psi^{(1)}$ such that $V_{2\psi^{(1)}}(f_1) \subseteq U_1$, then $\overline{V_{\psi^{(1)}}(f_1)} \subseteq U_1$ so (B1) holds. (C1) is empty. Now assume inductively that the data is chosen for all $j \leq i-1$. We will choose f_i satisfying (Ai) and (Ci) and not using any (Cj), $j < i$, and then we can easily find $\psi^{(i)}$ such that (Bi) holds. Consider the open set $V_\eta(f_{i-1})$ where $\eta = (0, 2^i, \ldots, 2^i, 0, 0, \ldots)$ with i-times 2^i.

Let $E_i = V_{\frac{1}{2}\varphi}(f_o) \cap \bigcap_{j=1}^{i-1} V_{\psi^{(j)}}(f_j) \cap V_\eta(f_{i-1})$, then E_i is open and $f_{i-1} \in E_i$ by (A_{i-1}), so $E_i \neq \emptyset$ and we may pick $f_i \in E_i \cap U_i$ by density of U_i. Then clearly (Ai) holds since we have $f_i \in V_{\frac{1}{2}\varphi}(f_o) \cap \bigcap_{j=1}^{i-1} V_{\psi^{(j)}}(f_j) \cap U_i$. Furthermore we have for $1 \leq s \leq i$ $d_s(j^s f_{i-1}(x), j^s f_i(x)) < 1/2^i$ by the form of η, so (Ci) is satisfied. Finally $f_i \in U_i$, U_i is open, so there is a family $\psi^{(i)}$ such that $V_{2\psi^{(i)}}(f_i) \subseteq U_i$, so $\overline{V_{\psi^{(i)}}(f_i)} \subseteq U_i$ and (Bi) holds too.

Now we use this data to prove the theorem. Define $g^s(x) = \lim_{i\to\infty} j^s f_i(x) \in J^s(X,Y)$. This limit exists since for each s d_s is a complete metric on $J^s(X,Y)$ and for each x the sequence $j^s f_i(x)$ is a Cauchy-sequence by (C). Since $j^o f_i(x) = (x, f_i(x))$, the graph of f_i, we can define $g: X \to Y$ by $g^o(x) = (x, g(x))$. We claim that g is smooth. This is a local question and in a chart-neighbourhood we see that all partial derivatives of f_i converge uniformly by (C), so g is smooth by a classical theorem of Fubini.

Now $f_i \in V_{\frac{1}{2}\varphi}(f_o)$ by (Ai), i.e. $\varphi_n(x)\, d_n(j^n f_o(x), j^n f_i(x)) < 2$ for all $x \in X$ and $n \geq 0$. Since $j^n f_i(x) \longrightarrow j^n g(x)$ for all x and n we conclude that $\varphi_n(x)\, d_n(j^n f_o(x), j^n g(x)) \leq 2$ for all x and n, so $g \in \overline{V_{\frac{1}{2}\varphi}(f_o)}$. By (Bi) $\psi^{(i)}$ was chosen so that $\overline{V_{\psi^{(i)}}(f_i)} \subseteq U_i$

and by (Ai) we have that $f_s \in V_{\psi^{(i)}}(f_i)$ for all $s > i$, i.e. $\psi_n^{(i)}(x)\, d_n(j^n f_i(x), j^n f_s(x)) < 1$ for all x and n. Since $j^n f_s(x) \longrightarrow j^n g(x)$ for all x and n we conclude that $\psi_n^{(i)}(x)\, d_n(j^n f_i(x), j^n g(x)) \leq 1$ for all x and n, i.e. $g \in \overline{V_{\psi^{(i)}}(f_i)}$. This holds for all i. So by (B) wehave $g \in \overline{V_{\frac{1}{2}\psi}(f_o)} \cap \bigcap_{i=1}^{\infty} \overline{V_{\psi^{(i)}}(f_i)} \subseteq \overline{V_{\frac{1}{2}\psi}(f_o)} \cap \bigcap_{i=1}^{\infty} U_i$. qed.

6. Examples: In his lecture Mather introduced a topology on $\mathrm{Diff}_c^r M$, the space of C^r-diffeomorphisms with compact support of a smooth manifold M, by the formula $\mathrm{Diff}_c^r M = \varinjlim \mathrm{Diff}_K^r M$, K compact in M. If $r = \infty$, then this is exactly the topology induced from the $\mathfrak{D}$-topology on $C^\infty(M,M)$, if $r < \infty$ then it is the topology induced from the Whitney-C^r-topology.
The same topology was used by Banyaga in his talk on the space of smooth symplectic diffeomorphisms with compact support.

7. We now introduce a refinement of the $\mathfrak{D}$-topology on $C^\infty(X,Y)$ which is needed for the manifold structure later on. It is called the $\mathfrak{D}^\infty$-topology in [7], not a very good name. It is given by the following process:
If $f,g \in C^\infty(X,Y)$ and the set $\{x \in X: f(x) \neq g(x)\}$ has compact closure in X we call f equivalent to g $(f \sim g)$. This is an equivalence relation. The $\mathfrak{D}^\infty$-topology is now the coarsest among all topologies on $C^\infty(X,Y)$, which are finer than the $\mathfrak{D}$-topology and for which all equivalence classes of the above relation are open. Another description is: equip each equivalence class with the trace of the $\mathfrak{D}$-topology and take their disjoint union.

The intrinsic descriptions of section 3 are still valid with alterations, just add $f \sim g$ to the definition of $V_\varphi(f)$ in (b) and intersect $M(m,U)$ resp. $M'(L,U)$ with equivalence classes. The properties 4(a) - 4(d) remain valid for the $\mathfrak{D}^\infty$-topology too, since the maps and constructions used there are compatibel with the equivalence relation.
$C^\infty(X,Y)$ is no longer a Baire space with the $\mathfrak{D}^\infty$-topology since it looks locally like the model space $\mathfrak{D}(f^*TY)$ as we shall see in the next section and functional analysis tells us, that this is no Baire space. But it is a Lindelöf space if X is second countable, so $C^\infty(X,Y)$ is normal and paracompact with the $\mathfrak{D}^\infty$-topology.

8. We now describe the manifold structure on $C^\infty(X,Y)$. Let $\tau : TY \longrightarrow Y$ be a smooth map such that for each $y \in Y$ the map $\tau_y : T_yY \longrightarrow Y$ is a diffeomorphism onto an open neighbourhood of y in Y. Such a map may be constructed by using a fibre-respecting diffeomorphism from TY onto an open neighbourhood of the zero-section in TY followed by an appropriate exponential map. If $f \in C^\infty(X,Y)$ consider the pullbach f^*TY which is a vectorbundle over X, and the space $\mathfrak{D}(f^*TY)$ of all smooth sections with compact support of this bundle, equipped with the $\mathfrak{D}^\infty$- (or $\mathfrak{D}$-) topology.
Let $\psi_f : \mathfrak{D}(f^*TY) \longrightarrow C^\infty(X,Y)$ be the mapping $\psi_f(s)(x) = \tau_{f(x)}s(x) \in Y$. Denote the image of ψ_f by U_f. $Z_f = \bigcup_x (\{x\} \times \tau_{f(x)}(T_{f(x)}Y))$ is an open neighbourhood of the graph $\{(x,f(x)),\ x \in X\}$ of f in $X \times Y = J^0(X,Y)$ (in fact a tubular neighbourhood), and U_f consists of all $g \in C^\infty(X,Y)$ such that the graph of g is contained in Z_f and $g \sim f$, so

U_f is open in the $\mathfrak{D}^{\infty}$-topology. ψ_f is continuous by 4(d) and has a continuous inverse $\varphi_f: U_f \longrightarrow \mathfrak{D}(f^*TY)$, given by $\varphi_f(g)(x) = \tau^{-1}_{f(x)}(g(x))$, as is easily checked up.

We use φ_f as coordinate map. Now let us check the form of the coordinate change: let $f, g \in C^{\infty}(X,Y)$ with $U_f \cap U_g \neq \emptyset$. For $s \in \varphi_f(U_f \cap U_g)$ we have $\varphi_g \psi_f(s)(x) = \tau^{-1}_{g(x)}(\psi_f(s)(x)) = \tau^{-1}_{g(x)} \circ \tau_{f(x)}(s(x))$, so the map

$\varphi_g \circ \psi_f : \varphi_f(U_f \cap U_g) \subseteq \mathfrak{D}(f^*TY) \longrightarrow \mathfrak{D}(g^*TY)$ is given by $(\tau^{-1}_g \circ \tau_f)_*$, by pushing forward sections by a fiber bundle diffeomorphism $\tau^{-1}_g \circ \tau_f$. This is clearly continuous.

So we have constructed on $C^{\infty}(X,Y)$ a structure of a topological manifold, where each $f \in C^{\infty}(X,Y)$ has a coordinate neighbourhood U_f homeomorphic to a whole space $\mathfrak{D}(f^*TY)$ of sections with compact support of the vector bundle f^*TY over X.

The construction we have given here is a simplified version of the one given in [7].

9. To make $C^{\infty}(X,Y)$ into a differentiable manifold we just have to take a suitable notion of C^r - mappings and to show that the coordinate change $(\tau^{-1}_g \circ \tau_f)$ is C^r. We remark that it is of class C^{∞}_{π} in the sense of [4], a rather strong notion, as is shown in [7], and probably of class C^{∞} for any notion of differentiability that has appeared in the literature until now. The tangent space at $f \in C^{\infty}(X,Y)$ turns out to be $\mathfrak{D}_f(X,TY) = \mathfrak{D}(f^*TY)$ and the whole tangent bundle is $\mathfrak{D}(X,TY)$, the space of all smooth maps $X \rightarrow TY$ which differ from zero only on a compact set. It is a vector bundle over $C^{\infty}(X,Y)$ (i.e. locally trivial) in the manifold structure it inherits from $C^{\infty}(X,TY)$ as an open subset. This tangent bundle seems to be independent of the notion of differentiation applied. See [7].

... The inverse function theorem I presented in my talk is wrong due to difficulties with chain-rule for the notion of differentiability applied. The statement that remains true is too special to be of any interest.

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