MOSER'S THEOREM ON MANIFOLDS WITH CORNERS

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Abstract. Moser's theorem [13] states that the diffeomorphism group of a compact manifold acts transitively on the space of all smooth positive densities with fixed volume. Here we describe the extension of this result to manifolds with corners. In particular we obtain Moser's theorem on simplices. The proof is based on Banyaga's paper [1], where Moser's theorem is proven for manifolds with boundary. A cohomological interpretation of Banyaga's operator is given, which allows a proof of Lefschetz duality using differential forms.

1. Introduction. In [13] Moser proved that on a connected compact oriented manifold M without boundary there exists for any two positive volume forms μ_0 and μ_1 with $\int_M \mu_0 = \int_M \mu_1$ an orientation preserving diffeomorphism φ with $\varphi^*\mu_1 = \mu_0$. In [1] Banyaga extended this to compact oriented manifolds with boundary and showed that the diffeomorphism can be chosen such that it restricts to the identity on the boundary. On a manifold with corners one cannot expect the diffeomorphism to be the identity on the boundary, since at a corner x of index 2 or higher the derivative of such a diffeomorphism would have to be the identity: in this case x lies in the boundary of at least two codimension 1 strata of ∂M. So the derivative at x restricted to two codimension 1 subspaces is the identity and thus it has to be the identity on the whole space; in particular the Jacobian determinant there equals 1.

Moser's theorem on manifolds with corners is needed for example in [2]. Even on simplices it does not seem to be known, but is highly desirable. In fact, Banyaga's method [1] gives just the desired result. But this is not immediately obvious and it took us a long time to realize it. Therefore we think that it is worth wile to write the proof with all details. Along the way we also prove Stokes' theorem on manifolds with corners.

For related results see [5] for a version of Moser's theorem on bounded domains in $\mathbb{R}^m$ with low differentiability requirements furnishing diffeomorphisms with only low regularity using PDE techniques; this does not imply the result given here. See also the recent book [4] for results on k-forms instead of volume forms. A version on non-compact manifolds is in [8] which also sketches a proof for non-compact manfolds with boundary which differs from Banyaga's proof.

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2. Manifolds with corners alias quadrantic (orthantic) manifolds. For more information we refer to [7], [11], [10]. Let $Q = Q^m = \mathbb{R}^m_{\geq 0}$ be the positive orthant or quadrant. By Whitney's extension theorem or Seeley's theorem, the restriction $C^\infty(\mathbb{R}^m) \to C^\infty(Q)$ is a surjective continuous linear mapping which admits a continuous linear section (extension mapping); so $C^\infty(Q)$ is a direct summand in $C^\infty(\mathbb{R}^m)$. A point $x \in Q$ is called a *corner of codimension (or index)* $q > 0$ if x lies in the intersection of q distinct coordinate hyperplanes. Let $\partial^q Q$ denote the set of all corners of codimension q.

A manifold with corners (recently also called a quadrantic manifold) M is a smooth manifold modeled on open subsets of Q^m. We assume that it is connected and second countable; then it is paracompact and each open cover admits a subordinated smooth partition of unity. Any manifold with corners M is a submanifold with corners of an open manifold $\tilde{M}$ of the same dimension, and each smooth function on M extends to a smooth function on $\tilde{M}$. We do not assume that M is oriented, but for Moser's theorem we will eventually assume that M is compact. Let $\partial^q M$ denote the set of all corners of codimension q. Then $\partial^q M$ is a submanifold without boundary of codimension q in M; it has finitely many connected components if M is compact. We shall consider ∂M as stratified into the connected components of all $\partial^q M$ for $q > 0$. Abusing notation we will call $\partial^q M$ the boundary stratum of codimension q; this will lead to no confusion. Note that ∂M itself is not a manifold with corners. We shall denote by $j_{\partial^q M} : \partial^q M \to M$ the embedding of the boundary stratum of codimension q into M, and by $j_{\partial M} : \partial M \to M$ the whole complex of embeddings of all strata.

Each diffeomorphism of M restricts to a diffeomorphism of ∂M and to a diffeomorphism of each stratum $\partial^q M$. The Lie algebra of $\mathrm{Diff}(M)$ consists of all vector fields X on M such that $X|\partial^q M$ is tangent to $\partial^q M$. We shall denote this Lie algebra by $\mathfrak{X}(M, \partial M)$.

3. Differential forms. There are several differential complexes on a manifold with corners. If M is not compact there are also the versions with compact support.

- Differential forms that vanish near ∂M. If M is compact, this is the same as the differential complex $\Omega_c(M \setminus \partial M)$ of differential forms with compact support in the open interior $M \setminus \partial M$.
- $\Omega(M, \partial M) = \{\alpha \in \Omega(M) : j^*_{\partial^q M}\alpha = 0 \text{ for all } q \geq 1\}$, the complex of differential forms that pull back to 0 on each boundary stratum.
- $\Omega(M)$, the complex of all differential forms. Its cohomology equals singular cohomology with real coefficients of M, since $\mathbb{R} \to \Omega^0 \to \Omega^1 \to \dots$ is a fine resolution of the constant sheaf on M; for that one needs existence of smooth partitions of unity and the Poincaré lemma which holds on manifolds with corners. The Poincaré lemma can be proved as in [12, 9.10] in each quadrant.

If M is an oriented manifold with corners of dimension m and if $\mu \in \Omega^m(M)$ is a nowhere vanishing form of top degree, then $\mathfrak{X}(M) \ni X \mapsto i_X\mu \in \Omega^{m-1}(M)$ is a linear isomorphism. Moreover, $X \in \mathfrak{X}(M, \partial M)$ (tangent to the boundary) if and only if $i_X\mu \in \Omega^{m-1}(M, \partial M)$.

4. Towards the long exact sequence of the pair $(M,\partial M)$**.** Let us consider the short exact sequence of differential graded algebras

$$0\to\Omega(M,\partial M)\to\Omega(M)\to\Omega(M)/\Omega(M,\partial M)\to 0\,.$$

The complex $\Omega(M)/\Omega(M,\partial M)$ is a subcomplex of the product of $\Omega(N)$ for all connected components N of all $\partial^q M$. The quotient consists of forms which extend continuously over boundaries to ∂M with its induced topology in such a way that one can extend them to smooth forms on M; this is contained in the space of 'stratified forms' as used in [15]. There Stokes' formula is proved for stratified forms.

5. Proposition (Stokes' theorem)**.** *For a connected oriented manifold M with corners of dimension $\dim(M)=m$ and for any $\omega\in\Omega^{m-1}_c(M)$ we have*

$$\int_M d\omega=\int_{\partial^1 M} j^*_{\partial^1 M}\omega\,.$$

Note that $\partial^1 M$ may have several components. Some of these might be non-compact.

We shall deduce this result from Stokes' formula for a manifold with boundary by making precise the fact that $\partial^{\ge 2}M$ has codimension 2 in M and has codimension 1 with respect to $\partial^1 M$. The proof also works for manifolds with more general boundary strata, like manifolds with cone-like singularities. A lengthy full proof can be found in [3].

Proof. We first choose a smooth decreasing function f on $\mathbb{R}_{\ge 0}$ such that $f=1$ near 0 and $f(r)=0$ for $r\ge\varepsilon$. Then $\int_0^\infty f(r)dr<\varepsilon$ and for $Q^m=\mathbb{R}^m_{\ge 0}$ with $m\ge 2$,

$$\begin{aligned}\Big|\int_{Q^m} f'(|x|)\,dx\Big| &= C_m\Big|\int_0^\infty f'(r)r^{m-1}\,dr\Big| = C_m\Big|\int_0^\infty f(r)(r^{m-1})'\,dr\Big|\\ &= C_m\int_0^\varepsilon f(r)(r^{m-1})'\,dr\le C_m\varepsilon^{m-1}\,,\end{aligned}$$

where C_m denotes the surface area of $S^{m-1}\cap Q^m$. Given $\omega\in\Omega^{m-1}_c(M)$ we use the function f on quadrant charts on M to construct a function g on M that is 1 near $\partial^{\ge 2}M=\bigcup_{q\ge 2}\partial^q M$, has support close to $\partial^{\ge 2}M$ and satisfies $\big|\int_M dg\wedge\omega\big|<\varepsilon$. Then $(1-g)\omega$ is an $(m-1)$-form with compact support in the manifold with boundary $M\setminus\partial^{\ge 2}M$, and Stokes' formula (cf. [12, 10.11]) now says

$$\int_{M\setminus\partial^{\ge 2}M} d((1-g)\omega)=\int_{\partial^1 M} j^*_{\partial^1 M}((1-g)\omega)\,.$$

But $\partial^{\ge 2}M$ is a null set in M and the quantities

$$\Big|\int_M d((1-g)\omega)-\int_M d\omega\Big|\quad\text{and}\quad\Big|\int_{\partial^1 M} j^*_{\partial^1 M}((1-g)\omega)-\int_{\partial^1 M} j^*_{\partial^1 M}\omega\Big|$$

are small if ε is small enough. □

6. Theorem (Moser's theorem for manifolds with corners)**.** *Let M be a compact connected smooth manifold with corners, possibly non-orientable. Let $\mu_0,\mu_1\in$* $\mathrm{Dens}_+(M)$ *be smooth positive densities with $\int_M\mu_0=\int_M\mu_1$. Then there exists* *a diffeomorphism $\varphi : M \to M$ such that $\mu_1 = \varphi^*\mu_0$. Moreover, φ can be chosen to be the identity on ∂M if and only if $\mu_0 = \mu_1$ on $\partial^{\geq 2}M$.*

Proof. We first prove the theorem for oriented M. In this case $\mathrm{Dens}_+(M)$ equals the space $\Omega^m_+(M)$ of positive m-forms for $m = \dim(M)$. Put $\mu_t := \mu_0 + t(\mu_1 - \mu_0)$ for $t \in [0,1]$; then each μ_t is a volume form on M since these form a convex set. We look for a curve of diffeomorphisms, $t \mapsto \varphi_t$, with $\varphi_t^*\mu_t = \mu_0$; this curve has to satisfy $\frac{\partial}{\partial t}(\varphi_t^*\mu_t) = 0$. Since $\int_M(\mu_1 - \mu_0) = 0$, we have $[\mu_1 - \mu_0] = 0 \in H^m(M)$. Fix $\omega \in \Omega^m_c(M \setminus \partial M)$ with $\int \omega = 1$. Using lemma **7** below we have

$$\begin{aligned}
&\psi := I^\omega(\mu_1 - \mu_0) \in \Omega^{m-1}(M, \partial M) \quad \text{ with}\\
&d\psi = dI^\omega(\mu_1 - \mu_0) = \mu_1 - \mu_0 - \omega \int_M (\mu_1 - \mu_0) = \mu_1 - \mu_0 \,.
\end{aligned}$$

Put $\eta_t := (\frac{\partial}{\partial t}\varphi_t) \circ \varphi_t^{-1}$; then by well known formulas (see [12, 31.11], e.g.) we have:

$$\begin{aligned}
0 &\overset{\text{wish}}{=} \tfrac{\partial}{\partial t}(\varphi_t^*\mu_t) = \varphi_t^*\mathcal{L}_{\eta_t}\mu_t + \varphi_t^*\tfrac{\partial}{\partial t}\mu_t = \varphi_t^*(\mathcal{L}_{\eta_t}\mu_t + \mu_1 - \mu_0)\,,\\
0 &\overset{\text{wish}}{=} \mathcal{L}_{\eta_t}\mu_t + \mu_1 - \mu_0 = di_{\eta_t}\mu_t + i_{\eta_t}d\mu_t + d\psi = di_{\eta_t}\mu_t + d\psi\,.
\end{aligned}$$

We can choose η_t uniquely by requiring that $i_{\eta_t}\mu_t = -\psi$, since μ_t is non-degenerate for all t. The time dependent vector field η_t is tangent to each boundary stratum $\partial^q M$, since $\psi \in \Omega(M, \partial M)$. Then the evolution operator $\varphi_t = \Phi^\eta_{t,0}$ exists for $t \in [0,1]$ since M is compact, by [12, 3.30]. Moreover, $\varphi_t : \partial M \to \partial M$. Thus φ_t restricts to a diffeomorphism of M for each t. On M we have, using [12, 31.11.2],

$$\tfrac{\partial}{\partial t}(\varphi_t^*\mu_t) = \varphi_t^*(\mathcal{L}_{\eta_t}\mu_t + d\psi) = \varphi_t^*(di_{\eta_t}\mu_t + d\psi) = 0\,,$$

so $\varphi_t^*\mu_t = \text{ constant } = \mu_0$. If $\mu_0 = \mu_1$ on $\partial^{\geq 2}M$, then $\psi = I^\omega(\mu_1 - \mu_0)$ vanishes along ∂M by lemma **7** and hence so does η_t, thus φ_t is the identity there.

If M is not orientable, we let $p : \mathrm{or}(M) \to M$ be the 2-sheeted orientable double cover of M: It is the $\mathbb{Z}_2$-principal bundle with cocycle of transition functions $\operatorname{sign}\det d(u_\beta \circ u_\alpha^{-1})(u_\alpha(x))$ where (U_α, u_α) is a smooth atlas for M. Each connected (thus orientable) chart of M appears twice as chart of $\mathrm{or}(M)$, once with each orientation. Thus $\mathrm{or}(M)$ is again a smooth manifold with corners. Let $\tau : \mathrm{or}(M) \to \mathrm{or}(M)$ be the orientation reversing deck-transformation; see [12, 13.1]. Pullback $p^* : \Omega(M) \to \Omega(\mathrm{or}(M))$ is an isomorphism onto the eigenspace $\Omega(\mathrm{or}(M))^{\tau^*=1}$ of τ^* with eigenvalue 1. The space $\mathrm{Dens}_+(M)$ of positive smooth densities on M is via p^* isomorphic to the space of positive m-forms in the eigenspace $\Omega^m(\mathrm{or}(M))^{\tau^*=-1}$ of τ^* with eigenvalue -1; these are the 'formes impaires' of de Rham. Note the abuse of notation here: p^* of a density differs (by local signs) from p^* of a form. See [12, 13.1 and 13.3] for more details.

We consider the pullback densities $p^*\mu_t$ as positive m-forms denoted by ν_t on $\mathrm{or}(M)$ which satisfy $\tau^*\nu_t = -\nu_t$, and for $\omega \in \Omega^m_c(\mathrm{or}(M)\setminus\partial\,\mathrm{or}(M))$ with $\int_{\mathrm{or}(M)}\omega = 1$ we choose

$$\begin{aligned}
&\tilde\psi = I^\omega(\nu_1 - \nu_0) \in \Omega^{m-1}(\mathrm{or}(M), \partial\,\mathrm{or}(M)) \quad \text{ which satisfies}\\
&d\tilde\psi = \nu_1 - \nu_0 - \omega \cdot \int_{\mathrm{or}(M)} (\nu_1 - \nu_0) = \nu_1 - \nu_0\,.
\end{aligned}$$

Let $\psi = \frac{1}{2}\tilde\psi - \frac{1}{2}\tau^*\tilde\psi$, then again $d\psi = \nu_1 - \nu_0$ and now also $\tau^*\psi = -\psi$. A time-dependent vector field η_t is uniquely given by $i_{\eta_t}\nu_t = -\psi$.

$$i_{\tau^*\eta_t}\nu_t = -i_{\tau^*\eta_t}\tau^*\nu_t = -\tau^*(i_{\eta_t}\nu_t) = \tau^*\psi = -\psi \qquad \Longrightarrow \qquad \tau^*\eta_t = \eta_t\,.$$

In particular, the time dependent vector field η_t is tangent to each boundary stratum $\partial^q \operatorname{or}(M)$, and it projects to a time dependent vector field on M whose evolution gives the curve of diffeomorphisms with all required properties. $\square$

7. Lemma. *Let M be an oriented connected manifold with corners of dimension $\dim(M) = m$. For each form $\omega \in \Omega_c^m(M\setminus \partial M)$ with $\int \omega = 1$ there exists a continuous linear operator*

$$I^\omega : \Omega_c^m(M) \to \Omega_c^{m-1}(M, \partial M) \qquad \textit{such that:}$$

- *$d\,I^\omega(\alpha) = \alpha - \omega\int\alpha$ for all $\alpha \in \Omega_c^m(M)$.*
- *If α vanishes on $\partial^{\ge 2}M$ then $I^\omega(\alpha)$ vanishes along ∂M.*

For a compact oriented manifold with boundary, this is due to Banyaga [1]. We call I^ω the *Banyaga operator.*

Proof. We first construct I^ω_m for the case when M is a partial quadrant $Q^m_p := \mathbb{R}^p_{\ge 0}\times\mathbb{R}^{m-p} = \{x\in\mathbb{R}^m : x^1\ge 0,\dots,x^p\ge 0\}$.

We construct I^ω_m by induction on the dimension m and start with $I^\omega_0 = 0$.

For $Q = Q^1_1 = \mathbb{R}_{\ge 0}$ we consider $\omega = g(u)du$ with $\operatorname{supp}(g)\subset\mathbb{R}_{>0}$ and $\int g(u)du = 1$. Then for $\alpha = a(u)du \in \Omega^1_c(Q)$ we put

$$I^\omega_1(\alpha)(u) := \int_0^u \Big(a(t) - g(t)\int_Q a(v)dv\Big)dt \qquad \text{so that}$$

$$dI^\omega_1(\alpha) = \alpha - \omega\int\alpha \qquad \text{and} \qquad I^\omega_1(\alpha)(0) = 0\,,$$

and thus $I^\omega_1(\alpha)\in\Omega^0(\mathbb{R}_{\ge 0},\{0\})$. For $Q = Q^1_0 = \mathbb{R}$ we just integrate from $-\infty$ to u. Note that $I^\omega_1(\alpha)(u)$ vanishes for large u, so it has compact support.

For general $Q = Q^m_p = \mathbb{R}^p_{\ge 0}\times\mathbb{R}^{m-p}$ we start with a specific form ω: Let $g(t)$ be a smooth function on $\mathbb{R}$ with compact support in $\mathbb{R}_{>0}$ and $\int g(t)\,dt = 1$. Then we put

$$\omega = g(u^1)du^1\wedge\cdots\wedge g(u^m)du^m =: \omega'\wedge g(u^m)du^m\,,$$

$$d:\Omega^{m-1}(Q)\to\Omega^m(Q), \quad d = \sum_{i=1}^{m-1} du^i\wedge\partial_{u^i} + du^m\wedge\partial_{u^m} =: d' + du^m\wedge\partial_{u^m}\,.$$

Any form $\alpha\in\Omega^m_c(Q^m_p)$ can be written as $\alpha = \alpha_1(u^m)\wedge du^m$ for a smooth curve

$$\alpha_1 : \begin{cases} \mathbb{R}\to\Omega^{m-1}_c(Q^{m-1}_p) & \text{if} \quad 0\le p\le m-1\,,\\ \mathbb{R}_{\ge 0}\to\Omega^{m-1}_c(Q^{m-1}_{p-1}) & \text{if} \quad p = m\,.\end{cases}$$

Following an idea of de Rham [6] used by [1], we put

$$I^\omega_m(\alpha) = I^{\omega'}_{m-1}(\alpha_1(u^m))\wedge du^m +$$

$$+\,(-1)^{m-1}\omega'\cdot\begin{cases}\int_{-\infty}^{u^m}\Big(\int_{Q_p^{m-1}}\alpha_1(t)-g(t)\int_{Q_p^m}\alpha\Big)\,dt & \text{if}\quad 0\le p\le m-1\,,\\ \int_0^{u^m}\Big(\int_{Q_{p-1}^{m-1}}\alpha_1(t)-g(t)\int_{Q_p^m}\alpha\Big)\,dt & \text{if}\quad p=m\,,\end{cases}$$

which is in $\Omega^{m-1}(Q_p^m,\partial Q_p^m)$: The first summand by induction on m and because it contains du^m. The second summand since the integral starts at 0 in the relevant case. $I_m^\omega(\alpha)$ has compact support: The first summand by induction, and the second summand since ω' has compact support in the first $m-1$ variables, and since the integral vanishes for large u^m. If α vanishes on $\partial^{\ge 2}Q^m$ then $\alpha_1(u^m)$ vanishes on $\partial^{\ge 2}Q^{m-1}$ in both cases. Then the first summand vanishes on ∂Q^m by induction, and second summand since the integral starts at 0 in the relevant case.

Next we compute the exterior derivative of $I_m^\omega(\alpha)$. For the first summand

$$\begin{aligned}d\big(I_{m-1}^{\omega'}(\alpha_1(u^m))\wedge du^m\big) &= d'I_{m-1}^{\omega'}(\alpha_1(u^m))\wedge du^m =\\ &= \Big(\alpha_1(u^m)-\omega'\int_{Q^{m-1}}\alpha_1(u^m)\Big)\wedge du^m \qquad\text{by induction}\\ &= \alpha-(\omega'\wedge du^m)\int_{Q^{m-1}}\alpha_1(u^m)\,.\end{aligned}$$

The exterior derivative of the second summand is

$$\begin{aligned}&(-1)^{m-1}du^m\wedge\omega'\Big(\int_{Q^{m-1}}\alpha_1(u^m)-g(u^m)\int_{Q^m}\alpha\Big)\\ &=(\omega'\wedge du^m)\int_{Q^{m-1}}\alpha_1(u^m)-(\omega'\wedge g(u^m)du^m)\int_{Q^m}\alpha\end{aligned}$$

which proves $dI_m^\omega(\alpha)=\alpha-\omega\int_{Q^m}\alpha$.

In order to change to another m-form $\tilde\omega\in\Omega_c^m(Q_p^m\setminus\partial Q_p^m)$ with $\int_{Q_p^m}\tilde\omega=1$ we put

$$I_m^{\tilde\omega}(\alpha)=I_m^\omega(\alpha)-I^\omega(\tilde\omega)\int_{Q_p^m}\alpha\,.$$

Now we extend the operators I_m^ω to the oriented manifold with corners M. We construct an oriented atlas similarly to [1, lemme 1] with the property that all charts contain a common chart U_0. Choose $x_0\in M\setminus\partial M$ and a closed neighborhood V_0 of x_0 in $M\setminus\partial M$ which is diffeomorphic to a closed ball in $\mathbb{R}^m$. For each $y\in M\setminus V_0$ choose an oriented open chart $\varphi_y: U_y\to Q_{p_y}^m$ centered at y onto some partial quadrant, $x_y\in U_y\setminus\partial U_y$ and a smooth embedded curve c_y in $M\setminus\partial M$ from x_y to x_0. Then choose a vector field X_y with $X_y(c_y(t))=c_y'(t)$ for each t that vanishes at y and on ∂M. The flow $\mathrm{Fl}_t^{X_y}$ moves x_y along c_y to x_0 and keeps y and ∂M fixed. $\mathrm{Fl}_1^{X_y}$ also maps an open neighborhood of x_y in U_y to an open neighborhood of x_0, which we may extend to an open neighborhood of V_0 via a diffeomorphism ψ_y of M that is the identity near y and near ∂M. Now consider the charts

$$\psi_y(\mathrm{Fl}_1^{X_y}(U_y))\xrightarrow{\psi_y^{-1}}\mathrm{Fl}_1^{X_y}(U_y)\xrightarrow{\mathrm{Fl}_{-1}^{X_y}}U_y\xrightarrow{\varphi_y}Q_{p_y}^m$$

and call the resulting atlas again (U_y,φ_y). We choose a smooth partition of unity λ_y with a locally finite family of supports subordinated to this atlas (most of the λ_y are 0). Finally we choose the chart (U_0,φ_0) inside $\bigcap_y U_y$ which is possible since the intersection contains the neighborhood V_0.

Choose $\omega \in \Omega_c^m(U_0)$ with $\int_M \omega = 1$ and let

$$
\begin{aligned}
I^\omega(\alpha) &:= \sum_y \varphi_y^* I_m^{(\varphi_y^{-1})^*\omega}\big((\varphi_y^{-1})^*(\lambda_y.\alpha)\big) \in \Omega_c^{m-1}(M,\partial M) \quad \text{ with}\\
dI^\omega(\alpha) &= \sum_y \varphi_y^* dI_m^{(\varphi_y^{-1})^*\omega}\big((\varphi_y^{-1})^*(\lambda_y.\alpha)\big)\\
&= \sum_y \varphi_y^*\Big((\varphi_y^{-1})^*(\lambda_y.\alpha) - (\varphi_y^{-1})^*\omega \cdot \int_{Q_y} (\varphi_y^{-1})^*(\lambda_y.\alpha)\Big) = \alpha - \omega \int_M \alpha\,.
\end{aligned}
$$

The sum is finite since α has compact support. The change to an arbitrary form $\tilde\omega \in \Omega_c^m(M \setminus \partial M)$ with $\int \tilde\omega = 1$ is as above. $\square$

8. Cohomological interpretation of the Banyaga operator. For a connected oriented manifold with corners M of dimension m (we assume that ∂M is not empty) we consider the following diagram where only the dashed arrow I^ω does not fit in commutingly. Here $\omega \in \Omega_c^m(M \setminus \partial M)$ is a fixed form with $\int \omega = 1$. All instances of $\mathbb{R}$ in the diagram are connected by identities which fit commutingly into the diagram. Each line is the definition of the corresponding top de Rham cohomology space. The integral in the first line induces an isomorphism in cohomology since $M \setminus \partial M$ is a connected oriented open manifold. The bottom triangle commutes by Stokes' theorem **5**.

$$
\begin{tikzcd}
\Omega_c^{m-1}(M\setminus\partial M) \arrow[r, "d"] \arrow[d, hook] & \Omega_c^m(M\setminus\partial M) \arrow[r, two heads] \arrow[d, hook] \arrow[rr, bend left, "\int_{M\setminus\partial M}"] & H_c^m(M\setminus\partial M) \arrow[r, equal] \arrow[d] & \mathbb{R} \\
\Omega_c^{m-1}(M,\partial M) \arrow[r, "d"] \arrow[d, hook] & \Omega_c^m(M,\partial M) \arrow[r, two heads] \arrow[d, equal] \arrow[rr, bend right, "\int_M"'] & H_c^m(M,\partial M) \arrow[r, equal] \arrow[d] & \mathbb{R} \\
\Omega_c^{m-1}(M) \arrow[r, "d"] \arrow[dr, "\int_{\partial^1 M}\circ j^*_{\partial^1 M}"'] & \Omega_c^m(M) \arrow[r, two heads] \arrow[d, "\int_M"] \arrow[ul, dashed, "I^\omega"'] & H_c^m(M) \arrow[r, equal] & 0 \\
& \mathbb{R} & &
\end{tikzcd}
$$

Claim. $\int_M : \Omega_c^m(M,\partial M) = \Omega_c^m(M) \to \mathbb{R}$ *induces* $H_c^m(M,\partial M) = \mathbb{R}$.
Namely, given $\alpha, \beta \in \Omega_c^m(M,\partial M)$ with $\int \alpha = \int \beta$, we have $\alpha - dI^\omega(\alpha) = \omega \int_M \alpha$ and similarly for β. This implies $\alpha - \beta = d\,(I^\omega(\alpha) - I^\omega(\beta))$ and hence $[\alpha] = [\beta]$ in $H_c^m(M,\partial M)$.

Claim. *If M has non-empty boundary then* $H_c^m(M) = 0$.
For any form $\alpha \in \Omega_c^m(M)$ we have $\alpha - dI^\omega(\alpha) = \omega \int_M \alpha$, so α equals a multiple of ω modulo an exact form with compact support. Now choose $\beta \in \Omega_c^{m-1}(M)$ with $\int_M d\beta \neq 0$; for example with $\int_{\partial^1 M} j^*_{\partial^1 M}\beta \neq 0$. Then $d\beta - dI^\omega(d\beta) = \omega \int_M d\beta$ shows that any multiple of ω is exact. Thus $H_c^m(M) = 0$.

9. Theorem (Poincaré–Lefschetz duality)**.** *For an oriented connected manifold with corners of dimension m the cohomological integral* $\int_* : H_c^m(M,\partial M) \to \mathbb{R}$

induces a non-degenerate bilinear form

$$P_M^k : H^k(M) \times H_c^{m-k}(M, \partial M) \to \mathbb{R} \qquad \textit{given by}$$
$$P_M^k([\alpha],[\beta]) = \int_* [\alpha] \wedge [\beta] = \int_M \alpha \wedge \beta\,.$$

This is in fact the special case for real coefficients of Lefschetz' duality [9]. In [14] Lefschetz duality is proven for piecewise linear stratified ∂-pseudomanifolds in terms of intersection homology. Here we can give a proof based completely on differential forms.

Proof. Note first that $\Omega_c(M, \partial M)$ is a graded ideal in $\Omega(M)$, thus the integral makes sense. The proof follows now, for example, [12, 12.14 – 12.16] with some obvious changes. □

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