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FRÖLICHER SPACES AS A SETTING FOR TREE SPACES AND STRATIFIED SPACES

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Frölicher spaces were introduced under the name 'espaces lisses' (smooth spaces) in [4] and [5]; they were called Frölicher spaces in [7, section 23]. They furnish a very simple vehicle for extending the notion of smooth mappings from manifolds to singular spaces and they give a cartesian closed category.

Frölicher spaces. A *Frölicher space*, also called a *smooth space* or a space with *smooth structure*, is a triple $(X, \mathcal{C}_X, \mathcal{F}_X)$ consisting of a set X, a subset $\mathcal{C}_X$ of the set of all mappings $\mathbb{R} \to X$, and a subset $\mathcal{F}_X$ of the set of all functions $X \to \mathbb{R}$, with the following two properties:

- A function $f : X \to \mathbb{R}$ belongs to $\mathcal{F}_X$ if and only if $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $c \in \mathcal{C}_X$.
- A curve $c : \mathbb{R} \to X$ belongs to $\mathcal{C}_X$ if and only if $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $f \in \mathcal{F}_X$.

Note that a set X together with any subset $\mathcal{F}$ of the set of functions $X \to \mathbb{R}$ generates a unique Frölicher space $(X, \mathcal{C}_X, \mathcal{F}_X)$, where we put in turn:

$$\begin{aligned} \mathcal{C}_X &:= \{c : \mathbb{R} \to X : f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } f \in \mathcal{F}\}, \\ \mathcal{F}_X &:= \{f : X \to \mathbb{R} : f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } c \in \mathcal{C}_X\}, \end{aligned}$$

so that $\mathcal{F} \subseteq \mathcal{F}_X$. The set $\mathcal{F}$ will be called a *generating set of functions for the Frölicher space.*

Likewise, a set X together with any subset $\mathcal{C}$ of the set of curves $\mathbb{R} \to X$ generates a unique Frölicher space $(X, \mathcal{C}_X, \mathcal{F}_X)$, where we put in turn:

$$\begin{aligned} \mathcal{F}_X &:= \{f : X \to \mathbb{R} : f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } c \in \mathcal{C}\}, \\ \mathcal{C}_X &:= \{c : \mathbb{R} \to X : f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } f \in \mathcal{F}_X\}, \end{aligned}$$

so that $\mathcal{C} \subseteq \mathcal{C}_X$. The set $\mathcal{C}$ will be called a *generating set of curves for the Frölicher space.*

Smooth mappings. A mapping $\phi : X \to Y$ between two Frölicher spaces is called *smooth* if one of the following three equivalent conditions hold:

- For each $c \in \mathcal{C}_X$ the composite $\phi \circ c$ is in $\mathcal{C}_Y$.
- For each $f \in \mathcal{F}_Y$ the composite $f \circ \phi$ is in $\mathcal{F}_X$.
- For each $c \in \mathcal{C}_X$ and for each $f \in \mathcal{F}_Y$ the composite $f \circ \phi \circ c$ is in $C^\infty(\mathbb{R}, \mathbb{R})$.

Note that $\mathcal{F}_Y$ can be replaced by any generating set, as well as $\mathcal{C}_X$. The set of all smooth mappings from X to Y will be denoted by $C^\infty(X, Y)$. Then we have $C^\infty(\mathbb{R}, X) = \mathcal{C}_X$ and $C^\infty(X, \mathbb{R}) = \mathcal{F}_X$. Obviously, Frölicher spaces and smooth mappings form a category.

Theorem. [7, 23.2] *The category of Frölicher spaces and smooth mappings has the following properties:*

- *Complete, i.e., arbitrary limits exist. The underlying set is formed as in the category of sets as a certain subset of the cartesian product, and the smooth structure is generated by the smooth functions on the factors.*
- *Cocomplete, i.e., arbitrary colimits exist. The underlying set is formed as in the category of sets as a certain quotient of the disjoint union, and the smooth functions are exactly those which induce smooth functions on the cofactors.*
- *Cartesian closedness, which means: The set $C^\infty(X,Y)$ carries a canonical smooth structure generated by all functions of the form*
$$C^\infty(X,Y) \xrightarrow{C^\infty(c,f)} C^\infty(\mathbb{R},\mathbb{R}) \xrightarrow{\lambda} \mathbb{R}$$
where $c \in C^\infty(\mathbb{R},X)$ and $f \in C^\infty(Y,\mathbb{R})$, or in a generating sets, and where $\lambda \in C^\infty(\mathbb{R},\mathbb{R})'$. With this structure the exponential law holds:
$$C^\infty(X\times Y,Z) \cong C^\infty(X, C^\infty(Y,Z)).$$

Corollary. *Canonical mappings are smooth, for Frölicher spaces X, Y, Z:*

$$\begin{aligned}
&\mathrm{ev} : C^\infty(X,Y)\times X \to Y, \quad \mathrm{ev}(f,x) = f(x)\\
&\mathrm{ins} : X \to C^\infty(Y, X\times Y), \quad \mathrm{ins}(x)(y) = (x,y)\\
&(\quad)^\wedge : C^\infty(X, C^\infty(Y,Z)) \to C^\infty(X\times Y, Z)\\
&(\quad)^\vee : C^\infty(X\times Y, Z) \to C^\infty(X, C^\infty(Y,Z))\\
&\mathrm{comp} : C^\infty(Y,Z)\times C^\infty(X,Y) \to C^\infty(X,Z)\\
&C^\infty(\quad,\quad) : C^\infty(Y,Y_1)\times C^\infty(X_1,X) \to\\
&\qquad\qquad \to C^\infty(C^\infty(X,Y), C^\infty(X_1,Y_1))\\
&\qquad (f,g) \mapsto (h \mapsto f\circ h\circ g)
\end{aligned}$$

Natural topologies on Frölicher spaces. [3, section 1] On a Frölicher space $(X, \mathcal{C}_X, \mathcal{F}_X)$ we consider the following two topologies:

- The final topology with respect to all smooth curves in $\mathcal{C}_X$; it is denoted by $\tau_\mathcal{C}$.
- The inital topology with respect to all smooth functions in $\mathcal{F}_X$; we denote it by $\tau_\mathcal{F}$

The identity mapping $(X,\tau_\mathcal{C}) \to (X,\tau_\mathcal{F})$ is obviously continuous. A Frölicher space is called *balanced* if these two topologies coincide and are Hausdorff.

Related concepts.

- *Holomorphic Frölicher spaces.* As curves one has to take mappings from the complex unit disk $\mathbb{D}$, and complex valued functions such that each composition is holomorphic $\mathbb{D}\to\mathbb{C}$. Stein manifolds are holomophic Frölicher spaces whereas compact complex manifolds are not. See [7, 23.5].
- *Sikosrki spaces.* Here one specifies an algebra of 'smooth' functions with certain properties. One can also specify sheafs of 'smooth' functions.
- *Diffeological spaces.* Here one specifies mappings from open sets in all $\mathbb{R}^n$'s with appropriate conditions. These were introduces by Sourieau, see the recent book [6].

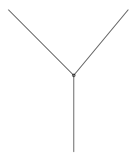
There are natural functors from the categories of Sikorski spaces and of diffeological spaces into the category of Frölicher spaces, which are right and left adjoints. See [9] for a comparison.

Theorem. *Tree spaces in the sense of* [1] *are balanced Frölicher spaces.*

This follows from the fact that a tree space $\mathcal{T}$ is always a closed subspace of $\mathbb{R}^N$, where different quadrants always meet at non-trivial angles. As generating set of functions one can take the restrictions of linear functions on $\mathbb{R}^N$. This is called the standard Frölicher structure.

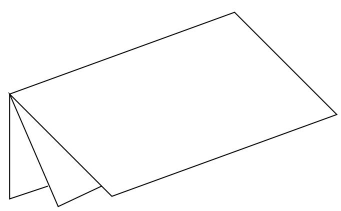
The following two examples are fundamental to understanding tree spaces, see e.g. [8].

Example of a treespace: the 3-spider.



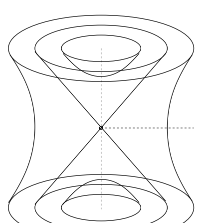
A generating set of functions consists of all linear functions on $\mathbb{R}^2$ or on $\mathbb{R}^3$. Smooth curves in $\mathcal{C}_X$ then have to stop in all derivatives when they change sheets. Functions $f \in \mathcal{F}_X$ are then smooth on each closed sheet.

The open book as part of tree space.

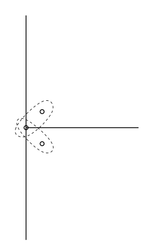


A generating set of functions consists again of all linear functions on $\mathbb{R}^n$. Smooth curves in $\mathcal{C}_X$ can meet the spine S only tangentially; more precisely, the first non-vanishing derivative of the normal component has to be of even order. Functions $f \in \mathcal{F}_X$ are smooth on each closed sheet.

Example of a non-Hausdorff orbit space: adjoint action of $SL(2,\mathbb{R})$.



The adjoint action of $SL(2,\mathbb{R})$ on its Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ has as orbits the connected components of the 'spheres' with respect to the Killing form, which is isomorphic to Minkowski space $\mathbb{R}^{1,2}$. The orbits are as follows. The double light cone decomposes in three orbits: the future light cone, the past one (these two are not closed orbits), and 0.



The other orbits are: The two parts of each two-sheeted hyperboloid, and the one sheeted hyperboloids. The orbit space X can be visualized as a vertical line, a horizontal half-line, and two further points (corresponding to the open light cones) which cannot be separated in the quotient topology from the intersection point depicting the equivalence class of 0.

The structure of a Frölicher space on X is generated by the set $\mathcal{C}$ of projections to X of all smooth curves in $\mathbb{R}^{1,2}$. A smooth curve can go from the vertical half-line through one of the nonclosed orbits to the horizontal half-line, but through 0 it can only go infinitely flat (in $\mathbb{R}^{1,2}$). The functions $f \in \mathcal{F}_X$ are those such that $f \circ \pi$ is in $C^\infty(\mathbb{R}^{1,2},\mathbb{R})$. The topology $\tau_{\mathcal{F}}$ is strictly coarser then the quotient topology: The closure of each non-closed point contains all 3 points. We get curves in $\mathcal{C}_X$

which are not in $\mathcal{C}$, namely, a curve in $\mathcal{C}_X$ can now also go smoothly with nontrivial speed through 0 from vertical to horizontal. The final topology $\tau_{\mathcal{C}}$ is finer: the two non-closed points become closed, too; so $\tau_{\mathcal{C}}$ is T_1 but still not T_2. The space X is not balanced.

The geodesic Frölicher structure on tree spaces. Then we can put the following Frölicher structure on X: Let us take as generating set $\mathcal{C}$ the union of the space $\mathcal{C}_X$ of smooth curves for the standard Frölicher structure on X with the set of all curves $\gamma : \mathbb{R} \to X$ such that $s \mapsto \gamma(\tan(s)) = \tilde{\gamma}(s)$ is a geodesic between the points $\tilde{\gamma}(-\pi/2)$ and $\tilde{\gamma}(\pi/2)$ which is parameterized proportional to arclength. That means, we put:

$$\begin{aligned}\mathcal{F}_X^{\text{geo}} &= \{f : X \to \mathbb{R} \,:\, f \circ \gamma \in C^\infty(\mathbb{R},\mathbb{R})\,\forall \gamma \in \mathcal{C}\},\\ \mathcal{C}_X^{\text{geo}} &= \{c : \mathbb{R} \to X \,:\, f \circ c \in C^\infty(\mathbb{R},\mathbb{R})\,\forall f \in \mathcal{F}_X^{\text{geo}}\}.\end{aligned}$$

Then $(X, \mathcal{C}_X^{\text{geo}}, \mathcal{F}_X^{\text{geo}})$ is a Frölicher space by the general construction.

We define $T_x^i X$ as the quotient of $\{c \in \mathcal{C}_X^{\text{geo}} \,:\, c(0) = x\}$ by the equivalence relation $c_1 \sim c_2 \iff (f \circ c_1)'(0) = (f \circ c_2)'(0)\ \forall\, f \in \mathcal{F}_X^{\text{geo}}$, and call this the inner tangent space at $x \in X$. For a tree-space $T_x^i X$ is the tangent space at x of the stratum containing x in its interior.

We may define $T_x^c X$ as the quotient of the set of all geodesics $\gamma : [0,1] \to X$ with $\gamma(0) = x$, parameterized proportional to arclength, modulo the equivalence relation $\gamma_1 \sim \gamma_2 \iff \gamma_1 = \gamma_2$ near 0. We call $T_x^c X$ the *conical tangent space*. It contains all vectors pointing from x into higher strata which are bounded by the stratum of x.

Geodesic Frölicher structures on certain metric spaces. Let X be a geodesic metric space, i.e., between any two points there exists a unique geodesic realizing the distance (see e.g. [2]).

If we generate a Frölicher structure only by the set $\mathcal{C}$ of geodesics, even in $\mathbb{R}^n$ we do not get the usual structure. Besides C^∞-function we also get homogeneous rational functions in $\mathcal{F}_X$, and more.

Let us take as generating set $\mathcal{F}$ of functions squares of geodesic distances $x \mapsto d(y_i, x)^2$, where y_i runs through a subset of points in X. If $X = \mathbb{R}^n$ and y_i are $n+1$ generic points, the resulting Frölicher structure is the usual one. If X is a tree-space, the resulting Frölicher structure seems to be the $(\mathcal{C}_X^{\text{geo}}, \mathcal{F}_X^{\text{geo}})$ structure.

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