ADDENDUM TO: "LIFTING SMOOTH CURVES OVER INVARIANTS FOR REPRESENTATIONS OF COMPACT LIE GROUPS, III" [J. LIE THEORY 16 (2006), NO. 3, 579–600.]

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Abstract. We improve the main results in the paper from the title using a recent refinement of Bronshtein's theorem due to Colombini, Orrú, and Pernazza. They are then in general best possible both in the hypothesis and in the outcome. As a consequence we obtain a result on lifting smooth mappings in several variables.

A recent refinement of Bronshtein's theorem [5] and of some of its consequences due to Colombini, Orrú, and Pernazza [6] (namely theorem 1(i) below) allows to essentially improve our main results in [10]; see theorem 2 and corollary 3 below. The improvement consists in weakening the hypothesis considerably: In [10] we needed a curve c to be of class

(i) C^k in order to admit a differentiable lift with locally bounded derivative,
(ii) C^{k+d} in order to admit a C^1-lift, and
(iii) C^{k+2d} in order to admit a twice differentiable lift.

It turns out that theorem 2 and corollary 3 are in general best possible both in the hypothesis and in the outcome. In theorem 4 and corollary 5 we deduce some results on lifting smooth mappings in several variables.

Refinement of Bronshtein's theorem. Bronshtein's theorem [5] (see also Wakabayashi's version [15]) states that, for a curve of monic hyperbolic polynomials

$$P(t)(x) = x^n + \sum_{j=1}^{n} (-1)^j a_j(t) x^{n-j}. \tag{1}$$

with coefficients $a_j \in C^n(\mathbb{R})$ $(1 \le j \le n)$, there exist differentiable functions λ_j $(1 \le j \le n)$ with locally bounded derivatives which parameterize the roots of P. A polynomial is called hyperbolic if all its roots are real.

The following theorem refines Bronshtein's theorem [5] and also a result of Mandai [14] and a result of Kriegl, Losik, and Michor [8]. In [14] the coefficients are required to be of class C^{2n} for C^1-roots, and in [8] they are assumed to be C^{3n} for twice differentiable roots.

1. **Theorem** ([6, 2.1]). *Consider a curve P of monic hyperbolic polynomials* (1). *Then:*

(i) *If $a_j \in C^n(\mathbb{R})$ $(1 \le j \le n)$, then there exist functions $\lambda_j \in C^1(\mathbb{R})$ $(1 \le j \le n)$ which parameterize the roots of P.*
(ii) *If $a_j \in C^{2n}(\mathbb{R})$ $(1 \le j \le n)$, then the roots of P may be chosen twice differentiable.*

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Counterexamples (e.g. in [6, section 4]) show that in this result the assumptions on P cannot be weakened.

Improvement of the results in [10]**.** Let $\rho : G \to \mathrm{O}(V)$ be an orthogonal representation of a compact Lie group G in a real finite dimensional Euclidean vector space V. Choose a minimal system of homogeneous generators $\sigma_1, \dots, \sigma_n$ of the algebra $\mathbb{R}[V]^G$ of G-invariant polynomials on V. Define

$$d = d(\rho) := \max\{\deg \sigma_i : 1 \le i \le n\},$$

which is independent of the choice of the σ_i (see [10, 2.4]).

If G is a finite group, we write $V = V_1 \oplus \cdots \oplus V_l$ as orthogonal direct sum of irreducible subspaces V_i. We choose $v_i \in V_i \backslash \{0\}$ such that the cardinality of the corresponding isotropy group G_{v_i} is maximal, and put

$$k = k(\rho) := \max\{d(\rho), |G|/|G_{v_i}| : 1 \le i \le l\}.$$

The mapping $\sigma = (\sigma_1, \dots, \sigma_n) : V \to \mathbb{R}^n$ induces a homeomorphism between the orbit space V/G and the image $\sigma(V)$. Let $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a smooth curve in the orbit space (smooth as a curve in $\mathbb{R}^n$). A curve $\bar{c} : \mathbb{R} \to V$ is called lift of c if $\sigma \circ \bar{c} = c$. The problem of lifting curves smoothly over invariants is independent of the choice of the σ_i (see [10, 2.2]).

2. **Theorem.** *Let $\rho : G \to \mathrm{O}(V)$ be a representation of a finite group G. Let $d = d(\rho)$ and $k = k(\rho)$. Consider a curve $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ in the orbit space of ρ. Then:*

(i) *If c is of class C^k, then any differentiable lift $\bar{c} : \mathbb{R} \to V$ of c (which always exists) is actually C^1.*

(ii) *If c is of class C^{k+d}, then there exists a global twice differentiable lift $\bar{c} : \mathbb{R} \to V$ of c.*

Proof. (i) Let $\bar{c}$ be any differentiable lift of c. Note that the existence of $\bar{c}$ is guaranteed for any C^d-curve c, by [9]. In the proof of [10, 8.1] we construct curves of monic hyperbolic polynomials $t \mapsto P_i(t)$ which have the regularity of c and whose roots are parameterized by $t \mapsto \langle v_i \mid g.\bar{c}(t)\rangle$ $(g \in G_{v_i} \backslash G)$.

If c is of class C^k, then theorem 1(i) provides C^1-roots of $t \mapsto P_i(t)$. By the proof of [10, 4.2] we obtain that the parameterization $t \mapsto \langle v_i \mid g.\bar{c}(t)\rangle$ is C^1 as well. Hence $\bar{c}$ is a C^1-lift of c. Alternatively, the proof of 1(i) in [6] actually shows that any differentiable choice of roots is C^1.

(ii) Let c be of class C^{k+d}. The existence of a global twice differentiable lift $\bar{c}$ of c follows from the proof of [10, 5.1 and 5.2], where we use (i) instead of [10, 4.2]. □

3. **Corollary.** *Let $\rho : G \to \mathrm{O}(V)$ be a polar representation of a compact Lie group G. Let $\Sigma \subseteq V$ be a section, $W(\Sigma) = N_G(\Sigma)/Z_G(\Sigma)$ its generalized Weyl group, and $\rho_\Sigma : W(\Sigma) \to \mathrm{O}(\Sigma)$ the induced representation. Let $d = d(\rho_\Sigma)$ and $k = k(\rho_\Sigma)$. Consider a curve $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ in the orbit space of ρ. Then:*

(i) *If c is of class C^k, then there exists a global orthogonal C^1-lift $\bar{c} : \mathbb{R} \to V$ of c.*

(ii) *If c is of class C^{k+d}, then there exists a global orthogonal twice differentiable lift $\bar{c} : \mathbb{R} \to V$ of c.* □

The examples which show that the hypothesis in 1 are best possible also imply that in general the hypothesis in 2 and 3 cannot be improved.

On the other hand the outcome of 2 and 3 cannot be refined either: A C^∞- curve c does in general not allow a $C^{1,\alpha}$-lift for any $\alpha > 0$. See [7], [1], [4]. But see also [3] and [10, remark 4.2].

Note that the improvement affects also [13, part 6].

Lifting smooth mappings in several variables. From theorem 2 we can deduce a lifting result for mappings in several variables.

4. **Theorem.** *Let $\rho : G \to \mathrm{O}(V)$ be a representation of a finite group G, $d = d(\rho)$, and $k = k(\rho)$. Let $U \subseteq \mathbb{R}^q$ be open. Consider a mapping $f : U \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ of class C^k. Then any continuous lift $\bar{f} : U \to V$ of f is actually locally Lipschitz.*

Proof. Let $c : \mathbb{R} \to U$ be a C^∞-curve. By theorem 2(i) the curve $f \circ c$ admits a C^1-lift $\overline{f \circ c}$. A further continuous lift of $f \circ c$ is formed by $\bar{f} \circ c$. By [12, 5.3] we can conclude that $\bar{f} \circ c$ is locally Lipschitz. So we have shown that $\bar{f}$ is locally Lipschitz along C^∞-curves. By Boman [2] (see also [11, 12.7]) that implies that $\bar{f}$ is locally Lipschitz. □

In general there will not always exist a continuous lift of f (for instance, if G is a finite rotation group and f is defined near 0). However, if G is a finite reflection group, then any continuous f allows a continuous lift (since the orbit space can be embedded homeomorphically in V).

5. **Corollary.** *Let $\rho : G \to \mathrm{O}(V)$ be a polar representation of a compact connected Lie group G. Let $\Sigma \subseteq V$ be a section, $W(\Sigma) = N_G(\Sigma)/Z_G(\Sigma)$ its generalized Weyl group, $\rho_\Sigma : W(\Sigma) \to \mathrm{O}(\Sigma)$ the induced representation, $d = d(\rho_\Sigma)$, and $k = k(\rho_\Sigma)$. Let $U \subseteq \mathbb{R}^q$ be open. Consider a mapping $f : U \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ of class C^k. Then there exists an orthogonal lift $\bar{f} : U \to V$ of f which is locally Lipschitz.*

Proof. The Weyl group $W(\Sigma)$ is a finite reflection group, since G is connected. □

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