A ZOO OF DIFFEOMORPHISM GROUPS ON $\mathbb{R}^n$

PETER W. MICHOR AND DAVID MUMFORD

Abstract. We consider the groups $\text{Diff}_{\mathcal{B}}(\mathbb{R}^n)$, $\text{Diff}_{H^\infty}(\mathbb{R}^n)$, and $\text{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ of smooth diffeomorphisms on $\mathbb{R}^n$ which differ from the identity by a function which is in either $\mathcal{B}$ (bounded in all derivatives), $H^\infty = \bigcap_{k\ge 0} H^k$, or $\mathcal{S}$ (rapidly decreasing). We show that all these groups are smooth regular Lie groups.

1. Introduction

The purpose of this article is to prove that the following groups of diffeomorphisms on $\mathbb{R}^n$ are regular (see 3.2) Lie groups:

- $\text{Diff}_{\mathcal{B}}(\mathbb{R}^n)$, the group of all diffeomorphisms which differ from the identity by a function which is bounded together with all derivatives separately; see 3.3.
- $\text{Diff}_{H^\infty}(\mathbb{R}^n)$, the group of all diffeomorphisms which differ from the identity by a function in the intersection H^∞ of all Sobolev spaces H^k for $k \in \mathbb{N}_{\ge 0}$; see 3.4.
- $\text{Diff}_{\mathcal{S}}(\mathbb{R}^n)$, the group of all diffeomorphisms which fall rapidly to the identity; see 3.5.

Since we are giving a kind of uniform proof, we also mention in 3.6 the group $\text{Diff}_c(\mathbb{R}^n)$ of all diffeomorphisms which differ from the identity only on a compact subset, where this result is known for many years, by [16] and [15]. The groups $\text{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ and partly $\text{Diff}_{H^\infty}(\mathbb{R}^n)$ have been used essentially in the papers [18], [10], [2], [3], [4], and [1]. In particular, $\text{Diff}_{H^\infty}(\mathbb{R}^n)$ is essential if one wants to prove that the geodesic equation of a right Riemannian invariant metric is well-posed with the use of Sobolov space techniques. The regular Lie groups $\text{Diff}_{\mathcal{B}}(E)$ and $\text{Diff}_{\mathcal{S}}(E)$ have been treated, using single derivatives iteratively, in [23], for a Banach space E. See [7] for the role of diffeomorphism groups in quantum physics. Andreas Kriegl, Leonard Frerick, and Jochen Wengenroth helped with discussions and hints.

2. Some words on smooth convenient calculus

Traditional differential calculus works well for finite dimensional vector spaces and for Banach spaces. For more general locally convex spaces, we sketch here the convenient approach as explained in [6] and [8]. The main difficulty is that composition of linear mappings stops to be jointly continuous at the level of Banach

Date: May 16, 2013.
2010 *Mathematics Subject Classification.* 58B20, 58D15.

spaces, for any compatible topology. We use the notation of [8] and this is the main reference for this section.

2.1. **The c^∞-topology.** Let E be a locally convex vector space. A curve $c : \mathbb{R} \to E$ is called *smooth* or C^∞ if all derivatives exist and are continuous — this is a concept without problems. Let $C^\infty(\mathbb{R}, E)$ be the space of smooth functions. It can be shown that the set $C^\infty(\mathbb{R}, E)$ does not depend on the locally convex topology of E, only on its associated bornology (system of bounded sets). The final topologies with respect to the following sets of mappings into E coincide:

(1) $C^\infty(\mathbb{R}, E)$.
(2) The set of all Lipschitz curves (so that $\{\frac{c(t)-c(s)}{t-s} : t \neq s, |t|, |s| \leq C\}$ is bounded in E, for each $C > 0$).
(3) The set of injections $E_B \to E$ where B runs through all bounded absolutely convex subsets in E, and where E_B is the linear span of B equipped with the Minkowski functional $\|x\|_B := \inf\{\lambda > 0 : x \in \lambda B\}$.
(4) The set of all Mackey-convergent sequences $x_n \to x$ (there exists a sequence $0 < \lambda_n \nearrow \infty$ with $\lambda_n(x_n - x)$ bounded).

This topology is called the c^∞-topology on E and we write $c^\infty E$ for the resulting topological space. In general (on the space $\mathcal{D}$ of test functions for example) it is finer than the given locally convex topology, it is not a vector space topology, since scalar multiplication is no longer jointly continuous. The finest among all locally convex topologies on E which are coarser than $c^\infty E$ is the bornologification of the given locally convex topology. If E is a Fréchet space, then $c^\infty E = E$.

2.2. **Convenient vector spaces.** A locally convex vector space E is said to be a *convenient vector space* if one of the following equivalent conditions is satisfied (called c^∞-completeness):

(1) For any $c \in C^\infty(\mathbb{R}, E)$ the (Riemann-) integral $\int_0^1 c(t)dt$ exists in E.
(2) Any Lipschitz curve in E is locally Riemann integrable.
(3) A curve $c : \mathbb{R} \to E$ is smooth if and only if $\lambda \circ c$ is smooth for all $\lambda \in E^*$, where E^* is the dual consisting of all continuous linear functionals on E. Equivalently, we may use the dual E' consisting of all bounded linear functionals.
(4) Any Mackey-Cauchy sequence (i. e. $t_{nm}(x_n - x_m) \to 0$ for some $t_{nm} \to \infty$ in $\mathbb{R}$) converges in E. This is visibly a mild completeness requirement.
(5) If B is bounded closed absolutely convex, then E_B is a Banach space.
(6) If $f : \mathbb{R} \to E$ is scalarwise Lip^k, then f is Lip^k, for $k \geq 0$.
(7) If $f : \mathbb{R} \to E$ is scalarwise C^∞ then f is differentiable at 0.
(8) If $f : \mathbb{R} \to E$ is scalarwise C^∞ then f is C^∞.

Here, a mapping $f : \mathbb{R} \to E$ is called Lip^k if all derivatives up to order k exist and are Lipschitz, locally on $\mathbb{R}$. That f is scalarwise C^∞ means $\lambda \circ f$ is C^∞ for all continuous linear functionals on E.

2.3. **Smooth mappings.** Let E, F, and G be convenient vector spaces, and let $U \subset E$ be c^∞-open. A mapping $f : U \to F$ is called *smooth* or C^∞, if $f \circ c \in C^\infty(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, U)$. *The main properties of smooth calculus are the following.*

(1) *For mappings on Fréchet spaces this notion of smoothness coincides with all other reasonable definitions. Even on $\mathbb{R}^2$ this is non-trivial.*
(2) *Multilinear mappings are smooth if and only if they are bounded.*
(3) *If $f : E \supseteq U \to F$ is smooth then the derivative $df : U \times E \to F$ is smooth, and also $df : U \to L(E, F)$ is smooth where $L(E, F)$ denotes the space of all bounded linear mappings with the topology of uniform convergence on bounded subsets.*
(4) *The chain rule holds.*
(5) *The space $C^\infty(U, F)$ is again a convenient vector space where the structure is given by the obvious injection*

$$C^\infty(U,F) \xrightarrow{C^\infty(c,\ell)} \prod_{c \in C^\infty(\mathbb{R},U), \ell \in F^*} C^\infty(\mathbb{R},\mathbb{R}), \quad f \mapsto (\ell \circ f \circ c)_{c,\ell},$$

where $C^\infty(\mathbb{R}, \mathbb{R})$ carries the topology of compact convergence in each derivative separately.
(6) *The exponential law holds: For c^∞-open $V \subset F$,*

$$C^\infty(U, C^\infty(V, G)) \cong C^\infty(U \times V, G)$$

is a linear diffeomorphism of convenient vector spaces.
(7) *A linear mapping $f : E \to C^\infty(V, G)$ is smooth (by* (2) *equivalent to bounded) if and only if $E \xrightarrow{f} C^\infty(V, G) \xrightarrow{\mathrm{ev}_v} G$ is smooth for each $v \in V$. This is called the smooth uniform boundedness theorem* [8, 5.26].
(8) *The following canonical mappings are smooth.*

$$\begin{aligned}
&\mathrm{ev} : C^\infty(E,F) \times E \to F, \quad \mathrm{ev}(f,x) = f(x)\\
&\mathrm{ins} : E \to C^\infty(F, E \times F), \quad \mathrm{ins}(x)(y) = (x,y)\\
&(\quad)^\wedge : C^\infty(E, C^\infty(F,G)) \to C^\infty(E \times F, G)\\
&(\quad)^\vee : C^\infty(E \times F, G) \to C^\infty(E, C^\infty(F,G))\\
&\mathrm{comp} : C^\infty(F,G) \times C^\infty(E,F) \to C^\infty(E,G)\\
&C^\infty(\quad,\quad) : C^\infty(F,F_1) \times C^\infty(E_1,E) \to C^\infty(C^\infty(E,F), C^\infty(E_1,F_1))\\
&\qquad (f,g) \mapsto (h \mapsto f \circ h \circ g)\\
&\prod : \prod C^\infty(E_i,F_i) \to C^\infty(\prod E_i, \prod F_i)
\end{aligned}$$

Note that the conclusion of (6) is the starting point of the classical calculus of variations, where a smooth curve in a space of functions was assumed to be just a smooth function in one variable more. It is also the source of the name convenient calculus. This and some other obvious properties already determine convenient calculus. There are, however, smooth mappings which are not continuous. This is unavoidable and not so horrible as it might appear at first sight. For example, the evaluation $E \times E^* \to \mathbb{R}$ is jointly continuous if and only if E is normable, but it is always smooth. Clearly smooth mappings are continuous for the c^∞-topology.

This ends our review of the standard results of convenient calculus. But we will need more.

2.4. **Theorem.** [6, 4.1.19] *Let $c : \mathbb{R} \to E$ be a curve in a convenient vector space E. Let $\mathcal{V} \subset E'$ be a subset of bounded linear functionals such that the bornology of E has a basis of $\sigma(E, \mathcal{V})$-closed sets. Then the following are equivalent:*

(1) *c is smooth*
(2) *There exist locally bounded curves $c^k : \mathbb{R} \to E$ such that $\ell \circ c$ is smooth $\mathbb{R} \to \mathbb{R}$ with $(\ell \circ c)^{(k)} = \ell \circ c^k$, for each $\ell \in \mathcal{V}$.*

If E is reflexive, then for any point separating subset $\mathcal{V} \subset E'$ the bornology of E has a basis of $\sigma(E, \mathcal{V})$-closed subsets, by [6, 4.1.23].

This theorem is surprisingly strong: Note that $\mathcal{V}$ does not need to recognize bounded sets.

2.5. **Faa di Bruno formula.** Let $g \in C^\infty(\mathbb{R}^n, \mathbb{R}^k)$ and let $f \in C^\infty(\mathbb{R}^k, \mathbb{R}^l)$. Then the p-th deivative of $f \circ g$ looks as follows where sym_p denotes symmetrization of a p-linear mapping:

$$\frac{d^p(f \circ g)(x)}{p!} = \operatorname{sym}_p \Big(\sum_{j=1}^{p} \sum_{\substack{\alpha \in \mathbb{N}^j_{>0} \\ \alpha_1 + \dots + \alpha_j = p}} \frac{d^j f(g(x))}{j!} \Big(\frac{d^{\alpha_1} g(x)}{\alpha_1!}, \dots, \frac{d^{\alpha_j} g(x)}{\alpha_j!} \Big) \Big)$$

The one-dimensional version is due to Faà di Bruno [5], the only beatified mathematician. The formula is seen by composing the Taylor series.

3. Groups of smooth diffeomorphisms

3.1. **Model spaces for Lie groups of diffeomorphism.** If we consider the group of all orientation preserving diffeomorphisms $\operatorname{Diff}(\mathbb{R}^n)$ of $\mathbb{R}^n$, it is not an open subset of $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with the compact C^∞-topology. So it is not a smooth manifold in the usual sense, but we may consider it as a Lie group in the cartesian closed category of Frölicher spaces, see [8, Section 23] with the structure induced by the injection $f \mapsto (f, f^{-1}) \in C^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Or one can use the theory of smooth manifolds based on smooth curves instead of charts from [11], [12], which agrees with the usual theory up to Banach manifolds.

We shall now describe regular Lie groups in $\operatorname{Diff}(\mathbb{R}^n)$ which are given by diffeomorphisms of the form $f = \operatorname{Id} + g$ where g is in some specific convenient vector spaces of bounded functions in $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Now, we discuss these spaces on $\mathbb{R}^n$, we describe the smooth curves in them, and we describe the corresponding groups.

3.2. **Regular Lie groups.** We consider a smooth Lie group G with Lie algebra $\mathfrak{g} = T_e G$ modelled on convenient vector spaces. The notion of a regular Lie group is originally due to Omori and collaborators (see [19], [20]) for Fréchet Lie groups, was weakened and made more transparent by Milnor [17] and carried over to convenient Lie groups in [9], see also [8, 38.4]. A Lie group G is called *regular* if the following holds:

- For each smooth curve $X \in C^\infty(\mathbb{R}, \mathfrak{g})$ there exists a curve $g \in C^\infty(\mathbb{R}, G)$ whose right logarithmic derivative is X, i.e.,

$$\begin{cases} g(0) &= e \\ \partial_t g(t) &= T_e(\mu^{g(t)})X(t) = X(t).g(t) \end{cases}$$

  where $\mu : G \times G \to G$ is multiplication with $\mu(g,h) = g.h = \mu_g(h) = \mu^h(g)$. The curve g is uniquely determined by its initial value $g(0)$, if it exists.
- Put $\operatorname{evol}^r_G(X) = g(1)$ where g is the unique solution required above. Then $\operatorname{evol}^r_G : C^\infty(\mathbb{R}, \mathfrak{g}) \to G$ is required to be C^∞ also.

3.3. **The group** $\operatorname{Diff}_{\mathcal{B}}(\mathbb{R}^n)$**.** The space $\mathcal{B}(\mathbb{R}^n)$ (called $\mathcal{D}_{L^\infty}(\mathbb{R}^n)$ by L. Schwartz [21]) consists of all smooth functions with all derivatives (separately) bounded. It is a Fréchet space. By [22], the space $\mathcal{B}(\mathbb{R}^n)$ is linearly isomorphic to $\ell^\infty \hat{\otimes}\, \mathfrak{s}$ for any completed tensor product between the projective one and the injective one, where $\mathfrak{s}$ is the nuclear Fréchet space of rapidly decreasing real sequences. Thus $\mathcal{B}(\mathbb{R}^n)$ is not reflexive and not nuclear.
The space $C^\infty(\mathbb{R}, \mathcal{B}(\mathbb{R}^n))$ of smooth curves in $\mathcal{B}(\mathbb{R}^n)$ consists of all functions $c \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R})$ satisfying the following property:

- *For all $k \in \mathbb{N}_{\geq 0}$, $\alpha \in \mathbb{N}^n_{\geq 0}$ and each $t \in \mathbb{R}$ the expression $\partial_t^k \partial_x^\alpha c(t,x)$ is uniformly bounded in $x \in \mathbb{R}^n$, locally in t.*

To see this, we use 2.4 for the set $\{\operatorname{ev}_x : x \in \mathbb{R}\}$ of point evaluations in $\mathcal{B}(\mathbb{R}^n)$. Here, $\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x^\alpha}$ and $c^k(t) = \partial_t^k f(t, \quad)$.
$\operatorname{Diff}^+_{\mathcal{B}}(\mathbb{R}^n) = \big\{ f = \operatorname{Id} + g : g \in \mathcal{B}(\mathbb{R}^n)^n, \det(\mathbb{I}_n + dg) \geq \varepsilon > 0 \big\}$ *denotes the corresponding group*, see Theorem 3.8 below.

3.4. **The group** $\operatorname{Diff}_{H^\infty}(\mathbb{R}^n)$**.** The space $H^\infty(\mathbb{R}^n) = \bigcap_{k\geq 1} H^k(\mathbb{R}^n)$ is the intersection of all Sobolev spaces which is a reflexive Fréchet space. It is called $\mathcal{D}_{L^2}(\mathbb{R}^n)$ by L. Schwartz in [21]. By [22], the space $H^\infty(\mathbb{R}^n)$ is linearly isomorphic to $\ell^2 \hat{\otimes}\, \mathfrak{s}$. Thus it is not nuclear, not Schwartz, not Montel, but still smoothly paracompact.
The space $C^\infty(\mathbb{R}, H^\infty(\mathbb{R}^n))$ of smooth curves in $H^\infty(\mathbb{R}^n)$ consists of all functions $c \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R})$ satisfying the following property:

- *For all $k \in \mathbb{N}_{\geq 0}$, $\alpha \in \mathbb{N}^n_{\geq 0}$ the expression $\|\partial_t^k \partial_x^\alpha f(t, \quad)\|_{L^2(\mathbb{R}^n)}$ is locally bounded near each $t \in \mathbb{R}$.*

The proof is literally the same as for $\mathcal{B}(\mathbb{R}^n)$, noting that the point evaluations are continuous on each Sobolev space H^k with $k > \frac{n}{2}$.
$\operatorname{Diff}^+_{H^\infty}(\mathbb{R}^n) = \big\{ f = \operatorname{Id} + g : g \in H^\infty(\mathbb{R}^n), \det(\mathbb{I}_n + dg) > 0 \big\}$ *denotes the correrponding group*, see Theorem 3.8 below.

3.5. **The group** $\operatorname{Diff}_{\mathcal{S}}(\mathbb{R}^n)$**.** The algebra $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing functions is a reflexive nuclear Fréchet space.
The space $C^\infty(\mathbb{R}, \mathcal{S}(\mathbb{R}^n))$ of smooth curves in $\mathcal{S}(\mathbb{R}^n)$ consists of all functions $c \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R})$ satisfying the following property:

- *For all $k, m \in \mathbb{N}_{\geq 0}$ and $\alpha \in \mathbb{N}^n_{\geq 0}$, the expression $(1+|x|^2)^m \partial_t^k \partial_x^\alpha c(t,x)$ is uniformly bounded in $x \in \mathbb{R}^n$, locally uniformly bounded in $t \in \mathbb{R}$.*

$\operatorname{Diff}^+_{\mathcal{S}}(\mathbb{R}^n) = \big\{f = \operatorname{Id} + g : g \in \mathcal{S}(\mathbb{R}^n)^n, \det(\mathbb{I}_n + dg) > 0\big\}$ *is the correponding group.*

3.6. **The group $\operatorname{Diff}_c(\mathbb{R}^n)$.** The algebra $C^\infty_c(\mathbb{R}^n)$ of all smooth functions with compact support is a nuclear (LF)-space. *The space $C^\infty(\mathbb{R}, C^\infty_c(\mathbb{R}^n))$ of smooth curves in $C^\infty_c(\mathbb{R}^n)$ consists of all functions $f \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R})$ satisfying the following property:*

- *For each compact interval $[a,b]$ in $\mathbb{R}$ there exists a compact subset $K \subset \mathbb{R}^n$ such that $f(t,x) = 0$ for $(t,x) \in [a,b] \times (\mathbb{R}^n \setminus K)$.*

$\operatorname{Diff}_c(\mathbb{R}^n) = \big\{f = \operatorname{Id} + g : g \in C^\infty_c(\mathbb{R}^n)^n, \det(\mathbb{I}_n + dg) > 0\big\}$ *is the correponding group.*

3.7. **Ideal properties of function spaces.** The function spaces are boundedly mapped into each other as follows:

$$C^\infty_c(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \longrightarrow H^\infty(\mathbb{R}^n) \longrightarrow \mathcal{B}(\mathbb{R}^n)$$

and each space is a bounded locally convex algebra and a bounded $\mathcal{B}(\mathbb{R}^n)$-module. Thus, each space is an ideal in each larger space.

3.8. **Theorem.** *The sets of diffeomorphisms $\operatorname{Diff}_c(\mathbb{R}^n)$, $\operatorname{Diff}_{\mathcal{S}}(\mathbb{R}^n)$, $\operatorname{Diff}_{H^\infty}(\mathbb{R}^n)$, and $\operatorname{Diff}_{\mathcal{B}}(\mathbb{R}^n)$ are all smooth regular Lie groups in the sense of 3.2. We have the following smooth injective group homomorphisms*

$$\operatorname{Diff}_c(\mathbb{R}^n) \longrightarrow \operatorname{Diff}_{\mathcal{S}}(\mathbb{R}^n) \longrightarrow \operatorname{Diff}_{H^\infty}(\mathbb{R}^n) \longrightarrow \operatorname{Diff}_{\mathcal{B}}(\mathbb{R}^n)\ .$$

Each group is a normal subgroup in any other in which it is contained, in particular in $\operatorname{Diff}_{\mathcal{B}}(\mathbb{R}^n)$.

The case $\operatorname{Diff}_c(\mathbb{R}^n)$ is well-known, see for example, [8, 43.1]. The one-dimensional version $\operatorname{Diff}_{\mathcal{S}}(\mathbb{R})$ was treated in [13, 6.4].

Proof. Let $\mathcal{A}$ denote any of $\mathcal{B}$, H^∞, $\mathcal{S}$, or c, and let $\mathcal{A}(\mathbb{R}^n)$ denote the corresponding function space as described in 3.3 - 3.6. Let $f(x) = x + g(x)$ for $g \in \mathcal{A}(\mathbb{R}^n)^n$ with $\det(\mathbb{I}_n + dg) > 0$ and for $x \in \mathbb{R}^n$. We have to check that each f as described is a diffeomorphism. By the inverse function theorem, f is a locally a diffeomorphism everywhere. Thus, the image of f is open in $\mathbb{R}^n$. We claim that it is also closed. So let $x_i \in \mathbb{R}^n$ with $f(x_i) = x_i + g(x_i) \to y_0$ in $\mathbb{R}^n$. Then $f(x_i)$ is a bounded sequence. Since $g \in \mathcal{A}(\mathbb{R}^n) \subset \mathcal{B}(\mathbb{R}^n)$, the x_i also form a bounded sequence, thus contain a convergent subsequence. Without loss let $x_i \to x_0$ in $\mathbb{R}^n$. Then $f(x_i) \to f(x_0) = y_0$. Thus, f is surjective. This also shows that f is a proper mapping (i.e., compact sets have compact inverse images under f). By [14, 17.2], a proper surjective submersion is the projection of a smooth fiber bundle. In our case here f has discrete fibers, so f is a covering mapping and a diffeomorphism since $\mathbb{R}^n$ is simply connected. In each case, the set of g used in the definition of $\operatorname{Diff}_{\mathcal{A}}$ is open in $\mathcal{A}(\mathbb{R}^n)^n$.

Let us next check that $\operatorname{Diff}_{\mathcal{A}}(\mathbb{R}^n)_0$ is closed under composition. We have

$$((\operatorname{Id}+f)\circ(\operatorname{Id}+g))(x) = x + g(x) + f(x+g(x)), \tag{1}$$

and we have to check that $x \mapsto f(x+g(x))$ is in $\mathcal{A}(\mathbb{R}^n)$ if $f, g \in \mathcal{A}(\mathbb{R}^n)^n$. For $\mathcal{A} = \mathcal{B}$ this follows by the Faà di Bruno formula 2.5. For $\mathcal{A} = \mathcal{S}$ we need furthermore the following estimate:

$$(\partial_x^\alpha f)(x+g(x)) = O\Big(\frac{1}{(1+|x+g(x)|^2)^k}\Big) = O\Big(\frac{1}{(1+|x|^2)^k}\Big) \tag{2}$$

which holds, since

$$\frac{1+|x|^2}{1+|x+g(x)|^2} \quad \text{is globally bounded.}$$

For $\mathcal{A} = H^\infty$ we also need that

$$\begin{aligned}\int_{\mathbb{R}^n} |(\partial_x^\alpha f)(x+g(x))|^2\,dx &= \int_{\mathbb{R}^n} |(\partial^\alpha f)(y)|^2 \frac{dy}{|\det(\mathbb{I}_n + dg)((\operatorname{Id}+g)^{-1}(y))|} \\ &\le C(g)\int_{\mathbb{R}^n} |(\partial^\alpha f)(y)|^2\,dy;\end{aligned} \tag{3}$$

this holds, since the denominator is globally bounded away from 0 since g and dg vanish at ∞ by the lemma of Riemann-Lebesque. The case $\mathcal{A}(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n)$ is easy and well known.

Let us check next that multiplication is smooth on $\operatorname{Diff}_{\mathcal{A}}(\mathbb{R}^n)$. Suppose that the curves $t \mapsto \operatorname{Id}+f(t, \quad)$ and $t \mapsto \operatorname{Id}+g(t, \quad)$ are in $C^\infty(\mathbb{R}, \operatorname{Diff}_{\mathcal{A}}(\mathbb{R}^n))$ which means that the functions $f, g \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R}^n)$ satisfy condition $\bullet$ of either 3.3, 3.4, 3.5, or 3.6. We have to check that $f(t, x+g(t,x))$ satisfies the same condition $\bullet$. For this, we reread the proof that composition preserves $\operatorname{Diff}_{\mathcal{A}}(\mathbb{R}^n)$ and pay attention to the further parameter t.

To check that the inverse $(\operatorname{Id}+g)^{-1}$ is again an element in $\operatorname{Diff}_{\mathcal{A}}(\mathbb{R}^n)$ for $g \in \mathcal{A}(\mathbb{R}^n)^n$, we write $(\operatorname{Id}+g)^{-1} = \operatorname{Id}+f$ and we have to check that $f \in \mathcal{A}(\mathbb{R}^n)^n$.

$$\begin{aligned}(\operatorname{Id}+f)\circ(\operatorname{Id}+g) = \operatorname{Id} &\implies x + g(x) + f(x+g(x)) = x \\ &\implies x \mapsto f(x+g(x)) = -g(x) \text{ is in } \mathcal{A}(\mathbb{R}^n)^n.\end{aligned} \tag{4}$$

We treat again first the case $\mathcal{A} = \mathcal{B}$. We know already that $\operatorname{Id}+g$ is a diffeomorphism. By Definition 3.3, we have $\det(\mathbb{I}_n + dg(x)) \ge \varepsilon > 0$ for some ε. This implies that

$$\|(\mathbb{I}_n + dg(x))^{-1}\|_{L(\mathbb{R}^n,\mathbb{R}^n)} \quad \text{is globally bounded,} \tag{5}$$

using the inequality $\|A^{-1}\| \le \frac{\|A\|^{n-1}}{|\det(A)|}$ for any linear $A : \mathbb{R}^n \to \mathbb{R}^n$. To see this, we write $A = US$ with U orthogonal and $S = \operatorname{diag}(s_1 \ge s_2 \ge \cdots \ge s_n \ge 0)$. Then $\|A^{-1}\| = \|S^{-1}U^{-1}\| = \|S^{-1}\| = 1/s_n$ and $|\det(A)| = \det(S) = s_1.s_2 \dots s_n \le s_1^{n-1}.s_n = \|A\|^{n-1}.s_n$.

Moreover,

$$\begin{aligned}&(\mathbb{I}_n + df(x+g(x)))(\mathbb{I}_n + dg(x)) = \mathbb{I}_n \\ &\quad\implies \det(\mathbb{I}_n + df(x+g(x))) = \det(\mathbb{I}_n + dg(x))^{-1} \ge \|\mathbb{I}_n + dg(x)\|^{-n} \ge \eta > 0\end{aligned}$$

for all x. For higher derivatives, we write the Faa di Bruno formula 2.5 in the following form:

$$
\begin{aligned}
&\frac{d^p(f\circ(\mathrm{Id}+g))(x)}{p!} = \\
&= \operatorname{sym}_p\Big(\sum_{j=1}^{p}\sum_{\substack{\alpha\in\mathbb{N}^j_{>0}\\ \alpha_1+\dots+\alpha_j=p}}\frac{d^jf(x+g(x))}{j!}\Big(\frac{d^{\alpha_1}(\mathrm{Id}+g)(x)}{\alpha_1!},\dots,\frac{d^{\alpha_j}(\mathrm{Id}+g)(x)}{\alpha_j!}\Big)\Big)\\
&= \frac{d^pf(x+g(x))}{p!}\Big(\mathrm{Id}+dg(x),\dots,\mathrm{Id}+dg(x)\Big)\\
&\quad + \operatorname{sym}_p\Big(\sum_{j=1}^{p-1}\sum_{\substack{\alpha\in\mathbb{N}^j_{>0}\\ \alpha_1+\dots+\alpha_j=p\\ (h_{\alpha_1},\dots,h_{\alpha_j})}}\frac{d^jf(x+g(x))}{j!}\Big(\frac{d^{\alpha_1}h_{\alpha_1}(x)}{\alpha_1!},\dots,\frac{d^{\alpha_j}h_{\alpha_j}(x)}{\alpha_j!}\Big)\Big)
\end{aligned}
\tag{6}
$$

where $h_{\alpha_i}(x)$ is $g(x)$ for $\alpha_i>1$ (there is always such an i), and where $h_{\alpha_i}(x)=x$ or $g(x)$ if $\alpha_i=1$. Now, we argue as follows: The left-hand side is globally bounded. By (5), we know that $\mathbb{I}_n+dg(x):\mathbb{R}^n\to\mathbb{R}^n$ is invertible with $\|(\mathbb{I}_n+dg(x))^{-1}\|_{L(\mathbb{R}^n,\mathbb{R}^n)}$ globally bounded. Thus, we can conclude by induction on p that $d^pf(x+g(x))$ is bounded uniformly in x, thus also uniformly in $y=x+g(x)\in\mathbb{R}^n$. For general $\mathcal{A}$ we note that the left-hand side is in $\mathcal{A}$. Since we already know that $f\in\mathcal{B}$, and since $\mathcal{A}$ is a $\mathcal{B}$-module, the last term is in $\mathcal{A}$. Thus, also the first term is in $\mathcal{A}$, and any summand there containing at least one $dg(x)$ is in $\mathcal{A}$, so the unique summand $d^pf(x+g(x))$ is also in $\mathcal{A}$ as a function of x. It is thus in rapidly decreasing or in L^2 as a function of $y=x+g(x)$, by arguing as in (2) or (3) above. Thus, inversion maps $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ into itself.

Next, we check that inversion is smooth on $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$. We retrace the proof that inversion preserves $\mathrm{Diff}_{\mathcal{A}}$ assuming that $g(t,x)$ satisfies condition • of either 3.3, 3.4, 3.5, or 3.6. We see again that $f(t,x+g(t,x))=-g(t,x)$ satisfies the condition 3.1 as a function of t,x, and we claim that f then does the same. We reread the proof paying attention to the parameter t and see that the same condition • is satisfied.

We claim that $\mathrm{Diff}_{A}(\mathbb{R})$ is also a regular Lie group in the sense of 3.2. So let $t\mapsto X(t,\ \ )$ be a smooth curve in the Lie algebra $\mathfrak{X}_{\mathcal{A}}(\mathbb{R}^n)=\mathcal{A}(\mathbb{R}^n)^n$, i.e., X satisfies condition • of either 3.3, 3.4, 3.5, or 3.6. The evolution of this time-dependent vector field is the function given by the ODE

$$
\begin{aligned}
&\mathrm{Evol}(X)(t,x)=x+f(t,x),\\
&\begin{cases}\partial_t(x+f(t,x))=f_t(t,x)=X(t,x+f(t,x)),\\ f(0,x)=0.\end{cases}
\end{aligned}
\tag{7}
$$

We have to show first that $f(t,\ \ )\in\mathcal{A}(\mathbb{R}^n)^n$ for each $t\in\mathbb{R}$, second that it is smooth in t with values in $\mathcal{A}(\mathbb{R}^n)^n$, and third that $X\mapsto f$ is also smooth. For $0\le t\le C$ we consider

$$
|f(t,x)|\le\int_0^t|f_t(s,x)|ds=\int_0^t|X(s,x+f(s,x))|\,ds. \tag{8}
$$

Since $\mathcal{A} \subseteq \mathcal{B}$, the vector field $X(t,y)$ is uniformly bounded in $y \in \mathbb{R}^n$, locally in t. So the same is true for $f(t,x)$ by (8).
Next, consider

$$\begin{aligned}
\partial_t d_x f(t,x) &= d_x(X(t, x + f(t,x))) \\
&= (d_x X)(t, x + f(t,x)) + (d_x X)(t, x + f(t,x)).d_x f(t,x) \qquad (9) \\
\|d_x f(t,x)\| &\le \int_0^t \|(d_x X)(s, x + f(s,x))\| ds \\
&\quad + \int_0^t \|(d_x X)(s, x + f(s,x))\|.\|d_x f(s,x)\| ds \\
&=: \alpha(t,x) + \int_0^t \beta(s,x).\|d_x f(s,x)\| ds
\end{aligned}$$

By the Bellman–Grönwall inequality,

$$\|d_x f(t,x)\| \le \alpha(t,x) + \int_0^t \alpha(s,x).\beta(s,x).e^{\int_s^t \beta(\sigma,x)\,d\sigma}\,ds,$$

which is globally bounded in x, locally in t. For higher derivatives in x (where $p > 1$) we use Faá di Bruno's formula in the form

$$\begin{aligned}
&\partial_t d_x^p f(t,x) = d_x^p(X(t, x + f(t,x))) = \operatorname{sym}_p \Big(\\
&\sum_{j=1}^p \sum_{\substack{\alpha \in \mathbb{N}_{>0}^j \\ \alpha_1 + \dots + \alpha_j = p}} \frac{(d_x^j X)(t, x + f(t,x))}{j!} \Big(\frac{d_x^{\alpha_1}(x + f(t,x))}{\alpha_1!}, \dots, \frac{d_x^{\alpha_j}(x + f(t,x))}{\alpha_j!} \Big) \Big) \\
&= (d_x X)(t, x + f(t,x)) \big(d_x^p f(t,x) \big) + \operatorname{sym}_p \Big(\\
&\sum_{j=2}^p \sum_{\substack{\alpha \in \mathbb{N}_{>0}^j \\ \alpha_1 + \dots + \alpha_j = p}} \frac{(d_x^j X)(t, x + f(t,x))}{j!} \Big(\frac{d_x^{\alpha_1}(x + f(t,x))}{\alpha_1!}, \dots, \frac{d_x^{\alpha_j}(x + f(t,x))}{\alpha_j!} \Big) \Big)
\end{aligned}$$

We can assume recursively that $d_x^j f(t,x)$ is globally bounded in x, locally in t, for $j < p$. Then, we have reproduced the situation of (9) (with values in the space of symmetric p-linear mappings $(\mathbb{R}^n)^p \to \mathbb{R}^n$) and we can repeat the argument above involving the Bellman–Grönwall inequality to conclude that $d_x^p f(t,x)$ is globally bounded in x, locally in t. To conclude the same for $\partial_t^m d_x^p f(t,x)$ we just repeat the last arguments for $\partial_t^m f(t,x)$. So we have now proved that $f \in C^\infty(\mathbb{R}, \mathfrak{X}_{\mathcal{B}}(\mathbb{R}^n))$. Since $x \mapsto x + f(t,x)$ is a diffeomorphism for each t as the solution of a flow equation, it is thus in $\operatorname{Diff}_{\mathcal{B}}(\mathbb{R}^n)$. In order to prove that $C^\infty(\mathbb{R}, \mathfrak{X}_{\mathcal{B}}(\mathbb{R}^n)) \ni X \mapsto \operatorname{Evol}(X)(1, \quad) \in \operatorname{Diff}_{\mathcal{B}}(\mathbb{R}^n)$ is smooth, we consider a smooth curve X in $C^\infty(\mathbb{R}, \mathfrak{X}_{\mathcal{B}}(\mathbb{R}^n))$; thus $X(t_1, t_2, x)$ is smooth on $\mathbb{R}^2 \times \mathbb{R}^n$, globally bounded in x in each derivative separately, locally in $t = (t_1, t_2)$ in each derivative. Or, we assume that t is two-dimensional in the argument above. But then it suffices to show that $(t_1, t_2) \mapsto X(t_1, t_2, \quad) \in \mathfrak{X}_{\mathcal{B}}(\mathbb{R}^n)$ is smooth along smooth curves in $\mathbb{R}^2$, and we are again in the situation we have just treated. Thus, $\operatorname{Diff}_{\mathcal{B}}(\mathbb{R}^n)$ is a regular Lie group.

If $\mathcal{A} = \mathcal{S}$, we already know that $f(s,x)$ is globally bounded in x, locally in s. Thus, we may insert $X(s, x + f(s,x)) = O(\frac{1}{(1+|x+f(s,x)|^2)^k}) = O(\frac{1}{(1+|x|^2)^k})$ into (8)

and can conclude that $f(t,x) = O(\frac{1}{(1+|x|^2)^k})$ globally in x, locally in t, for each k. Using this argument, we can repeat the proof for the case $\mathcal{A} = \mathcal{B}$ from above and conclude that $\operatorname{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ is a regular Lie group.
If $\mathcal{A} = H^\infty$ we first consider the differential version of (8),

$$\begin{aligned} \|d_x f(s,x)\| &= \Big\| \int_0^t d_x(X(s,\quad))(x + f(s,x)).(\mathbb{I}_n + df(s,\quad)(x))\,ds \Big\| \\ &\le \int_0^t \big\| d_x(X(s,\quad))(x+f(s,x))\big\|.C\,ds \end{aligned} \tag{10}$$

since $d_x f(s,x)$ is globally bounded in x, locally in s, by the case $\mathcal{A} = \mathcal{B}$. The same holds for $f(s,x)$. Moreover, $X(s,x)$ vanishes at $x = \infty$ by the lemma of Riemann–Lebesque for each x and it is continuous in all variables, so that the same holds for $f(s,x)$ by (8). Now we consider

$$\int_{\mathbb{R}^n} \|(d_x^p f)(t,x)\|^2\,dx = \int_{\mathbb{R}^n} \Big\| \int_0^t d_x^p\big(X(s, \operatorname{Id} + f(s,\quad))\big)(x)\,ds \Big\|^2\,dx. \tag{11}$$

We apply the Faá di Bruno formula in the form (6) to the integrand, remember that we already know that each $d^{\alpha_i}(\operatorname{Id} + f(s,\quad))(x)$ is globally bounded, locally in s, thus (11) is

$$\begin{aligned} &\le \int_{\mathbb{R}^n} \Big(\int_0^t \sum_{j=1}^p \|(d_x^j X)(s, x+f(s,x))\|.C_j\,ds \Big)^2\,dx \\ &= \int_{\mathbb{R}^n} \Big(\int_0^t \sum_{j=1}^p \|(d_x^j X)(s,y)\|.C_j\,ds \Big)^2 \frac{dy}{|\det(\mathbb{I}_n + df(s,\quad))((\mathbb{I}_n + f(s,\quad))^{-1}(y)|} \end{aligned}$$

which is finite, since $X(s,\quad) \in H^\infty$ and since the determinand in the denominator is bounded away from zero – we just checked that $d_x f(s,\quad)$ vanishes at infinity. Then, we repeat this for $\partial_t^m d_x^p f(t,x)$. This shows that $\operatorname{Evol}(X)(t,\quad) \in \operatorname{Id} + H^\infty(\mathbb{R}^n)^n$. As solution of an evolution equation for a bounded non-autonomous vector field it is a diffeomorphism, and thus in $\operatorname{Diff}_{H^\infty}(\mathbb{R}^n)$ for each t. By the same trick as in the case $\mathcal{A} = \mathcal{B}$ we can conclude that $\operatorname{Diff}_{H^\infty}(\mathbb{R}^n)$ is a regular Lie group.

We prove now that $\operatorname{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ is a normal subgroup of $\operatorname{Diff}_{\mathcal{B}}(\mathbb{R}^n)$. So let $g \in \mathcal{B}(\mathbb{R}^n)^n$ with $\det(\mathbb{I}_n + dg(x)) \ge \varepsilon > 0$ for all x, and $s \in \mathcal{S}(\mathbb{R}^n)^n$ with $\det(\mathbb{I}_n + ds(x)) > 0$ for all x. We consider

$$\begin{aligned} &(\operatorname{Id} + g)^{-1}(x) = x + f(x) \quad \text{for } f \in \mathcal{B}(\mathbb{R}^n)^n \iff f(x+g(x)) = -g(x) \\ &\big((\operatorname{Id}+g)^{-1} \circ (\operatorname{Id}+s) \circ (\operatorname{Id}+g)\big)(x) = \big((\operatorname{Id}+f) \circ (\operatorname{Id}+s) \circ (\operatorname{Id}+g)\big)(x) = \\ &\qquad = x + g(x) + s(x+g(x)) + f\big(x + g(x) + s(x+g(x))\big) \\ &\qquad = x + s(x+g(x)) - f(x+g(x)) + f\big(x+g(x)+s(x+g(x))\big). \end{aligned}$$

Since $g(x)$ is globally bounded, we get $s(x+g(x)) = O((1+|x+g(x)|)^{-k}) = O((1+|x|)^{-k})$ for each k. For $d_x^p(s \circ (\operatorname{Id}+g))(x)$ this follows from the Faá die Bruno formula in the form of (6). Moreover we have

$$\begin{aligned} &f\big(x+g(x)+s(x+g(x))\big) - f(x+g(x)) = \\ &\qquad\qquad = \int_0^1 df\big(x+g(x)+ts(x+g(x))\big)(s(x+g(x)))\,dt \end{aligned}$$

which is in $\mathcal{S}(\mathbb{R}^n)^n$ as a function of x since df is in $\mathcal{B}$ and $s(x+g(x))$ is in $\mathcal{S}$.

Finally we prove that $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$ is a normal subgroup of $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$. We redo the last proof under the assumption that $s \in H^\infty(\mathbb{R}^n)^n$. By the argument in (3) we see that $s(x+g(x))$ is in H^∞ as a function of x. The rest is as above, since H^∞ is an ideal in $\mathcal{B}$ as noted in 3.7. □

3.9. **Corollary.** $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$ *acts on* Γ_c, $\Gamma_{\mathcal{S}}$ *and* Γ_{H^∞} *of any tensorbundle over* $\mathbb{R}^n$ *by pullback. The infinitesimal action of the Lie algebra* $\mathfrak{X}_{\mathcal{B}}(\mathbb{R}^n)$ *on these spaces by the Lie derivative thus maps each of these spaces into itself. A fortiori,* $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$ *acts on* $\Gamma_{\mathcal{S}}$ *of any tensor bundle by pullback.*

Proof. Since $\mathrm{Diff}_c(\mathbb{R}^n)$, $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$, and $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$ are normal subgroups in $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$, their Lie algebras $\mathfrak{X}_{\mathcal{A}}(\mathbb{R}^n) = \Gamma_{\mathcal{A}}(T\mathbb{R}^n)$ are all invariant under the adjoint action of $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$. This extends to all tensor bundles. The Lie derivatives are just the infinitesimal versions of the adjoint actions. □

References

[1] M. Bauer, M. Bruveris, P. Harms, and P. W. Michor. Vanishing geodesic distance for the Riemannian metric with geodesic equation the KdV-equation. *Ann. Global Anal. Geom.*, 41(4):461–472, 2012.

[2] M. Bauer, M. Bruveris, P. Harms, and P. W. Michor. Geodesic distance for right invariant Sobolev metrics of fractional order on the diffeomorphism group. *To appear in Ann. Global Anal. Geom.*, 2013. doi:10.1007/s10455-012-9353-x

[3] M. Bauer, M. Bruveris, and P. W. Michor. Geodesic distance for right invariant Sobolev metrics of fractional order on the diffeomorphism group II. *To appear in Ann. Glob. Anal. Geom.*, 2013. doi:10.1007/s10455-013-9370-4.

[4] M. Bauer, M. Bruveris, and P. W. Michor. The homogeneous Sobolev metric of order one on diffeomorphism groups on the real line. 2012.

[5] C. F. Faà di Bruno. Note sur une nouvelle formule du calcul différentielle. *Quart. J. Math.*, 1:359–360, 1855.

[6] A. Frölicher and A. Kriegl. *Linear spaces and differentiation theory*. Pure and Applied Mathematics (New York). John Wiley & Sons Ltd., Chichester, 1988. A Wiley-Interscience Publication.

[7] G. A. Goldin. Lectures on diffeomorphism groups in quantum physics. In *Contemporary problems in mathematical physics*, pages 3–93. World Sci. Publ., Hackensack, NJ, 2004.

[8] A. Kriegl and P. W. Michor. *The convenient setting of global analysis*, volume 53 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.

[9] A. Kriegl and P. W. Michor. Regular infinite-dimensional Lie groups. *J. Lie Theory*, 7(1):61–99, 1997.

[10] M. Micheli, P. W. Michor, and D. Mumford. Sobolev metrics on diffeomorphism groups and the derived geometry of spaces of submanifolds. *Izvestiya: Mathematics, to appear*, 2012.

[11] P. W. Michor. A convenient setting for differential geometry and global analysis. *Cahiers Topologie Géom. Différentielle*, 25(1):63–109, 1984.

[12] P. W. Michor. A convenient setting for differential geometry and global analysis. II. *Cahiers Topologie Géom. Différentielle*, 25(2):113–178, 1984.

[13] P. W. Michor. Some geometric evolution equations arising as geodesic equations on groups of diffeomorphisms including the Hamiltonian approach. In *Phase space analysis of partial differential equations*, volume 69 of *Progr. Nonlinear Differential Equations Appl.*, pages 133–215. Birkhäuser Boston, Boston, MA, 2006.

[14] P. W. Michor. *Topics in differential geometry*, volume 93 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[15] P. W. Michor. *Manifolds of differentiable mappings*. Shiva Mathematics Series 3, Orpington, 1980.

[16] P. W. Michor. Manifolds of smooth maps II: The Lie group of diffeomorphisms of a non compact smooth manifold. *Cahiers Topol. Geo. Diff.*, 21:63–86, 1980. MR 82b:58022b.
[17] J. Milnor. Remarks on infinite-dimensional Lie groups. In *Relativity, groups and topology, II (Les Houches, 1983)*, pages 1007–1057. North-Holland, Amsterdam, 1984.
[18] D. Mumford and P. W. Michor. On Euler's equation and 'EPDiff'. *arXiv:1209.6576*, 2012.
[19] H. Omori, Y. Maeda, and A. Yoshioka. On regular Fréchet Lie groups IV. Definitions and fundamental theorems. *Tokyo J. Math.*, 5:365–398, 1982.
[20] H. Omori, Y. Maeda, and A. Yoshioka. On regular Fréchet Lie groups V. Several basic properties. *Tokyo J. Math.*, 6:39–64, 1983.
[21] L. Schwartz. *Théorie des distributions.* Hermann, Paris, 1966. Nouvelle édition.
[22] D. Vogt. Sequence space representations of spaces of test functions and distributions. In *Functional analysis, holomorphy, and approximation theory (Rio de Janeiro, 1979)*, volume 83 of *Lecture Notes in Pure and Appl. Math.*, pages 405–443. Dekker, New York, 1983.
[23] B. Walter. Weighted diffeomorphism groups of Banach spaces and weighted mapping groups. *Dissertationes Math. (Rozprawy Mat.)*, 484:128, 2012.

Peter W. Michor: Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090 Wien, Austria

E-mail address: Peter.Michor@univie.ac.at

David Mumford: Division of Applied Mathematics, Brown University, Box F, Providence, RI 02912, USA

E-mail address: David_Mumford@brown.edu