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BOUNDS ON THE MULTIPLICITY OF EIGENVALUES FOR FIXED MEMBRANES

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Abstract. For a membrane in the plane the multiplicity of the k-th eigenvalue is known to be not greater than $2k-1$. Here we prove that it is actually not greater than $2k-3$, for $k \geq 3$.

1. Introduction and Statement of the Result

Let $D \subset \mathbb{R}^2$ be a bounded domain with smooth boundary ∂D. We consider the corresponding Dirichlet eigenvalue problem

$$\begin{cases} -\Delta u = \lambda_k u, \quad k = 1, 2, \dots, \quad \lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \dots \\ u|\partial D = 0. \end{cases} \tag{1.1}$$

For this problem we investigate the multiplicity of the eigenvalues λ_j, where λ_j is said to have *multiplicity*

$$m(\lambda_k) = l \quad \text{if} \quad \lambda_{k-1} < \lambda_k = \lambda_{k+1} = \dots = \lambda_j = \dots = \lambda_{k+l-1} < \lambda_{k+l}.$$

It is the dimension of the eigenspace $U(\lambda_k) = U(\lambda_{k+1}) = \dots = U(\lambda_{k+l-1})$ of the eigenvalue $\lambda_k = \dots = \lambda_{k+l-1}$.

Our goal is to find universal upper bounds for $m(\lambda_k)$.

From basic spectral theory it is known that λ_1 is simple. Cheng showed in a celebrated paper [4] that

$$m(\lambda_2) \leq 3$$

for membranes and surfaces of genus 0. This is sharp for membranes, see [9], where an example with $m(\lambda_2) = 3$ is given; note that then also $m(\lambda_3) = 3$. There is very interesting work about $m(\lambda_2)$ for surfaces with genus > 0, [2], [5], and [6]. It is

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known that in higher dimensions no universal bound to multiplicities can exist, [6]. About 10 years ago one of us [10] showed that

$$m(\lambda_k) \leq 2k-1$$

not only for the membrane case but also for Laplacians on surfaces with genus 0.

In a recent paper [8] it was shown for eigenvalues of Laplace Beltrami operators on smooth compact surfaces without boundary with genus 0 that

$$m(\lambda_k) \leq 2k-3 \quad \text{for} \quad k \geq 3.$$

Here we prove the same result for the membrane case.

Theorem A. *Let $k \geq 3$. Then the multiplicity of the k-th eigenvalue λ_k for the Dirichlet problem on D satisfies*

$$m(\lambda_k) \leq 2k-3.$$

This will follow from the sharper Theorem B below.

1.2. Remarks. The proof in [8] and the present one are quite different. Now we have a boundary and this requires a different approach though both proofs are based on a combination of Courant's nodal theorem, a suitable version of Euler's polyhedral theorem, and a detailed investigation of the zero sets of solutions u of (1.1). Here we have to investigate the zero sets near the boundary.

The Laplacian in (1.1) can be replaced by a strictly elliptic operator of second order in divergence form with smooth coefficients and one can also allow for a potential, so that theorem A holds also for the multiplicities of the eigenvalues of the following problem:

$$\begin{cases} \Big(-\sum_{i,j=1}^2 \frac{\partial}{\partial x_i} a_{i,j} \frac{\partial}{\partial x_j} + V(x)\Big)u = \lambda_k u & \text{in } D \\ u = 0 & \text{in } \partial D \end{cases}$$

We consider the principal symbol as the inverse of a Riemann metric on D and use it to express angles etc. in the proofs below.

Our result can be shown to carry over to the free membrane case, i.e. $-\Delta u = \lambda_k u$ in D with Neumann boundary conditions, but we do not go into details here.

Probably one can relax the smoothness conditions considerably. It would be interesting to allow for unbounded regions, in particular for Schrödinger equations in $\mathbb{R}^2$.

For the membrane case there is an extensive literature on the asymptotics of eigenvalues. It is interesting to investigate the asymptotics of the following quantity:

$$\mathcal{M}(k) = \max\{m(\lambda_k): \text{ all membranes } D\}$$

In section 2 we shall recall some well known properties of eigenfunctions of membrane problems, state theorem B (a generalization of theorem A), give a suitable version of Euler's theorem on polyhedra, and prove that $m(\lambda_k) \leq 2k-2$ for $k \geq 3$. In section 3 we complete the proof of theorem B.

2. Basics and the proof that $m(\lambda_k) \leq 2k-2$ for $k \geq 3$.

2.1. Nodal sets. Let D be a bounded domain in $\mathbb{R}^2$ with smooth boundary which decomposes into connected components as

$$\partial D = \bigcup_{i=1}^{N} (\partial D)^i.$$

We consider a solution u of (1.1), and define its *nodal set* by

$$\mathcal{N}(u) := \overline{\{x \in D : u(x) = 0\}}.$$

It is well known (and follows from 2.2, 2.4, 2.5, and 2.6 below) that:

(1) $\mathcal{N}(u)$ is a union of smoothly immersed circles in D and immersed arcs connecting points of ∂D. Each of these is called a *nodal line*. Note that self intersections are allowed. Maximal embedded pieces of nodal lines will be called *nodal arcs.*
(2) If $u(x) = 0$ but $du(x) \neq 0$ then x lies on exactly one nodal line and is no point of self intersection of this nodal line.
(3) If $u(x) = 0, du(x) = 0, \dots d^l u(x) = 0$, but $d^{l+1}u(x) \neq 0$ then exactly $l+1$ nodal lines go through x whose tangents at x dissect the full circle into $2l+2$ equal angles. In particular the intersections are transversal. See 2.4 below.
(4) If x is a zero of order l as in (3) and lies in ∂D, then one of the nodal lines lies in ∂D. Here we use the fact the locally near x the eigenfunction u may be extended to the outside of D and is there a solution of the extended Laplace operator.
(5) Each component of the boundary is hit by an even number of nodal lines (since u changes sign at the nodal lines).

2.2. The *nodal domains* of u are the connected components of $D \backslash \mathcal{N}(u)$. We denote the number of nodal domains by $\mu(u) = \mu(\mathcal{N}(u))$.

Courant's Nodal Theorem. [3] *For each function u in the eigenspace $U(\lambda_k)$ we have $\mu(u) \leq k$.*

Remarks. It has been observed by Pleijel [11] that for the membrane case we have $\limsup \frac{\mu(u_k)}{k} < 1$ for any choice of $u_k \in U(\lambda_k)$. Or, in other words, Courant's nodal theorem is sharp only for finitely many eigenvalues. In view of this the following result is a substantial improvement of theorem A.

Theorem B. *Let U be a linear subspace of an eigenspace $U(\lambda)$ and let $1 < l \in \mathbb{N}$ be such that for each $u \in U$ we have $\mu(u) \leq l$. Then* $\dim U \leq \max\{3, 2l-3\}$.

Note that now the inequality $m(\lambda_k) \leq 2k-3$ for $k \geq 3$ in theorem A follows immediately from theorem B using Courant's nodal theorem 2.2.

2.3. Remarks. It would be interesting to investigate for which l theorem B is sharp. For large l it is not at all clear whether $\dim U$ can be estimated by something better than $O(l)$, such as $O(l^{1/2})$. There is a severe lack of examples with high multiplicities.

2.4. Proposition. [1], [4] *For an eigenfunction u and $x_0 \in D$ there exists an integer $n \geq 0$ such that*

$$u(x) = P_n(x - x_0) + O(|x - x_0|^{n+1})$$

for a harmonic homogeneous polynomial $P_n \not\equiv 0$ of degree n.

Actually for the membrane case we even have

$$u(x) = P_n(x - x_0) + P_{n+1}(x - x_0) + O(|x - x_0|^{n+2})$$

for harmonic homogeneous polynomials $P_n \neq 0$ and P_{n+1} of degrees n and $n+1$, respectively, but we shall not need this sharper result.

Harmonic homogeneous polynomials P_n of degree n have a particularly simple representation in polar coordinates (r, θ):

$$P_n(r\cos\theta, r\sin\theta) = ar^n \cos(n\theta) + br^n \sin(n\theta).$$

Obviously the set of zeros of such a P_n consists of n straight lines which meet at equal angles.

2.5. Proposition. *For an eigenfunction u and $x_0 \in \partial D$ there exists a harmonic homogeneous polynomial $P_n \not\equiv 0$ of degree $n \geq 1$ with*

$$u(x) = P_n(x - x_0) + O(|x - x_0|^{n+1})$$

such that one of the nodal lines of P_n is tangent to ∂D at x_0.

Proof. This follows from the smoothness of ∂D, see e.g. [7]. □

2.6. Note that for $x_0 \in D$ the leading harmonic homogeneous polynomial P_n of an eigenfunction u lies in the 2-dimensional (if $n > 0$) vector space of all harmonic homogeneous polynomials of degree n, whereas for $x_0 \in \partial D$ it lies in the 1-dimensional subspace of those polynomials which vanish on the tangent $T_{x_0}(\partial D)$.

2.7. Definition. An *(abstract) nodal set* is a set $\mathcal{N}$ satisfying 2.1, (1)-(5), where we do not require that it is the nodal set of an eigenfunction.

An *isotopy* of nodal sets is a curve of nodal sets such that each immersed circle or arc moves along a smooth isotopy which respects nodal arcs. So intersection points can move but not change the multiplicity.

A *nodal pattern* is an isotopy class of nodal sets. We shall often draw a clearly recognizable representative of a nodal pattern, see below.

2.8. Proposition. *Let $\mathcal{N}$ be an abstract nodal set in a domain D. Then we have*

$$\mu(\mathcal{N}) = b_0(\mathcal{N} \cup \partial D) - b_0(\partial D) + \sum_{x \in \mathcal{N} \cap D} (\nu(x) - 1) + \sum_{y \in \partial D \cap \mathcal{N}} \frac{\rho(y)}{2} + 1, \quad \textit{where}$$

$$\begin{aligned}
\mu(\mathcal{N}) &= \textit{ number of (nodal) domains of } D \setminus \mathcal{N}, \\
b_0(\mathcal{N} \cup \partial D) &= \textit{ number of connected components of } \mathcal{N} \cup \partial D, \\
b_0(\partial D) &= \textit{ number of connected components of the boundary } \partial D, \\
\nu(x) &= \textit{ number of nodal lines containing } x \in D, \\
\rho(y) &= \textit{ number of nodal lines hitting } \partial D \textit{ in } y.
\end{aligned}$$

Moreover,

$$b_0(\mathcal{N} \cup \partial D) - b_0(\partial D) + \sum_{y \in \partial D \cap \mathcal{N}} \frac{\rho(y)}{2} \geq 1,$$

so that for $\mathcal{N} \neq \emptyset$ we get

$$\mu(\mathcal{N}) \geq \sum_{x \in \mathcal{N} \cap D} (\nu(x) - 1) + 2.$$

Proof. Suppose that ∂D has k components. Then for the Euler characteristic of $\overline{D}$ we have $\chi(\overline{D}) = 2 - b_0(\partial D)$, which can be seen from a simple cell decomposition of $\bar{D}$. See e.g. [12].

We consider $\mathcal{N} \subset \overline{D}$ and extend it to a cell decomposition of $\overline{D}$ with

- c_0 many 0-cells, namely the points $x \in \mathcal{N} \subset D$ through which $\nu(x) > 1$ nodal lines pass (i.e. $2\nu(x) > 2$ 1-cells emanate), the points $y \in \mathcal{N} \cap \partial D$ in which $\rho(y) > 0$ nodal lines hit the boundary, and one extra point z on a smooth part of each of the $b_{\mathcal{N}} = b_0(\mathcal{N} \cup \partial D)$ connected components of $\mathcal{N} \cup \partial D$.
- c_1 many 1-cells. First the $\frac{1}{2}(\sum_x 2\nu(x) + \sum_y \rho(y))$ nodal arcs of $\mathcal{N}$ connecting the intersection points and boundary hitting points of $\mathcal{N}$. Second, the $\sum_y 1$ smooth pieces of ∂D lying between the hitting points y. Moreover, $b_{\mathcal{N}}$ more 1-cells, namely for each z either a smooth arc coming from subdividing the smooth arc by choosing z, or a 1-cell corresponding to a component of ∂D which is hit by no nodal line. Finally, $b_{\mathcal{N}} - 1$ extra 1-cells connecting the $b_{\mathcal{N}}$ points z on the components of $\mathcal{N} \cup \partial D$ in a suitable way to each other or to some of the y's.
- c_2 many 2-cells. Note that none of the extra 1-cells dissects a nodal domain, so we have $c_2 = \mu(\mathcal{N})$.

Thus for the Euler characteristic we have

$$\begin{aligned} 2 - b_0(\partial D) &= \chi(\overline{D}) = c_0 - c_1 + c_2 \\ &= \sum_x 1 + \sum_y 1 + b_{\mathcal{N}} - \sum_x \nu(x) - \sum_y \frac{\rho(y)}{2} - \sum_y 1 - b_{\mathcal{N}} - (b_{\mathcal{N}} - 1) + \mu, \end{aligned}$$

and thus

$$\mu = \sum_x (\nu(x) - 1) + \sum_y \frac{\rho(y)}{2} + b_0(\mathcal{N} \cup \partial D) - b_0(\partial D) + 1.$$

The last assertion follows by treating each connected component $(\partial D)^i$ of the boundary separately, if $\mathcal{N} \neq \emptyset$:

$$\begin{aligned} b_0(\mathcal{N} \cup \partial D) - b_0(\partial D) + \sum_{y \in \partial D \cap \mathcal{N}} \frac{\rho(y)}{2} &= \\ &= b_0(\mathcal{N} \cup \partial D) + \sum_i \Bigg(\sum_{y \in (\partial D)^i \cap \mathcal{N}} \frac{\rho(y)}{2} - 1 \Bigg) \geq 1. \quad \square \end{aligned}$$

2.9. Lemma. *Let U be a linear subspace of dimension $m \geq 1$ of an eigenspace $U(\lambda)$.*

Then for each $x_0 \in D$ there exists an eigenfunction $0 \neq u \in U$ such that $d^l u(x_0) = 0$ for $0 \leq l < [m/2]$, where $[m/2]$ is the largest integer $\leq m/2$. If $d^{[m/2]}u(x_0) \neq 0$ we have

$$u(x) = P_{[m/2]}(x - x_0) + O(|x - x_0|^{[m/2]+1}),$$

On the boundary, for any choice of points $y_1, \dots, y_{m-1} \in \partial D$ there exists an eigenfunction $0 \neq u \in U$ such that at each y_i at least one nodal line of u hits ∂D. Some points y_i might coincide, in which case the corresponding number of nodal lines hit there.

Proof. This is linear algebra using 2.4 and 2.5. □

2.10. Let $U(\lambda)$ be an m-dimensional eigenspace for an eigenvalue λ. Consider the unit sphere $S^{m-1} \subset U(\lambda)$ with respect to the L^2-inner product, say. For each $u \in S^{m-1}$ we may consider its nodal set $\mathcal{N}(u)$. We get a disjoint decomposition of S^{m-1} (actually of $P^{m-1}(\mathbb{R})$) according to the nodal patters. This should actually be a stratification into smooth manifolds.

2.11. Lemma. *Let $\varphi : \mathbb{R} \to S^{m-1} \subset U(\lambda)$ be smooth.*

Then for each multiindex α we have

$$\sup_{y \in \overline{D}} |(\partial_x)^\alpha (\varphi_t - \varphi_s)| \leq C_\alpha |t - s|$$

for some constant C_α.

Proof. This follows from the assumptions that all data are smooth. □

2.12. Already from lemma 2.9 we can prove the main result from [10] that $m(\lambda_k) \leq 2k - 1$ as follows: Suppose that $m(\lambda_k) \geq 2k$ and pick $x_0 \in D$, then there is an eigenfunction $u \in U(\lambda_k)$ with $\nu_u(x_0) = k$, by lemma 2.9. Hence by lemma 2.8 we get $\mu(u) \geq 2 - 1 + k - 1 + 1 = k + 1$, a contradiction to Courant's nodal theorem 2.2.

Actually we even proved: *If U is a linear subspace of an eigenspace $U(\lambda)$ and if $\sup\{\mu(u) : u \in U\} = l > 1$, then $\dim(U) \leq 2l - 1$.*

But we can do even better:

2.13. Lemma. *Let U be a linear subspace of an eigenspace $U(\lambda)$ and suppose that $\sup\{\mu(u) : u \in U \setminus 0\} = l > 1$.*

Then we have $\dim(U) \leq 2l - 2$.

Proof. Assume for contradiction that there is some $U \subseteq U(\lambda)$ with $\dim(U) = 2l - 1$. We first assume that D is simply connected. We pick $y, z \in \partial D$. By lemma 2.9 there exists an eigenfunction $u = u_{y,z} \in U$ such that $\rho_u(y) = 2l - 3$ and $\rho_u(z) = 1$. By lemma 2.8 and by 2.2 we get $\mu(u) = l$, $\nu(x) - 1 = 0$ for all $x \in \mathcal{N} \cap D$, and $b_0(\mathcal{N} \cup \partial D) = 1$.

Consider now $\mathcal{N}(u)$: there are $2l - 3$ nodal lines emanating from y and one from z, and there is no intersection point in D. Let $\tilde{\mathcal{N}} = \mathcal{N}(u) \setminus \partial D$, which consists of one smooth arc with endpoints y and z, and of $l-2$ non-intersecting loops starting at y. For $l = 4$ e.g., we have one of the following 5 nodal patterns:

We also note that for given $y, z \in \partial D$ the eigenfunction $u_{y,z}$ is unique up to multiplication by a constant. Indeed, if there are two linearly independent eigenfunctions $u^1_{y,z}$ and $u^2_{y,z}$, then by 2.5 there is an eigenfunction $v \in \operatorname{span}(u^1, u^2)$ with $\rho_v(z) \geq 2$. Via 2.8 (see 2.1.(5)) we get as above for $l > 1$ a contradiction to $\sup\{\mu(u) : u \in U\} = l$.

Now we move z towards y, once clockwise and once anticlockwise. Since we work at the maximal number of nodal domains, $\mu(u_{y,z}) = l$, no additional intersection points in D nor additional hitting points in ∂D may appear during these moves. Hence the arc from y to z will eventually become a loop as $z \to y$. But the limit nodal patterns differ, which is obvious from the figures above. For example

will eventually tend to:

(clockwise) (anticlockwise)

Hence there are two linearly independent functions with $2l-2$ nodal lines hitting at y. So again there is a function v in their span with $2l-1$ nodal lines hitting at y, by 2.5 a contradiction to $\mu(v) \leq l$.

The case of non simply connected domains is similar. We pick y and z as above on the outer component of the boundary and we proceed as above. Any other component of the boundary can be hit at most twice by nodal lines, according to 2.8, since we work at the maximal number l of nodal domains. □

3. Proof of theorem B

3.1. Let U be a linear subspace of an eigenspace $U(\lambda)$ and let $\sup\{\mu(u) : u \in U\} = l \geq 3$. We already know from 2.13 that then $\dim(U) \leq 2l-2$, so let us assume for contradiction that $\dim(U) = 2l-2$, throughout this section. In all our constructions below we will use only eigenfunctions u for which the number of nodal domains has to be maximal, i.e. $\mu(u) = l$.

Before going into details we sketch the main ideas of the proof. We shall show that for each $x \in D$ there is a unique (up to multiplication by a constant) eigenfunction u_x which vanishes of order $l-1$ at x. On the boundary, for each $y \in \partial D$ there exists also a unique function u_y (up to a multiplicative constant) which vanishes of order $\geq 2l-2$. We will show that these combine to a continuous mapping from $\overline{D}$ to the projective space $P(U)$. Then we shall use a winding number argument to get a contradiction.

We start by giving a list of possible nodal patterns which hit ∂D in two points x, y with $\rho(y) = 2l-4$ or $2l-3$ and $\rho(x) = 2$ or $\rho(x) = 1$: We give all possible configurations at x, but just a sample of those possible at y, and we assume that D is simply connected. All of these configurations look similar. We split each nodal pattern into two parts, namely into 'the loops hitting the boundary only at y', and the rest, which can be either one nodal arc from y to x, or a loop hitting only at x, called a *drop*, or two nodal arcs from y to x, called a *banana*.

All pictures and most arguments below will be given in the case that D is simply connected. But since we always work with eigenfunctions which have the maximal number of nodal domains allowed by 2.8, everything remains valid in the non simply connected case: Then we have further boundary components each of which can be hit at most twice by nodal lines: otherwise we get too many nodal domains. Furthermore all boundary components are equivalent for our arguments (put D into S^2), and we shall treat each of them separately.

3.2. By 2.9, for each $y \in \partial D$ there is a function $u_y \in U$ such that at least $2l-3$ nodal lines hit ∂D at y. The nodal pattern of u_y consists thus of loops at y and one or no nodal line from y to another point $x \neq y$, in the simply connected case. In the general case it is similar with the changes described at the end of 3.1.

Lemma. *There are no two linearly independent functions $u, v \in U$ such that $\rho_u(y) = 2l-3$ and $\rho_v(y) = 2l-2$. Moreover, the set of points $y \in \partial D$ where* *there exists a u such that $\mathcal{N}(u)$ has just $l-1$ loops at y, i.e. $\rho_y(u) = 2l-2$, is discrete.*

Proof. The nodal pattern $\mathcal{N}(u)$ consists of $l-2$ loops at y and one nodal line from y to some point $x \neq y$, whereas $\mathcal{N}(v)$ consists only of $l-1$ loops at y. By a linear combination of u and v we may move x anticlockwise or clockwise to y and produce a function $w \in U$ such that $\mathcal{N}(w)$ consists of loops at y which is different from $\mathcal{N}(v)$ (see 2.13 for a similar argument). But by 2.5 the leading terms of w and v at y are multiples of the same harmonic polynomial, so a suitable linear combination of w and v has a zero of order at least $2l$ at y which contradicts our assumption on U.

For the proof of the second assertion, suppose that there is an open arc I in ∂D such that for each $y \in I$ there exists $u_y \in U$ with $\rho_{u_y}(y) = 2l-2$. Then by the argument above for the first assertion, u_y is uniquely determined by y and the mapping $I \ni y \mapsto u_y$ into the projective space of U is smooth, since u_y is given (up to a multiplicative constant) by solving a system of linear equations of maximal rank. Let $y(t) = (y_1(t), y_2(t))$ be a unit speed parametrisation of I. Then near $y(t)$ the eigenfunction $u_{y(t)}$ with $2l-2$ nodal lines hitting $y(t)$ can be written as

$$u_{y(t)}(x) = f(t)\Big(c_1(t)P^1_{2l-1}(x-y(t)) + c_2(t)P^2_{2l-1}(x-y(t))\Big) + O(|x-y(t)|^{2l})$$

where $P^1_{2l-1} = r^{2l-1}\cos((2l-1)\theta)$ and $P^2_{2l-1} = r^{2l-1}\sin((2l-1)\theta)$ span the 2-dimensional space of harmonic polynomials of degree $2l-1$, where $f(t)$ is a normalizing function, and where $(c_1(t), c_2(t)) \in S^1$ is chosen in such a way that the leading term vanishes along the tangent $T_{y(t)}(\partial D)$ spanned by $\dot{y}(t)$. We have $\partial_t u_{y(t)} \in U$, and we compute this for a point t_0 where we may assume without loss that $\dot{y}(t_0) = (1,0)$, so that $c_1(t_0) = 0$ and $c_2(t_0) = 1$. Then

$$(\partial_t u_{y(t)})|_{t=t_0}(x) = f(t_0)\frac{\partial P^2_{2l-1}}{\partial x_1}(x-y(t_0)) + O(|x-y(t_0)|^{2l-1}),$$

where the leading term of order $2l-2$ does not vanish. So in $\mathcal{N}(\partial_t(u_{y(t)}|_{t=t_0}))$ we have $2l-3$ nodal lines hitting $y(t_0)$, in contradiction to the first assertion of the lemma. □

3.3. Lemma. *For each $y \in \partial D$ there exists a unique (up to a nonzero constant) function $u_y \in U$ such that $\rho_{u_y}(y) \geq 2l-3$. Moreover, $y \mapsto u_y$ is a smooth map from ∂D into the projective space $P(U)$ of U. Put*

$$\begin{aligned}(\partial D)_{2l-3} &:= \{y \in \partial D : \rho_{u_y}(y) = 2l-3\},\\ (\partial D)_{2l-2} &:= \{y \in \partial D : \rho_{u_y}(y) = 2l-2\},\end{aligned}$$

then we have the disjoint union $\partial D = (\partial D)_{2l-3} \cup (\partial D)_{2l-2}$, where $(\partial D)_{2l-2}$ is discrete. Thus $(\partial D)_{2l-3}$ is a union of open arcs and the nodal pattern of u_y is constant for y in one of these arcs.

Below is a list of such nodal patterns. Note that if y moves to one of the endpoints y_i of an interval of $(\partial D)_{2l-3}$ then the last hitting point $z(y)$ of u_y has to move towards y_i too.

Proof. If there are two linearly independent functions with $2l-3$ nodal lines hitting at y, a suitable linear combination has leading term of order one higher, so $2l-2$ nodal lines hitting at y, thus $y \in (\partial D)_{2l-3} \cap (\partial D)_{2l-2}$ which is empty by 3.2. If there are two linearly independent functions with $\rho(y) = 2l-2$, a suitable linear combination has $\rho(y) = 2l-1$ and thus too many nodal domains.

The map $y \mapsto u_y$ is smooth by uniqueness and 2.9, since we solve there a linear system which has maximal rank by uniqueness.

The rest is clear from 3.2. $\square$

3.4. Lemma. *For each $y \in (\partial D)_{2l-3}$ and $s \in \partial D$ there exist a function $v_{y,s} \in U$, unique up to a constant, with $\rho_{v_{y,s}}(y) \geq 2l-4$ and $\rho_{v_{y,s}}(s) \geq 1$. Moreover, $v_{y,z(y)} = u_y$, and $(y,s) \mapsto v_{y,s}$ is a smooth mapping from $(\partial D)_{2l-3} \times \partial D$ to $P(U)$.*

Proof. Existence follows from 2.9. As explained in the proof of 3.3, smoothness follows from uniqueness, which we prove now.

Suppose that $s \neq z(y)$ and that there are two linearly independent functions of this kind. Then a suitable linear combination has $\rho(y) = 2l-3$ and $\rho(s) = 1$ which contradicts the uniqueness of u_y.

If $s = z(y)$ and there exists a second function $v_{y,z(y)}$ which is linearly independent of u_y, then the nodal pattern of $v_{y,z(y)}$ is typically of the form

since the third possibility (a nodal line from $z(y)$ to y and all others loops at y) contradicts the uniqueness of u_y. In case (a) above, we arrange the signs of the functions $v_{y,z(y)}$ and u_y as follows

and look at the nodal pattern of $w_t := tu_y + (1-t)v_{y,z(y)}$. This can be viewed as follows: put the two drawings above each other and start at $t = 0$, at $\mathcal{N}(v_{y,z(y)})$. With growing t, domains where both functions are negative or where both functions are positive grow, whereas domains with mixed signs shrink. Thus the hitting point r moves towards $z(y)$ and we get eventually, at some $0 < t_1 < 1$, a nodal domain with a drop at $z(y)$. Further increasing t this drop has to open but one nodal line has to stay at $z(y)$. If it opens to the right we can never get the nodal pattern of u_y. If it opens to left the nodal line would eventually get to the point r again, but then we would have two linearly independent functions w_0 and w_{t_2} with $0 < t_1 < t_2 < 1$ with $2l-4$ nodal lines hitting at y and one each at r and $z(y)$. By a suitable linear combination we can then produce a function with $\rho(y) \geq 2l-3$, and nodal lines hitting at r and $z(y)$, contradicting the uniqueness of u_y.

Case (b) above is similar. These are the most complicated cases. Similar but more obvious methods apply if r and $z(y)$ change position. If the nodal pattern of u_y is different, such that the nodal line from y to $z(y)$ has loops to the left and to the right, then the argument is even easier. □

3.5. Lemma. *Let the open arc I be a connected component of $(\partial D)_{2l-3}$ with endpoints y_1 and y_2 as in the drawing below. Let $y \in I$ and let $z(y)$ be the ultimate hitting point of u_y. Then $z(y) \notin I$.*

Proof. Let us assume for contradiction that $z \in I$.

We consider the function $v_{y,s}$ from lemma 3.4. Then $v_{y,z(y)} = u_y$, and we move s inside I towards y (down right in the drawing). Then one of the nodal lines hitting at y must move away from y (otherwise we get a contradiction to the uniqueness of u_y): If this is the leftmost we must get eventually because s cannot move to y for this would lead to a nodal type different from that of u_y. Call the corresponding function v. The functions v and u_y cannot coexist since in their span there is a function with nodal pattern

which contradicts the nodal type valid on I.

Hence the rightmost nodal arc must move away from y. If it moves down to y we have already a contradiction. Thus it must eventually hit the downcoming s at s_1 so that we have the the following nodal domain.

Then we move y towards s_1 and consider $\mathcal{N}(v_{y,s_1})$. One of the nodal lines hitting at s_1 must move away before y hits s_1 since there is no point of $(\partial D)_{2l-2}$ in between, and it must move eventually towards y so that at some $x_1 \in I$ between y and s_1 we get $v_{x_1,s_1} = u_{x_1}$, since the nodal type of u_y is constant in I. We have then the same situation as at the beginning, and we start to move again s from s_1 to x_1 and consider $\mathcal{N}(v_{x_1,s})$, and so on. We get a sequence of points x_i and $s_i = z(x_i)$ in I which move together. Even if they accumulate, at any accumulation point we must have the same nodal type, and we can continue the procedure. So we assume that finally $x_i \to x$ in I and also $s_i \to x$. But since $s_i = z(x_i)$ we finally get $x = z(x)$ so that $x \in (\partial D)_{2l-2}$, a contradiction. $\square$

3.6. Lemma. *Let the open arc I be a connected component of $(\partial D)_{2l-3}$ with endpoints y_1 and y_2 as in the drawing below. Let $y \in I$ and let $z(y)$ be the last hitting point of u_y.*

Then the following holds: If y moves clockwise to y_1 then $z(y)$ moves anticlockwise to y_1. If y moves anticlockwise to y_2 then $z(y)$ moves clockwise to y_2. In particular the nodal patterns $\mathcal{N}(u_{y_1})$ and $\mathcal{N}(u_{y_2})$ are different. Moreover $(\partial D)_{2l-3}$ consists of only finitely many open arcs, and $(\partial D)_{2l-2}$ is a finite subset of ∂D.

Here is a sample of the nodal patterns of u_x for $x = y_1$, $x \in I$, and $x = y_2$.

Proof. Since u_y depends smoothly on y, $z(y)$ also depends smoothly on y. If y moves to y_1, $z(y)$ has to go to y_1 too since $\mathcal{N}(u_{y_1})$ consists of loops at y_1 only. But it cannot come through I, by 3.5, so it must come from the outside.

Looking at the possibilities for $\mathcal{N}(u_{y_i})$ one sees that there must be different nodal patterns at both ends of the arc I.

If $(\partial D)_{2l-2}$ were not finite, its points would accumulate at y_0, say. But then for y_0 there have to be two functions u_{y_0} with different nodal patterns, a contradiction. $\square$

3.7. Lemma. *For each point $x \in D$ there exists a function $u_x \in U$, unique up to a multiplicative constant, such that $\nu_{u_x}(x) = l - 1$. Moreover, $x \mapsto u_x$ induces a smooth mapping $D \to P(U)$ into the projective space of U.*

Proof. Existence of u_x follows from lemma 2.9 and $\dim U = 2l-2$. From 2.8 we see that $l \geq \mu(u_x) \geq \sum_{z\in D}(\nu_{u_x}(z) - 1) + 2 \geq l$ so that x is the only intersection point of $\mathcal{N}(u_x)$ in D. *At most two nodal lines can connect x to the (outer) boundary.*

If there are two linearly independent functions with the properties of u_x, we may choose functions u_0 and u_1 in their span such that in local Riemannian polar coordinates (r, θ) centered at x_0 we have

$$u_0 = r^{l-1}\cos((l-1)\theta) + O(r^l), \quad u_1 = r^{l-1}\sin((l-1)\theta) + O(r^l).$$

Let $v_\alpha := \cos(\alpha).u_1 + \sin(\alpha).u_2$, then $v_0 = u_0$, and the regular $(l-1)$-gon consisting of the tangents to the $l-1$ nodal lines through x_0 in $T_{x_0}M$ rotates with α and is the same at the angle $\alpha = \pi/(l-1)$. Thus $v_{\pi/(l-1)}$ has the same leading term at x_0 as $-u_0$, thus $v_{\pi/(l-1)} = -u_0$ since otherwise $v_{\pi/(l-1)} - u_1$ would have $\nu(x_0) \geq l$ and thus more than l nodal domains. Since no intersection points outside x_0 are possible for functions in U, the nodal set $\mathcal{N}(v_\alpha)$ moves smoothly to itself by this rotation. If the first nodal ray (counting from the angle 0) not leading to the outer boundary is connected by a smooth loop to the one with number i, then it follows that the second one is connected to the one with number $(i+1)$. But this is not possible without further intersection point, a contradiction. Thus u_x is unique up to a multiplicative constant.

Finally we get a smooth mapping $D \to P(U)$, since u_x is the solution of a linear system which has maximal rank by uniqueness. $\square$

3.8. Lemma. *The mapping $x \mapsto u_x$ is continuous from $\overline{D}$ into the real projective space $P(U)$.*

Proof. Inside D the function u_x, suitably normalized, depends smoothly on $x \in D$, by 3.7. On the boundary ∂D the function u_y depends smoothly on $y \in \partial D$ by 3.3.

So it remains to show that $u_{x_n} \to u_y$ in the projective space $P(U)$ if the sequence x_n in D converges to $y \in \partial D$. Since $P(U)$ is compact it suffices to show that each accumutation point of the sequence u_{x_n} in $P(U)$ coincides with u_y.

Thus let $v \in P(U)$ be a cluster point, then there is a subsequence of $u_{x_{n_k}}$ which converges to v.

Let C be a closed disk of small radius $\varepsilon > 0$ with center y intersected with $\overline{D}$. Choose n_k such that x_{n_k} is still in the interior of C. Then of the $2l-2$ nodal rays of $u_{x_{n_k}}$ leaving x_{n_k} all but one have to leave C, since otherwise there would exist a nodal domain which is completely contained in C. Since ε is small, this is not possible by energy reasons.

But since $u_{x_{n_k}}$ converges to v in $P(U)$, also at least $2l-3$ nodal lines of v lead into C. Since ε was arbitrary, $2l-3$ nodal lines of v hit ∂D at y. But the eigenfunction with this property is unique in $P(U)$ and is called u_y, by 3.3. Thus $v = u_y$ in $P(U)$. $\square$

3.9 Proof of theorem B. Suppose that D is a simply connected domain and that ∂D is its boundary. In 3.8 we proved that the mapping $x \mapsto u_x$ is continuous $\overline{D} \to P(U)$. Let $c : [0, 2\pi] \to D$ be a closed smooth curve following ∂D anti-clockwise close enough so that all arguments below work. We want to analyze how the star of tangents at $c(t)$ to the $l-1$ nodal lines of $u_{c(t)}$ crossing at $c(t)$ turns if we follow t from 0 to 2π.

To make this precise, we consider the continuous function $f : D \to S^1$, given by $f(x) = (2l-2)\alpha(x)$ modulo 2π, where

$$t \mapsto te^{2\pi ik/(2l-2)+i\alpha(x)}, \quad k = 0, \dots, 2l-3, \quad t \geq 0$$

are the tangents rays of the nodal lines through x in $\mathcal{N}(u_x)$. We want to analyze $f(c(t))$.

We consider again the sets the disjoint partition of ∂D into the sets

$$(\partial D)_{2l-3} : = \{y \in \partial D : \rho_{u_y}(y) = 2l-3\},$$
$$(\partial D)_{2l-2} : = \{y \in \partial D : \rho_{u_y}(y) = 2l-2\}.$$

We note first the following fact:

(1) If $x \in D$ is near enough to some point y in the open set $(\partial D)_{2l-3}$ then the nodal pattern $\mathcal{N}(u_x)$ can be read off the the nodal pattern $\mathcal{N}(u_y)$, since

the nodal domains move continuously. The nodal line leaving ∂D vertically stays connected to ∂D near y. For example:

This is seen as follows: $\mathcal{N}(u_x)$ for $x \in D$ can have at most two nodal lines connecting x to ∂D, since otherwise there would be too many nodal domains by 2.8. Moreover nodal lines can move off ∂D only in pairs. Thus, if x moves from $y \in (\partial D)_{2l-3}$ into D, the nodal lines of $\mathcal{N}(u_x)$ move away from ∂D in pairs and one of them stays connected to ∂D, since there was an odd number of them at y. Let us call this nodal arc from x to ∂D the *short arc* of $\mathcal{N}(u_x)$: it exists if x is near $(\partial D)_{2l-3}$. Since loops have to stay loops, the result follows.

This already implies that $f(c(t))$ follows the angle of ∂D along each arc in $(\partial D)_{2l-3}$.

What happens at a point in $(\partial D)_{2l-2}$? Without loss we assume that this point is $0 \in (\partial D)_{2l-2}$, that ∂D has horizontal tangent at 0, and that D lies above. Then there are two connected components of $(\partial D)_{2l-3}$ to the left and to the right of 0: the open arcs I_1 and I_2. We claim that:

(2) When passing over 0 from above I_1 to above I_2, the angle $\alpha(c(t))$ increases by an amount of $2\pi/(2l-2)$.

To see this, from 3.6 we conclude that for y to the left, in I_1, the last hitting point $z(y)$ of the nodal pattern $\mathcal{N}(u_y)$ has to move through I_2 towards 0 if y moves right to 0; if y then continues to move right, away from 0, then the leftmost nodal line emanating from 0 has to move off to the left and to slide away through I_1.

If $c(t)$ moves from the left over 0 the nodal pattern $\mathcal{N}(u_{c(t)})$ follows closely the behavious above. Well to the left, two nodal arcs connect $c(t)$ to ∂D, the short one directly to I_1 below, and another one, call it the long one, far away, but moving through I_2 towards the short one.

Eventually, near 0, they have to meet, to lift off ∂D, and the next loop in clockwise direction of the short arc has to touch ∂D, so that the short arc is replace by its next neighbor in clockwise direction and the new long arc then has to move away through I_1. Namely, this is the only continuous behaviour which connects the behaviour to the left to the one at the right of 0, where we know exactly what happens. An illustration of the behaviour is the following:

Thus statement (2) is proved.

So finally the smooth mapping $t \mapsto f(c(t))$, $S^1 \to S^1$ has mapping degree $2\pi\#(\partial D)_{2l-2} + 2\pi(2l-2) > 0$ and cannot be null homotopic. But by construction it is continuously extended into the interior of the circle and thus is nullhomotopic, a contradiction. This finishes the proof for the simply connected case.

If D is not simply connected, let $(\partial D)^i$ for $i = 1, \dots, p$ be the connected components of ∂D. Choose a point x_1 near $(\partial D)^1$. Then we choose a smooth curve $c : S^1 \to D$ which starts at x^1 and follows $(\partial D)^1$ closely back to x_1, then from x_1 along a smooth path e_2 to a point x_2 near $(\partial D)^2$, then follows $(\partial D)^2$ closely back to x_2, then back along e_2 to x_1. Then it follows a path e_3 not intersecting e_2 to some point x_3 and Ḍ^3, and so on until we end again at x_1.

Note that all results above also work for non simply connected domains, since we always worked with eigenfunctions which have the maximal number of nodal domains allowed by 2.8: each of the further (inner) boundary components can be hit by at most one nodal line twice, otherwise we get too many nodal domains.

Furthermore, all boundary components are equivalent for our arguments (put D into S^2), and we treat each of them separately.

We consider again $f(c(t))$. At each boundary component the contribution to the mappingg degree of f is a positive integer, by the arguments given above. The contributions from the parts going along the e_i cancel each other. So the mapping $f \circ c : S^1 \to S^1$ has positive mapping degree, and thus cannot be nullhomotopic. But the curve $c(t)$ bounds a simply connected region, thus the mapping $f \circ c : S^1 \to S^1$

has a continuous extension f to the 2 cell in the interior. So it is nullhomotopic, a contradiction. □

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