

THE FRÖLICHER-NIJENHUIS BRACKET

Basic information. Let M be a smooth manifold and let $\Omega^k(M; TM) = \Gamma(\wedge^k T^*M \otimes TM)$. We call $\Omega(M, TM) = \bigoplus_{k=0}^{\dim M} \Omega^k(M, TM)$ the space of all *vector valued differential forms*. The *Frölicher-Nijenhuis bracket* $[\ , \] : \Omega^k(M; TM) \times \Omega^l(M; TM) \rightarrow \Omega^{k+l}(M; TM)$ is a \mathbb{Z} -graded Lie bracket:

$$\begin{aligned} [K, L] &= -(-1)^{kl}[L, K], \\ [K_1, [K_2, K_3]] &= [[K_1, K_2], K_3] + (-1)^{k_1 k_2} [K_2, [K_1, K_3]]. \end{aligned}$$

It extends the *Lie bracket of smooth vector fields*, since $\Omega^0(M; TM) = \Gamma(TM) = \mathfrak{X}(M)$. The identity on TM generates the 1-dimensional center. It is called the Frölicher-Nijenhuis bracket since it appeared with its full properties for the first time in [1], after some indication in [8]. One formula for it is

$$\begin{aligned} [\varphi \otimes X, \psi \otimes Y] &= \varphi \wedge \psi \otimes [X, Y] + \varphi \wedge \mathcal{L}_X \psi \otimes Y - \mathcal{L}_Y \varphi \wedge \psi \otimes X \\ &\quad + (-1)^k (d\varphi \wedge i_X \psi \otimes Y + i_Y \varphi \wedge d\psi \otimes X), \end{aligned}$$

where X and Y are vector fields, φ is a k -form, and ψ is an l -form. It is a bilinear differential operator of bidegree $(1, 1)$.

The Frölicher-Nijenhuis bracket is natural in the same way as the Lie bracket for vector fields: if $f : M \rightarrow N$ is smooth and $K_i \in \Omega^{k_i}(M; TM)$ are f -related to $L_i \in \Omega^l(N; TN)$ then also $[K_1, K_2]$ is f -related to $[L_1, L_2]$.

More details. A convenient source is [3], section 8. The basic formulas of calculus of differential forms extend naturally to include the Frölicher-Nijenhuis bracket: Let $\Omega(M) = \bigoplus_{k \geq 0} \Omega^k(M) = \bigoplus_{k=0}^{\dim M} \Gamma(\wedge^k T^*M)$ be the algebra of differential forms. We denote by $\text{Der}_k \Omega(M)$ the space of all (*graded*) *derivations* of degree k , i.e. all bounded linear mappings $D : \Omega(M) \rightarrow \Omega(M)$ with $D(\Omega^l(M)) \subset \Omega^{k+l}(M)$ and $D(\varphi \wedge \psi) = D(\varphi) \wedge \psi + (-1)^{kl} \varphi \wedge D(\psi)$ for $\varphi \in \Omega^l(M)$. The space $\text{Der} \Omega(M) = \bigoplus_k \text{Der}_k \Omega(M)$ is a \mathbb{Z} -graded Lie algebra with the graded commutator $[D_1, D_2] := D_1 \circ D_2 - (-1)^{k_1 k_2} D_2 \circ D_1$ as bracket.

A derivation $D \in \text{Der}_k \Omega(M)$ with $D \mid \Omega^0(M) = 0$ satisfies $D(f \cdot \omega) = f \cdot D(\omega)$ for $f \in C^\infty(M, \mathbb{R})$, thus D is of tensorial character and induces a derivation $D_x \in \text{Der}_k \wedge T_x^* M$ for each $x \in M$. It is uniquely determined by its restriction to 1-forms $D_x \mid T_x^* M : T_x^* M \rightarrow \wedge^{k+1} T_x^* M$ which we may view as an element $K_x \in \wedge^{k+1} T_x^* M \otimes T_x M$ depending smoothly on $x \in M$; we express this by writing

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$D = i_K$, where $K \in C^\infty(\bigwedge^{k+1} T^*M \otimes TM) =: \Omega^{k+1}(M; TM)$, and we have

$$\begin{aligned} (i_K \omega)(X_1, \dots, X_{k+\ell}) &= \\ &= \frac{1}{(k+1)!(\ell-1)!} \sum_{\sigma \in \mathfrak{S}_{k+\ell}} \text{sign } \sigma \cdot \omega(K(X_{\sigma_1}, \dots, X_{\sigma_{(k+1)}}), X_{\sigma_{(k+2)}}, \dots) \end{aligned}$$

for $\omega \in \Omega^\ell(M)$ and $X_i \in \mathfrak{X}(M)$ (or $T_x M$).

By putting $i([K, L]^\wedge) = [i_K, i_L]$ we get a bracket $[\ , \]^\wedge$ on $\Omega^{*+1}(M, TM)$ which defines a graded Lie algebra structure with the grading as indicated, and for $K \in \Omega^{k+1}(M, TM)$, $L \in \Omega^{\ell+1}(M, TM)$ we have

$$[K, L]^\wedge = i_K L - (-1)^{k\ell} i_L K,$$

where $i_K(\omega \otimes X) := i_K(\omega) \otimes X$. The bracket $[\ , \]^\wedge$ is called the the *Nijenhuis-Richardson bracket*, see [6] and [7]. If viewed on a vector space V , it recognizes Lie algebra structures on V : A mapping $P \in L_{\text{skew}}^2(V; V)$ is a Lie bracket if and only if $[P, P]^\wedge = 0$. This can be used to study *deformations of Lie algebra structures*: $P+A$ is again a Lie bracket on V if and only if $[P+A, P+A]^\wedge = 2[P, A]^\wedge + [A, A]^\wedge = 0$; this can be written in *Maurer-Cartan equation* form as $\delta_P(A) + \frac{1}{2}[A, A]^\wedge = 0$, since $\delta_P = [P, \]^\wedge$ is the coboundary operator for the *Chevalley cohomology* of the Lie algebra (V, P) with values in the adjoint representation V . See [4] for a multigraded elaboration of this.

The exterior derivative d is an element of $\text{Der}_1 \Omega(M)$. In view of the formula $\mathcal{L}_X = [i_X, d] = i_X d + d i_X$ for vector fields X , we define for $K \in \Omega^k(M; TM)$ the *Lie derivation* $\mathcal{L}_K = \mathcal{L}(K) \in \text{Der}_k \Omega(M)$ by $\mathcal{L}_K := [i_K, d]$. The mapping $\mathcal{L} : \Omega(M, TM) \rightarrow \text{Der} \Omega(M)$ is injective. We have $\mathcal{L}(\text{Id}_{TM}) = d$.

For any graded derivation $D \in \text{Der}_k \Omega(M)$ there are unique $K \in \Omega^k(M; TM)$ and $L \in \Omega^{k+1}(M; TM)$ such that

$$D = \mathcal{L}_K + i_L.$$

We have $L = 0$ if and only if $[D, d] = 0$. Moreover, $D|\Omega^0(M) = 0$ if and only if $K = 0$.

Let $K \in \Omega^k(M; TM)$ and $L \in \Omega^\ell(M; TM)$. Then obviously $[[\mathcal{L}_K, \mathcal{L}_L], d] = 0$, so we have

$$[\mathcal{L}(K), \mathcal{L}(L)] = \mathcal{L}([K, L])$$

for a uniquely defined $[K, L] \in \Omega^{k+\ell}(M; TM)$. This vector valued form $[K, L]$ is the *Frölicher-Nijenhuis bracket* of K and L .

For $K \in \Omega^k(M; TM)$ and $L \in \Omega^{\ell+1}(M; TM)$ we have

$$[\mathcal{L}_K, i_L] = i([K, L]) - (-1)^{k\ell} \mathcal{L}(i_L K).$$

The space $\text{Der} \Omega(M)$ is a graded module over the graded algebra $\Omega(M)$ with the action $(\omega \wedge D)\varphi = \omega \wedge D(\varphi)$, because $\Omega(M)$ is graded commutative. Let the degree

of ω be q , of φ be k , and of ψ be ℓ . Let the other degrees be as indicated. Then we have:

$$\begin{aligned} [\omega \wedge D_1, D_2] &= \omega \wedge [D_1, D_2] - (-1)^{(q+k_1)k_2} D_2(\omega) \wedge D_1. \\ i(\omega \wedge L) &= \omega \wedge i(L) \\ \omega \wedge \mathcal{L}_K &= \mathcal{L}(\omega \wedge K) + (-1)^{q+k-1} i(d\omega \wedge K). \\ [\omega \wedge L_1, L_2]^\wedge &= \omega \wedge [L_1, L_2]^\wedge - \\ &\quad - (-1)^{(q+\ell_1-1)(\ell_2-1)} i(L_2)\omega \wedge L_1. \\ [\omega \wedge K_1, K_2] &= \omega \wedge [K_1, K_2] - (-1)^{(q+k_1)k_2} \mathcal{L}(K_2)\omega \wedge K_1 \\ &\quad + (-1)^{q+k_1} d\omega \wedge i(K_1)K_2. \end{aligned}$$

For $K \in \Omega^k(M; TM)$ and $\omega \in \Omega^\ell(M)$ the Lie derivative of ω along K is given by:

$$\begin{aligned} (\mathcal{L}_K \omega)(X_1, \dots, X_{k+\ell}) &= \\ &= \frac{1}{k!\ell!} \sum_{\sigma} \text{sign } \sigma \mathcal{L}(K(X_{\sigma_1}, \dots, X_{\sigma_k}))(\omega(X_{\sigma(k+1)}, \dots, X_{\sigma(k+\ell)})) \\ &\quad + \frac{-1}{k!(\ell-1)!} \sum_{\sigma} \text{sign } \sigma \omega([K(X_{\sigma_1}, \dots, X_{\sigma_k}), X_{\sigma(k+1)}], X_{\sigma(k+2)}, \dots) \\ &\quad + \frac{(-1)^{k-1}}{(k-1)!(\ell-1)!2!} \sum_{\sigma} \text{sign } \sigma \omega(K([X_{\sigma_1}, X_{\sigma_2}], X_{\sigma_3}, \dots), X_{\sigma(k+2)}, \dots). \end{aligned}$$

For $K \in \Omega^k(M; TM)$ and $L \in \Omega^\ell(M; TM)$ the Frölicher-Nijenhuis bracket $[K, L]$ is given by:

$$\begin{aligned} [K, L](X_1, \dots, X_{k+\ell}) &= \\ &= \frac{1}{k!\ell!} \sum_{\sigma} \text{sign } \sigma [K(X_{\sigma_1}, \dots, X_{\sigma_k}), L(X_{\sigma(k+1)}, \dots, X_{\sigma(k+\ell)})] \\ &\quad + \frac{-1}{k!(\ell-1)!} \sum_{\sigma} \text{sign } \sigma L([K(X_{\sigma_1}, \dots, X_{\sigma_k}), X_{\sigma(k+1)}], X_{\sigma(k+2)}, \dots) \\ &\quad + \frac{(-1)^{k\ell}}{(k-1)!\ell!} \sum_{\sigma} \text{sign } \sigma K([L(X_{\sigma_1}, \dots, X_{\sigma_\ell}), X_{\sigma(\ell+1)}], X_{\sigma(\ell+2)}, \dots) \\ &\quad + \frac{(-1)^{k-1}}{(k-1)!(\ell-1)!2!} \sum_{\sigma} \text{sign } \sigma L(K([X_{\sigma_1}, X_{\sigma_2}], X_{\sigma_3}, \dots), X_{\sigma(k+2)}, \dots) \\ &\quad + \frac{(-1)^{(k-1)\ell}}{(k-1)!(\ell-1)!2!} \sum_{\sigma} \text{sign } \sigma K(L([X_{\sigma_1}, X_{\sigma_2}], X_{\sigma_3}, \dots), X_{\sigma(\ell+2)}, \dots). \end{aligned}$$

The Frölicher-Nijenhuis bracket expresses obstructions to integrability in many different situations: If $J : TM \rightarrow TM$ is an almost complex structure, then J is complex structure if and only if the Nijenhuis tensor $[J, J]$ vanishes (theorem of Newlander and Nirenberg, [5]). If $P : TM \rightarrow TM$ is a fiberwise projection on the tangent spaces of a fiber bundle $M \rightarrow B$ then $[P, P]$ is a version of the curvature (see [3], sections 9 and 10). If $A : TM \rightarrow TM$ is fiberwise diagonalizable with all eigenvalues real and of constant multiplicity, then eigenspace of A is integrable if and only if $[A, A] = 0$.

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