Lie Theory and Applications. II `P 14195-MAT`

We are going to present three selected results obtained within this research project.

Choosing roots differentiably. Consider a family of polynomials

$$P_t(x) = x^n + \sigma_1(t)x^{n-1} + \cdots + \sigma_n(t)$$

whose coefficients $\sigma_i(t)$ depend smoothly on a real parameter t. One can ask if it possible to find n smooth functions $x_1(t), \ldots, x_n(t)$ parametrizing the roots of P_t. Simple quadratic examples show that this cannot be accomplished if the polynomial is allowed to have complex roots. However, assuming all roots of P_t are real, we could show that this is indeed possible, provided the coefficients depend real analytically on t, or provided no two roots meet of infinite order. In general one can arrange the roots twice differentiably, and this is best possible. As an application we find that for a smooth curve of unbounded self adjoint operators in Hilbert space with compact resolvent the eigenvalues can always be arranged twice differentiable, and C^∞ if no two of them meet of infinite order.

Pattern recognition and shape theory. We tried to find the simplest distance function of Riemannian type on the space of 2-dimensional shapes, for which we take the orbit space of immersions mod diffeomorphisms: $\mathrm{Imm}(S^1, \mathbb{R}^2)/\operatorname{Diff}(S^1)$. This action is not quite free hence this orbit space is an orbifold and not quite a manifold. In particular we investigate the metric for a constant $A > 0$:

$$G_c^A(h, k) := \int_{S^1} (1 + A\kappa_c(\theta)^2)\langle h(\theta), k(\theta)\rangle |c'(\theta)|\, d\theta$$

where κ_c is the curvature of the curve c and h, k are normal vector fields to c. For $A = 0$, the geodesic distance between any 2 distinct curves is 0, while for $A > 0$ the distance is always positive. We give some lower bounds for the distance function, derive the geodesic equation and the sectional curvature, solve the geodesic equation with simple endpoints numerically, and pose some open questions. The space has an interesting split personality: among large smooth curves, all its sectional curvatures are ≥ 0, while for curves with high curvature or perturbations of high frequency, the curvatures are ≤ 0.

Dynamics and spectral geometry. Suppose we have a closed manifold M equipped with a vector field X. Consider the flow of X as dynamics on M. Besides assumptions of rather technical nature we assume that there is a circle valued Lyapunov function $f : M \to S^1$ which is strictly decreasing along non-constant flow lines. Vector fields of this type occur e.g. in the physics of electron transport in metals exposed to magnetic fields. For every pair of rest points x and y one can consider the instantons from x to y, i.e. isolated flow lines connecting x with y. In general there can be infinitely many such instantons, and one is forced to count them in a more subtle way. It turns out that for every integer n there is only a finite number of instantons from x to y which, when composed with f, perform n full twists in S^1. Counting them with appropriate signs one obtains Novikov's incidence numbers $I_n(x, y)$. Following Hutchings one can count the number of periodic flow lines which, when composed with f, perform n full twists in S^1. This leads to rational numbers Z_n. On the other hand we can pick a Riemannian metric on M, and use f to produce a one parameter family Δ_s of zero order Witten type perturbations of the Laplacian on M. One of our main results tells how the numbers $I_n(x, y)$ and Z_n can be recovered from the spectral geometry of Δ_s.