LIE THEORY AND APPLICATIONS. III

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This project is planned as a continuation of the project P 17108-N04 which was finished in September 2007.

The research in the new project will follow different lines which are explained below.

Papers cited as [Mxy] can be found (fulltext) via the homepage of Peter Michor

`http://www.mat.univie.ac.at/~michor/listpubl.html`

under the number [xy]

Geometry and analysis of shape space. The core of shape space in one of its simplest forms is the orbit space of the action of the group of diffeomorphisms of the circle S^1 (the reparametrization group) on the space of immersions of the circle into the plane $\mathbb{R}^2$. The aim is to find good Riemannian metrics which allow applications in pattern recognition and visualization. The contributions of the PI were obtained mainly in collaboration with David Mumford. In the paper [M107] via the Hamiltonian approach many metrics were investigated, together with their conserved quantities (one of them is the reparameterization momentum) and their sectional curvatures. Recall from a result of the predecessor of this project, that the L^2-metric on the space of immersions has zero geodesic distance on the orbit space under the reparameterization group. These metrics come in 3 flavors: Some are derived from the the L^2-metric by multiplying it with a function of the length of the curve (a conformal change) or by multiplying the L^2-integrand with a function of length and curvature (this is called almost local). Of the second flavor are the metrics which come from the Sobolev H^n-inner product on the space of immersions. The last flavor comes from using a suitable Sobolov metric on the diffeomorphism group of the plane and treating shape space as a homogeneous space. This is the metric that has found already many applications, particularly in medicine. This metric was pioneered by Michael Miller, Alain Trouve, and Laurent Younes, see [3, 4, 9] These applications are done at the 'Center for imaging sciences' at the Johns Hopkins University, and they are used to recognize diseases from the the tomographical data of hearts and brains, [6, 7]. A version of one of the metrics in [M107] turns out to be isometric to a quotient of (an open subset in) an infinite dimensional Grassmannian of 2-planes in Hilbert space. Here recent

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formulas of Neretin allow explicit geodesics and explicit formulas for the geodesic distance. This is done in [M111].

The geodesic equations arising in vision are relatives of some well known PDE's, like the Burgers' hierarchy. Some of them might be completely integrable systems.

In [M108] the homotopy type of the rotation degree 0 immersed plane curves is determined: $\pi_1 = \mathbb{Z}$ and $\pi_2 = \mathbb{Z}$, all others vanish. [M109] is a review article on Riemannian metrics on infinite dimensional regular Lie groups, containing detailed presentations of the Hamiltonian approach. This are the notes for a lecture course which the PI gave in order to prepare for the existence results for geodesics equations in [M107].

The aims of this part are as follows:

- To complete the geometric information on the shape space of plane curves, in particular to compute the sectional curvature for the metrics derived from Sobolev metrics on the space of immersions, and and derived from right invariant Sobolov metrics on the diffeomorphism group of the the 2-plane. The latter one is tied to the formula due to Arnol'd for the curvature of the right invariant Sobolov metric on the group of diffeomorphisms.
- To develop the theory for surfaces in 3-space in great detail, because this has many applications.
- To develop the theory in the general situation where shape space is the non-linear Grassmannian $\mathrm{Imm}(M,N)/\operatorname{Diff}(M)$, for a compact (template) manifold M, and Riemannian manifold (N,g) of higher dimension. The metric here comes from invariant metrics on the space of immersions. This is interesting from a pure mathematical point of view.
- To develop the theory in the setting, where $\mathrm{Emb}(M,N)/\operatorname{Diff}(M)$ is a homogeneous of the diffeomorphism group $\operatorname{Diff}(N)$. Here submanifolds are transported around and deformed by diffeomorphism of N. But one can deform not only submanifold, one can deform also currents or measure or tensors on N (like the fiber structure of the muscles in the heart). To compute the sectional curvature is quite difficult.
- To further develop the theory of manifolds of mappings which lies at the basis of any kind of shape space. Also to analyse and solve the partial differential equations and their variants which appear as geodesic equations for various metrics on shape spaces.

Why is it so important to compute curvature on shape space? Let me start with a quotation from David Mumford:

> Pattern Theory started in the 70's with the ideas of Ulf Grenander and his school at Brown. The aim is to analyze from a statistical point of view the patterns in all 'signals' generated by the world, whether they be images, sounds,

> written text, DNA or protein strings, spike trains in neurons, time series of prices or weather, etc. Pattern theory proposes that the types of patterns – and the hidden variables needed to describe these patterns – found in one class of signals will often be found in the others and that their characteristic variability will be similar. The underlying idea is to find classes of stochastic models which can capture all the patterns that we see in nature, so that random samples from these models have the same 'look and feel' as the samples from the world itself. Then the detection of patterns in noisy and ambiguous samples can be achieved by the use of Bayes's rule, a method that can be described as 'analysis by synthesis'.

So eventually one has to do statistics on shape space of a certain kind, and has to group patterns by geodesic distance to some 'template'. But this depends tremendously on curvature. In particular, positvie curvature is difficult because shortest geodesics to a template can jump when moving a shape.

How will this part of the project be run? There will be a lot of collaboration with David Mumford (emeritus at Brown), the group at the Center for Imaging Sciences at Johns Hopkins (Michael Miller, Laurent Younes), with Alain Trouve at the Ecole Normale Superieure de Cachan in France, with Daniel Cremer at the computer vision group in Bonn, and with Darryl Holm at Imperial college. Collaboration (less focused on vision) is planned with Tudor Ratiu in Lausanne and with Thomas Kappeler in Zürich. Graduate students and post docs can rotate through some of this places and become aquainted with many aspects of vision. So in this project I ask for support for 2 graduate students and 2 postdocs and some money for travel, invitations, and workshops.

Choosings roots smoothly alias lifting of mappings over orbit mappings, and invariant theory. In [M105] the following is proved: Any sufficiently often differentiable curve in the orbit space V/G of a real finite dimensional orthogonal representation $G \to O(V)$ of a finite group G admits a differentiable lift into the representation space V with locally bounded derivative. As a consequence any sufficiently often differentiable curve in the orbit space V/G can be lifted twice differentiably which is in general best possible. These results are then generalized to arbitrary polar representations.

Question: Can one lift a sufficiently often differentiable curve twice differentiable for any orthogonal representation of a compact group?

In [8] the the regularity of the roots of complex monic polynomials $P(t)$ of degree n depending smoothly on a real parameter t is studied. If $P(t)$ is C^∞ and no two of the continuously chosen roots meet of infinite order of flatness, then there exists a locally absolutely continuous parameterization

of the roots. Provided that $P(t)$ is C^n, the roots may be parameterized differentiably if and only if whenever roots meet they meet of order at least 1. Applications to the perturbation theory of normal matrices and unbounded normal operators with compact resolvents and common domain of definition are given. The eigenvalues and eigenvectors of a C^∞ curve of such operators can be arranged locally in an absolutely continuous way, provided that no two of the eigenvalues meet of infinite order of flatness.
[M110] shows the following: If $u \mapsto A(u)$ is a $C^{1,\alpha}$-mapping having as values unbounded self-adjoint operators with compact resolvents and common domain of definition, parametrized by u in an (even infinite dimensional) space then any continuous arrangement of the eigenvalues $u \mapsto \lambda_i(u)$ is $C^{0,1}$ in u. If $u \mapsto A(u)$ is $C^{0,1}$, then the eigenvalues may be chosen $C^{0,1/N}$ (even $C^{0,1}$ if $N = 2$), locally in u, where N is locally the maximal multiplicity of the eigenvalues.
Question: Can the roots of a many parameter family of complex polynomials be chosen absolutily continuously in the parameters? A positive answer would have strong consequences for well-posedness of linear PDE's.

In this research direction, collaboration with Armin Rainer from Vienna and with Ferruccio Colombini and Sergio Spagnolo from Pisa is expected. Since Armin Rainer is an independent mathematician now, he will employed in this project only for short periods, if necessary.

Symplectic and Poisson geometry. Here I expect mainly results in connections with the Hamiltonian approach for geodesic equations on various shape spaces which arise from invariance under various reparametrization groups.

Actions of Lie groups and structures of orbit spaces. Collaboration of Peter Michor with D. Alekseevsky from Moscow, M. Losik from Saratov, Andreas Kriegl and Armin Rainer from Vienna. The project is to continue the investigations in the direction of obtaining a better understanding of the geometry of the orbit space of an isometric Lie group action. Paper [M85] gives a good description of the orbit space and of the image of geodesics on it: Short geodesics which are orthogonal to the orbits map to distance minimizing curves in the orbit space. Longer orthogonal geodesics map to curves which are reflected at singular strata. Non-orthogonal geodesics map in special examples to solutions of very interesting dynamical systems generalizing the Calogero-Moser system. The symplectic reduction side of this has been vastly extended by Simon Hochgerner, and is now outside of the scope of this project.
But many other **questions** are still open: Describe the orbit space stratification in infinite dimension (this has relevance for shape spaces). Work out in detail the orbit space of some class of finite dimensional representations of compact groups.

Paper [M97] develops reflection groups on Riemannian manifolds and shows how to reconstruct them from the orbit space (a generalized Weyl chamber). Research in this direction will go on.

The generalized Cayley transform for a representation. The paper [M87] with Bert Kostant contains the costruction of the generalized Cayley transform from a Lie group to its Lie algebra, induced by a representation. Many properties were deduced, mostly of algebraic geometric type. For the spin representation this equals the classical Cayley transform $A \mapsto (A - \mathrm{Id})(A + \mathrm{Id})^{-1}$ for matrices, multiplied by a rational function which kills the polar divisor of the classical Cayley transform. This has been extended to a classification of all algebraic groups which are birational to its Lie algebra, see [2]. In this project I want to carry over the construction of the Cayley transform to the case of super Lie groups and super Lie algebras. Attempts to do this cry out for some basic facts of super algebraic geometry, which do not seem to have been worked out in detail yet. One could also look for the quantum group version of this construction.

Also the Cayley transform from a finite group of Lie type to its Lie algebra should be investigated. It is the first connection between the group and its (finite) Lie algebra (the exponential mapping does not make sense). An attempt to do this in a thesis in a a predecessor of this project failed.

Convenient setting for Denjoy-Carleman ultradifferentiable mappings. Let $M = (M_k)_{k\in\mathbb{N}_0}$ be a non-decreasing sequence of real numbers with $M_0 = 1$. Let $U \subseteq \mathbb{R}^n$ be open. We denote by $C^M(U)$ the set of all $f \in C^\infty(U)$ such that, for all compact $K \subseteq U$, there exist positive constants C and ρ such that

$$|\partial^\alpha f(x)| \le C\, \rho^{|\alpha|}\, |\alpha|!\, M_{|\alpha|}$$

for all $\alpha \in \mathbb{N}_0^n$ and $x \in K$. The set $C^M(U)$ is the *Denjoy–Carleman class* of functions on U. If $M_k = 1$, for all k, then $C^M(U)$ coincides with the ring $C^\omega(U)$ of real analytic functions on U. In general, $C^\omega(U) \subseteq C^M(U) \subseteq C^\infty(U)$. Gevray differentiable functions are a particular case which have many applications in proving wellposedness for certain linear PDE's.

It seems that for certain classes of weight sequences $M = (M_k)$ one can obtain the convenient setting of C^M-calculus: A mapping is C^M if it maps C^M-curves to C^M-curves, and in general we have $C^M(E, C^M(F,G)) = C^M(E \times F, G)$ for locally convex spaces. This allows to treat the theory of manifolds of C^M-mappings, and in particular to prove that the group of all C^M-diffeomorphisms is a regular C^M-Lie group, but not better. Such results have applications for choosing roots of polynomials in a C^M-way, and for perturbation of eigenvalues of C^M-parameterized curves of unbounded operators on Hilbert space with compact resolvent and common domain of defintion. This is a collaboration with Andreas Kriegl and Armin Rainer.

Personnel and budget.

Name	Position	per year
N.N.	Post Doc position	54,180.00
N.N.	Post Doc position	54,180.00
N.N.	graduate student	31,670.00
Project money	Guests, travels	10,000.00
	Sum per year	150,030.00
	Sum for 3 years, in EURO	450,090.00

The project money is for inviting guests to Vienna, among them the scientists mentioned in the report.

Start of the program: October 2008.

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