FINAL REPORT FOR THE PROJECT 'LIE THEORY AND APPLICATIONS III', P21030-N13

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Summary

One of the centers of this project concerns the analysis of shape space. Here a shape is a certain kind of silhuette or form of fixed type in n-dimensional space. We investigated several Riemannian metrics on the space of shapes; the geodesic distance with respect to these metrics can be used to distinguish shapes and to compare them to a given set of templates; if possible in an automated computational way for data sets. Some of these distances have found applications in "Vision" and in computatinal anatomy.

Another field is the attempt to arrange the roots of a parameterized family of polynomials as differentiable in the parameter as possible. This allows then to arrange eigenvalues of parameterized families of unbounded operators on Hilbert space with compact resolvent and common domain of definition in a differentiable way. Examples of these operators are Schrödinger operators in quantum mechanics.

A third topic is the "convenient setting" for ultradifferentiable functions of Denjoy-Carleman type, which leads to a description of these functions which is correct in the sense of category theory.

Zusammenfassung

Ein zentrales Forschungsgebiet ist the Analyse des Raumes der Gestalten oder Formen (shape space). Für uns sind Gestalten gewisse Umrisse von festem Typ im n-dimensionalen Raum. Wir versuchen, Riemann Metriken am Raum der Gestalten zu finden, welche gute Eigenschaften haben: Die geodätische Distanz bezüglich einer solchen Metrik soll Gestalten gut unterscheiden können, automatisiert am Computer. Manche Metriken, die wir untersuchten, haben Anwendungen in der Bildanalyse und der rechnerischen Anatomie.

Ein weiterer Schwerpunkt ist der Versuch, die Wurzeln eines Polynoms, das von weiteren Parameter glatt abhängt, so differenzierbar wie möglich zu wählen. Das kann dann auch auf die Eigenwerte einer parametrisierten Familie von unbeschränkten Operatoren angewandt werden, wie sie etwa in der Quantenmechanik vorkommen.

Ein dritter Schwerpunkt ist die Beschreibung der ultradifferenzierbaren Funktionen vom Denjoy-Carleman Typ, die zu einer kartesisch abgeschlossenen Kategorie führt.

2. Brief project report, english

This report first follows the list of topics of the project application, which is available as

`http://www.mat.univie.ac.at/~michor/fwf08appl.pdf.`

All articles are on my homepage, including the 3 theses.

Geometry and analysis of shape space. The research in this direction was very successful.

Two excellent doctoral thesis [B] and [H] were prepared whose results were published in [M118], [M119], [M120], and [M123]. In some detail: Let M and N be connected manifolds without boundary with $\dim(M) < \dim(N)$, and let M compact. Then shape space is either the manifold $\operatorname{Emb}(M,N)/\operatorname{Diff}(M)$ of submanifolds of N that are diffeomorphic to M, or the orbifold $\operatorname{Imm}(M,N)/\operatorname{Diff}(M)$ of unparametrized immersions of M in N. We investigate weighted Sobolev Riemannian metrics on shape space: These are induced by metrics of the following form on the space $\operatorname{Imm}(M,N)$ of immersions:

$$G^P_f(h,k) = \int_M \Phi.\overline{g}(P^f h, k)\operatorname{vol}(f^*\overline{g})$$

where: $\overline{g}$ is some fixed metric on N. $f^*\overline{g}$ is the induced metric on M. $h,k \in \Gamma(f^*TN)$ are tangent vectors at f to the space of embeddings or immersions. P^f is a positive, selfadjoint, bijective scalar pseudo differential operator of order $2p$ depending smoothly and $\operatorname{Diff}(M)$-equivariantly on f. and where Φ is a smooth function depending on the total volume $\operatorname{Vol}(f) = \int_M \operatorname{vol}(f^*\overline{g})$ of $f(M)$, and on various extrinsic curvature expressions of $F^*\overline{g}$. In [M118] and [M120] we consider the special case where $N = \mathbb{R}^{\dim(M)+1}$ and $P^f = \operatorname{Id}$. In [M119] we take $\Phi = 1$ and consider as main example the operator $P^f = 1 + A\Delta^p$, where Δ is the Bochner-Laplacian on M induced by the metric $f^*\overline{g}$. The paper [M124] treats the general case, where Φ is used mainly to make Sobolev metrics scale invariant. For these metrics we compute the geodesic equations both on the space of immersions and on shape space, and also the conserved momenta arising from the obvious symmetries. We also show that the geodesic equation is well-posed on spaces of immersions and also on diffeomorphism groups, if the order $2p$ of P_f is > 2, We give many examples of numerical solutions.

The paper [M123] treats natural Sobolev metrics on the space of all Riemannian metrics on a manifold.

The paper [M121] arose in connection with the thesis of M.Micheli at Brown-University. It deals with the computation of sectional curvature for the manifolds of N landmarks (or feature points) in D dimensions, endowed with the (homogenous)Riemannian metric induced by the group action of diffeomorphisms. It turns out that the matrices of partial derivatives of the cometric (on the cotangent bundle) very sparse in nature, thus suggesting solving the highly non-trivial problem of developing a formula that expresses sectional curvature in terms of the cometric and its first and second partial derivatives (we call this

Mario's formula). We apply such formula to the manifolds of landmarks and in particular we fully explore the case of geodesics on which only two points have non-zero momenta and compute the sectional curvatures of 2-planes spanned by the tangents to such geodesics. The latter example gives insight to the geometry of the full manifolds of landmarks.

This was generalized in the paper [M127] as follows: Given a finite dimensional manifold N, the group $\mathrm{Diff}_{\mathcal{S}}(N)$ of diffeomorphism of N which fall suitably rapidly to the identity, acts on the manifold $B(M,N)$ of submanifolds on N of diffeomorphism type M where M is a compact manifold with $\dim M < \dim N$. For a right invariant weak Riemannian metric on $\mathrm{Diff}_{\mathcal{S}}(N)$ induced by a quite general operator $L : \mathfrak{X}_{\mathcal{S}}(N) \to \Gamma(T^*N \otimes \mathrm{vol}(N))$, we consider the induced weak Riemannian metric on $B(M,N)$ and we compute its geodesics and sectional curvature. For that we derive a covariant formula for curvature in finite and infinite dimensions, we show how it makes O'Neill's formula very transparent, and we use it finally to compute sectional curvature on $B(M,N)$.

The paper [M124] studies Sobolev-type metrics of fractional order on the group of compactly supported diffeomorphisms $\mathrm{Diff}_c(M)$, where M is a Riemannian manifold of bounded geometry. We prove that the geodesic distance, induced by the Riemannian metric, vanishes if the order s is between $0 \leq s < \frac{1}{2}$. For $M \neq \mathbb{R}$ we show the vanishing of the geodesic distance also for $s = \frac{1}{2}$ and for $\dim(M) = 1$ we show that the distance doesn't vanish for $\frac{1}{2} < s$. For $M = \mathbb{R}^n$ and specific s, the geodesic equations for these metrics corespond well known PDEs of hydrodynamics: Burgers' equation for $s = 0$, the modified Constantin-Lax-Majda equation for $s = \frac{1}{2}$ or the Camassa-Holm equation for $s = 1$.

Choosings roots smoothly alias lifting of mappings over orbit mappings, and invariant theory. Also here many results were obtained, in collaboration with Mark Losik (Saratov), Andreas Kriegl, and Armin Rainer: In [M112] a former lifting result was essentially improved. In [M114] resolution via blow ups was used to lift curves - a generalization of Pusieux's theorem. In [M110], many parameter Hölder perturbation of unbounded operators with compact resolvents was shown to lead to Hölder continuity of the eigenvalues. The main question stated in the application is still unsolved, but Armin Rainer seems very near to a solution.

Symplectic and Poisson geometry. Symplectic methods were heavily used for shape spaces. See there.

Actions of Lie groups and structures of orbit spaces. Not much was done here besides the results on shape spaces and the lifting results.

The generalized Cayley transform for a representation. The thesis [W] generalized the Cayley transform to super groups. Here a Russian mathematician published the big result before the thesis was finished.

Convenient setting for Denjoy-Carleman ultradifferentiable mappings. Collaboration with Andreas Kriegl and Armin Rainer. This was a complete success. Let $M = (M_k)_{k\in\mathbb{N}_0}$ be a non-decreasing sequence of real numbers with $M_0 = 1$. Let $U \subseteq \mathbb{R}^n$ be open. We denote by $C^M(U)$ the set of all $f \in C^\infty(U)$ such that, for all compact $K \subseteq U$, there exist positive constants C and ρ such that $|\partial^\alpha f(x)| \le C\,\rho^{|\alpha|}\,|\alpha|!\,M_{|\alpha|}$ for all $\alpha \in \mathbb{N}_0^n$ and $x \in K$. The set $C^M(U)$ is the *Denjoy–Carleman class* of functions on U. In [M112] convenient setting was obtained for non-quasianalytic mappings if M is log-convex and of moderate growth. In [M115] convenient setting was obtained for those quasianalytic mappings C^Q of moderate growth which can be described as intersections of non-quasianalytic ones. Both papers used a curve based approach. Finally, in [M126], for all classes of moderate growth, the convenient setting was obtained with a new approach, starting from Banach spaces. These results were applied to manifolds of mapping and diffeomorphism groups, and to choosing roots and perturbation of unbounded operators.

Personel development

3 Doctorates given. Dennis Bouke Westra defended his thesis successfully. Martin Bauer and Philipp Harms defended their (very collaboratively written) theses very successfully. Harms is a postdoc Harvard University, and Bauer has applied for a FWF-project.

Effects of the project outside the field

Many applications of the shape space theory in computer vision, computational anatomy, etc. Nearly all papers were presented in various lectures.

Publications, peer-refereed

[M110] Andreas Kriegl, Peter W. Michor, Armin Rainer, *Many parameter Hölder perturbation of unbounded operators.* Math. Ann. (2011) DOI:10.1007/s00208-011-0693-9. arXiv:math/0611506. ESI Preprint 1921. .

[M112] Andreas Kriegl, Peter W. Michor, Armin Rainer, *The convenient setting for non-quasianalytic Denjoy–Carleman differentiable mappings.* J. Functional Analysis 256 (2009), 3510-3544. arXiv:0804.2995. .

[M113] Andreas Kriegl, Mark Losik, Peter W. Michor, Armin Rainer, *Addendum to: 'Lifting smooth curves over invariants for representations of compact Lie groups, III' [J. Lie Theory 16 (2006), No. 3, 579–600].* J. Lie Theory 22, 1 (2012), 245–249. arXiv:1106.6041.

[M114] Mark Losik, Peter W. Michor, Armin Rainer, *A generalization of Puiseux's theorem and lifting curves over invariants.* Revista Matemática Complutense 25, 1 (2012), 139-155. arXiv:0904.2068.

[M115] Andreas Kriegl, Peter W. Michor, Armin Rainer, *The convenient setting for quasianalytic Denjoy–Carleman differentiable mappings.* J. Functional Analysis 261, 7 (2011) 1799-1834. arXiv:0909.5632.

[M116] Andreas Kriegl, Peter W. Michor, Armin Rainer, *Denjoy-Carleman differentiable perturbation of polynomials and unbounded operators.* Integral Equations and Operator Theory 71,3 (2011), 407-416. arXiv:0910.0155.

[M118] Martin Bauer, Philipp Harms, Peter W. Michor, *Almost local metrics on shape space of hypersurfaces in n-space.* SIAM J. Imaging Sci. 5 (2012), pp. 244-310. arXiv:1001.0717.

[M119] Martin Bauer, Philipp Harms, Peter W. Michor, *Sobolev metrics on shape space of surfaces.* Journal of Geometric Mechanics 3, 4 (2011), 389-438. arXiv:1009.3616.

[M120] Martin Bauer, Philipp Harms, Peter W. Michor, *Curvature weighted metrics on shape space of hypersurfaces in n-space.* Differential Geometry and its Applications. 30 (2012), 33-41. arXiv:1102.0678.

[M121] Mario Micheli, Peter W. Michor, David Mumford, *Sectional curvature in terms of the cometric, with applications to the Riemannian manifolds of landmarks.* SIAM J. Imaging Sci. 5, 1 (2012), 394-433. arXiv:1009.2637. .

[M122] Martin Bauer, Martins Bruveris, Philipp Harms, Peter W. Michor, *Vanishing geodesic distance for the Riemannian metric with geodesic equation the KdV-equation.* Ann. Glob. Anal. Geom. 41, 4 (2012) 461-472. arXiv:1102.0236. .

[M125] Martin Bauer, Philipp Harms, Peter W. Michor, *Sobolev Metrics on Shape Space, II: Weighted Sobolev Metrics and Almost Local Metrics.* 19 pages. To appear in: Journal of Geometric Mechanics. arXiv:1109.0404.

[BB] Martin Bauer, Martins Bruveris, *A New Riemannian Setting for Surface Registration*, Proceedings of the Third International Workshop on Mathematical Foundations of Computational Anatomy - Geometrical and Statistical Methods for Modelling Biological Shape Variability (2011) 182-193. arXiv:1106.0620

Publications, not yet peer-refereed

[M117] Dmitri V. Alekseevsky, Peter W. Michor, Yurii A. Neretin, *Rolling of Coxeter polyhedra along mirrors.* 19 pages. arXiv:0907.3502. .

[M123] Martin Bauer, Philipp Harms, Peter W. Michor, *Sobolev metrics on the Riemannian manifold of all Riemannian metrics.* 14 pages. Submitted. arXiv:1102.3347.

[M124] Martin Bauer, Martins Bruveris, Philipp Harms, Peter W. Michor, *Geodesic distance for right invariant Sobolev metrics of fractional order on the diffeomorphism group.* 16 pages. Submitted. arXiv:1105.0327.

[M126] Andreas Kriegl, Peter W. Michor, Armin Rainer, *The Convenient Setting for Denjoy–Carleman Differentiable Mappings of Beurling and Roumieu Type.* 40 pages. Submitted. arXiv:1111.1819.

[M127] Mario Micheli, Peter W. Michor, David Mumford, *Sobolev Metrics on Diffeomorphism Groups and the Derived Geometry of Spaces of Submanifolds.* 28 pages. Submitted. arXiv:1202.3677

[B] Martin Bauer *Almost local metrics on shape space.* Thesis, University of Vienna, Defense Dec.16, 2010. 117 pages. External examiner: David Mumford (Brown Univ.), Otmar Scherzer (Wien).

[H] Philipp Harms *Sobolev metrics on shape space of surfaces.* Doctoral Thesis, University of Vienna. Defense Dec.15, 2010. External examiner: David Mumford (Brown Univ.), Alain Trouvé (Paris).

[W] Dennis Bouke Westra *Superrings and supergroups.* Doctoral Thesis, University of Vienna, Defense May 2009. External examiner: Arkadij Onishchik (Jaroslawl, RU), Herwig Hauser (Wien).

Some lectures

Lectures of P. Michor

227. A metric on shape space with computable geodesics. Winter school in Geometry and Physics, Srni, January 12-18, 2008.

228. Riemannian geometries on shape space. Conference 'Problemi Attuali di Fisica Teorica' 14 - 19 Marzo 2008, Vietri sul Mare (Italy).

229. Riemannian Geometries on Shape Space. International Workshop in honour of Michel Dubois-Violette: Differential Geometry, Noncommutative Geometry, Homology and Fundamental Interactions. Laboratoire de Physique Théorique d'Orsay, 8-10th April, 2008.

230. Riemannian Geometries on shape spaces. Tag der Geometrie. Universitaet Wien mit TU Wien und Hochschule fuer darstellende Kunst. 25. April 2008.

231. The Hamiltonian approach to Riemannian Geometries of Shape Spaces. Minikolloquium
"uber Geometrie, Institut f
"ur Diskrete Mathematik und Geometrie, Technische Universitaet Wien, 30. Mai 2008

232. Geometry of landmark space and spaces of currents. Workshop: Geometry and Statistics of Shapes. Hausdorff Center for Mathematics, Bonn, June 9–14, 2008.

233. Riemannian Geometries on Shape Space. Seminar, SISSA, Trieste, I, June 17, 2008.

234. Geometries on Shape Space. Lecture to Charly Peskin's research group, Courant Institute, NY, October 21, 2008.

235. Infinite dimensional Lie groups via convenient calculus, in particular Denjoy-Carleman ultradifferentiable diffeomorphism groups. Workshop on infinite-Dimensional Lie Groups and Related Functional Analysis, Paderborn, D, November 6-8, 2008.

236. The convenient setting for Denjoy-Carleman ultradifferentiable functions and its applications in infinite dimensional differential geometry. Annual Eduard Cech Center Meeting, November 14 – 15, 2008, Trest (Třešť), CZ.

237. Analysis and geometry of shape spaces, Plenary lecture series (5 lectures). Winterschool Geometry and Physics, Srni, CZ. Jan. 17–24, 2009

238 Was Sie immer schon ueber Mšbius-Bänder, Euler's Charakter und Skelette wissen wollten. Vortrag im Rahmen der Austellung 'Imaginary – mit den Augen der Mathematik', Aula der Universitaet Wien, March 17, 2009.

239. A metric on shape space with explicit geodesics which run through singularities in finite time. Workshop on Singularities in Generic Geometry and Applications Valencia, March 23-27, 2009.

240. Curvature on the group of diffeomorphisms and on landmark space. Opening lecture of the Retreat of the NSF-FRG, Annapolis, April 20-22 2009.

241. Curvature on the group of diffeomorphisms and on landmark space. Central European Seminar on Differential Geometry, Mikulov, May 29-30, 2009.

242. Ueber das Finden ertragreicher Forschungsthemen. Vortrag zur Tagung der Österreichischen Forschungsgemeinschaft: Qualität und Wirkung: Von der guten zur signifikanten Forschung. 5.-6. Juni 2009, Hotel Sauerhof, Baden bei Wien.

243. Curvature on diffeomorphism groups and nonlinear Grassmannians alias differentiable Chow varieties alias shape spaces. Seminar, SISSA, Trieste, June 16, 2009

244. Introduction to shape spaces. Seminar at ISTA. September 10, 2009.

245. Convenient setting for some quasianalytic Denjoy-Carleman classes: setup and applications. Conference: Spaces of analytic and smooth functions, III. Banach center at Bedlewo, September 13–19, 2009.

246. Riemannian geometry of shape spaces. Jornada de homenaje a Angel Montesinos Amilibia Universidad de Valencia, Septiembre 25, 2009.

247. The Wasserstein metric on shape space of plane curves is sandwiched between two almost local Riemannian metrics. Seminar at IST Austria, October 13, 2009.

248. Geometry on the diffeomorphism group $\mathrm{Diff}(N)$ and the induced geometry on the shape space
$\mathrm{Emb}(M,N)/\mathrm{Diff}(M)$. 30th Winterschool on Geometry and Physics, Srni (CZ), January 16-23, 2010

249. PDE's arising as geodesic equations on shape spaces. Workshop: Linear and Nonlinear Hyperbolic Equations. Centro Ennio de Giorgi, Pisa, February 9-12, 2010

250. Curvature for metrics of diffeomorphic matching. Workshop: Shape FRG, Imperial College London, England May 23-26, 2010

251. Riemannian metrics on diffeomorphism groups and induced metrics on associated homogeneous spaces. Conference: Geometry, Mechanics and Dynamics (Tudor Ratiu 60 years), CIRM, Marseille, July 12 - 16th, 2010

252. A shape space of curves with explicit geodesics. Shape focused research group Workshop, Imperial College, London, May 16-19, 2011.

253. The Lagrangian tomographic transform. Folding and Unfolding: Interactions from Geometry. Workshop in honour of Giuseppe Marmo's 65th birthday. 8-12 June 2011. Ischia (NA) Italy.

254. Sobolev metrics H^s for $1 \leq s \leq 1/2$ on diffeomorphism groups. Joint workshop of the Eduard Cech Center and the AmeGA project (and also within the realm of the Central European Seminars in Differential Geometry), Masaryk University center in Telc, CZ, October 21-22, 2011

Lectures by Martin Bauer

1. Variational formulas on shape space of hyper-surfaces in n-space. The 30th Winter School: Geometry and Physics Srni, Czech RepublicJanuary 21, 2010

2. Well-poedness of the geodesic equation of a Sobolev type metric on shape space . Imaging sciences Seminar Johns Hopkins University, USAMarch 4, 2010

3. Sobolev type metrics on shape space. Shape FRG Imperial College London, EnglandMay 26, 2010

4. Almost local metrics on shape space. Public defensio University of Vienna, AustriaDecember 16, 2010

5. Fractional Sobolev metrics on diffeomorphism groups. EquaDiff Loughborough University, EnglandAugust 2, 2011