Smoothness of the action of the gauge transformation group on connections

M. C. Abbati, R. Cirelli, and A. Manià
Dipartimento di Fisica, Università di Milano, Milan, Italy and Istituto Nazionale di Fisica Nucleare, Sezione di Milano, Milan, Italy

P. Michor[a)]
Institut für Mathematik, Universität Wien, Vienna, Austria

(Received 18 November 1985; accepted for publication 28 May 1986)

The NLF–Lie group structure of the group $\mathcal{G}$ of the gauge transformations, defined as the group of sections of the bundle $P[G]$ associated to the principal bundle $P(M,G)$, is discussed. Other current definitions of the group of gauge transformations are shown to admit a nontrivial smooth structure only in the case of compact G. The space $\mathcal{C}$ of principal connections, as well, is given the structure of local affine NLF-manifold, after identifications of connections with sections of a convenient vector bundle on M. Finally, the smoothness of the action of $\mathcal{G}$ on $\mathcal{C}$ is proved in general. In the case of compact M, the group $\mathcal{G}$ becomes a tame Fréchet–Lie group and the action a tame smooth action.

I. INTRODUCTION

In gauge theories a first and important step is the study of the action of the group $\mathcal{G}$ of the gauge transformations of a principal bundle $P(M,G)$ on the set $\mathcal{C}$ of principal connections. In fact, according to the gauge principle, physical objects are the classes of gauge equivalent connections rather than connections themselves. In a natural way physicists are virtually forced to look at $\mathcal{G}$ as a smooth group acting on a smooth manifold $\mathcal{C}$.

The problem of endowing these objects with appropriate smoothness structures has been approached essentially on the basis of projective limit techniques (see Ref. 1 and references therein), making use of a rather indirect notion of smoothness and of very reductive assumptions like compactness of the base space M and of the structure group G. A new approach of the Japanese school[2] to infinite continuous groups introduces the "regular" Fréchet–Lie groups. Even in this approach one cannot avoid the compactness hypothesis for M in the treatment of the group $\mathcal{G}$ as a Lie group.

In a previous paper[3] the group $\mathcal{G}$, defined as the group of sections of the associated bundle $P[G]$, has been given the structure of the "Schwartz–Lie" group, i.e., of a Lie group modeled on a Schwartz space, without any assumption of compactness for M and G. In this paper we analyze two other current definitions of the gauge transformation group and show that they are not quite satisfactory from the point of view of smoothness properties, at least in the general case. However, assuming compactness of G we are able to show that the three definitions give isomorphic Lie groups (Sec. II).

In Sec. III we identify the principal connections with sections of a convenient vector bundle on M and again without any assumption of compactness we give $\mathcal{C}$ the structure of a local affine manifold model on a Schwartz space.

In Sec. IV we give the proof of the smoothness of the action of $\mathcal{G}$ on $\mathcal{C}$, in the case of compact M the group $\mathcal{G}$ becomes a tame Fréchet–Lie group and the action a tame smooth action.

The results of this paper, in our opinion interesting by themselves, are a necessary tool for the study of the orbit space $\mathcal{C}/\mathcal{G}$ and its stratification structure. This will be the content of a forthcoming paper.

II. THE GROUP OF GAUGE TRANSFORMATIONS

Our basic object is a principal bundle $P(M,G)\equiv(P,p,M;G)$, where M is an *ordinary* manifold (ordinary manifold means Hausdorff, second countable, and locally compact C^∞-manifold, hence finite-dimensional paracompact and metricizable) and G an *ordinary* Lie group. Throughout the paper we will denote by A the principal action, $A: P\times G\to P$, and by A^a and A_u the partial maps

$$A^a: P\to P,\quad A^a(u)=A(u,a),\quad a\in G,$$
$$A_u: G\to P,\quad A_u(a)=A(u,a),\quad u\in P.$$

We consider the associated bundles $P[G]\equiv(P\times_G G, p_G,M)$ (with fiber G and action of G on it given by inner automorphisms, $a\rightsquigarrow bab^{-1}$) and $P[\mathfrak{g}]\equiv(P\times_G\mathfrak{g},p_{\mathfrak{g}},M)$ (with fiber the Lie algebra $\mathfrak{g}$ of G and action of G on it given by the adjoint representation).

We recall that the total space $P\times_G F$ of an associated bundle $P[F]$ $(P\times_G F,p_F,M)$ with fiber F consists of equivalence classes on $P\times F$ relative to the joint action of G.

We will denote by $[(u,f)]_G$ the equivalence class of the point $(u,f)\in P\times F$. Thus, in the case of $P[G]$,

$$[(u,a)]_G := \{(u',a')\in P\times G\,|\,\exists b\in G:\ (u',a')=(ub,b^{-1}ab)\}$$

and similarly for $[(u,\alpha)]_G$ in the case of $P[\mathfrak{g}]$.

The group $\mathcal{G}$ of gauge transformations of $P(M,G)$ is, by definition, the set Sec $P[G]$ of the (smooth) sections of $P[G]$ with pointwise defined composition law.

It has been proved in Ref. 3 that $\mathcal{G}$ is an NLF–Lie group, that is, a Lie group modeled on a complete locally convex nuclear space, strict inductive limit of a countable

a) Corresponding author.

 0022-2488/86/102469-06$02.50

family of separable Fréchet spaces. More precisely, the results of Ref. 3 can be summarized in the following statements.

(i) $\mathscr{G}$ is an NLF–Lie group.

(ii) The set $\text{Sec}_c\, P[\mathscr{g}]$ of the compact support sections of $P[\mathscr{g}]$ with pointwise defined operations is an NLF–Lie algebra, in fact the Lie algebra $L_{\mathscr{G}}$ of $\mathscr{G}$.

(iii) An exponential map $\text{Exp}: \text{Sec}_c\, P[\mathscr{g}] \to \text{Sec}\, P[G]$ is defined by $(\text{Exp}\,\sigma)\ (x): = \exp \sigma(x)\ \forall x \in M$, where exp: $P \times_G \mathscr{g} \to P \times_G G$ is the fiberwise defined exponential map, which is a local diffeomorphism at 0.

According to these results we more simply say that $\mathscr{G}$ is a Schwartz–Lie group and its Lie algebra a Schwartz–Lie algebra.

From the algebraic point of view, it is well known (see, for instance, Ref. 4) that the group $\mathscr{G}$ is isomorphic with the group $\mathscr{G}^\#$ of those diffeomorphisms f of the total space P of $P(M,G)$ such that

(a) $p \circ f = p$,

(b) $f(ua) = f(u)a, \quad \forall u \in P, \quad \forall a \in G.$

The group $\mathscr{G}^\#$, in turn, is isomorphic with the group $\hat{\mathscr{G}}$ of those maps $\hat{f}: P \to G$ such that

$$\hat{f}(ua) = a^{-1}\hat{f}(u)a, \quad \forall u \in P, \quad \forall a \in G.$$

The isomorphisms are

$$\iota: \mathscr{G} \to \mathscr{G}^\#, \quad \iota(s)(u) = ua,$$

where $a \in G$ is such that $(s \circ p)\ (u) = [(u,a)]_G$, and

$$\hat{\ }: \mathscr{G}^\# \to \hat{\mathscr{G}}, \quad f \rightsquigarrow \hat{f},$$

where $\hat{f}$ is defined by $f(u) = u\hat{f}(u)$.

Obviously, $\mathscr{G}^\#$ is a subgroup of the group Diff P of the diffeomorphisms of P. Now, as shown by Michor,[5] Diff P can be given the structure of NLF–Lie group with Lie algebra the NLF–Lie algebra $\mathscr{X}_c(P)$ of vector fields on P with compact support. It is easy to see that $\mathscr{G}^\#$ is closed in the FD-topology, which is the toplogy underlying the differential structure of Diff P. Under this topology the connected component of the identity contains only diffeomorphisms with compact support. If G is not compact, the only element of $\mathscr{G}^\#$ with compact support is the identity itself, owing to the equivariance property, therefore, in this case $\mathscr{G}^\#$ is a discrete subgroup of Diff P.

Analogously, $\hat{\mathscr{G}}$ is a closed subgroup of the Schwartz–Lie group $C^\infty(P,G)$ (see Ref. 3) and again, if G is not compact, $\hat{\mathscr{G}}$ is a discrete subgroup of $C^\infty(P,G)$.

From these remarks it clearly appears that to consider the gauge transformations as diffeomorphisms of P can be unsatisfactory. Indeed they are bundle automorphisms and only the group $\mathscr{G}$ fits completely this character, since, from the categorical point of view, bundle morphisms must be looked at as sections of a suitable bundle.

If the structure group G is compact, however, $\mathscr{G}^\#$ is a Lie subgroup of Diff P and $\hat{\mathscr{G}}$ a Lie subgroup of $C^\infty(P,G)$ as shown in the following theorems.

Theorem 2.1: If G is compact, $\mathscr{G}^\#$ is a splitting Lie subgroup of Diff P and its Lie algebra $\mathscr{X}_c^\#(P)$ is the splitting subalgebra of $\mathscr{X}_c(P)$ consisting of the vertical G-invariant vector fields on P with compact support.

Proof: First we note that the subspace $\mathscr{X}_c^V(P)$ of the vertical vector fields splits $\mathscr{X}_c(P)$. We introduce then the linear operator[6]

$$\int_G: \mathscr{X}_c^V(P) \to \mathscr{X}_c^V(P),$$

$$\left(\int_G X\right)(u): = \int_G (X \cdot a)(u)\, d\mu(a),$$

where $X \cdot a$ is the induced right action of G on vector fields of P and μ is the normalized Haar measure on the compact group G. It is immediate that $\int_G$ is a continuous projection onto $\mathscr{X}_c^\#(P)$, the subspace of $\mathscr{X}_c^V(P)$ consisting of the G-invariant elements. This shows that $\mathscr{X}_c^\#(P)$ is a splitting subspace of $\mathscr{X}_c(P)$; moreover, by standard arguments, $\mathscr{X}_c^\#(P)$ turns out to be a Lie subalgebra of $\mathscr{X}_c(P)$.

Now we recall that Diff P is a NLF–Lie group and that a chart at the identity $\mathbf{e}$ is given by $(U_{\mathbf{e}}^\tau, \chi, \mathscr{X}_c(P))$, where

(1) τ is a local addition on P,

(see, for definition, Ref. 3 or Ref. 5);

(2) $U_{\mathbf{e}}^\tau = \{f \in \text{Diff}\, P \mid f \sim \mathbf{e},\ f(u) \in \tau_u(T_uP)\}$

[$f_1 \sim f_2$ means that the set $\{u \in P \mid f_1(u) \neq f_2(u)\}$ is relatively compact]; and

(3) $\chi: U_{\mathbf{e}}^\tau \to \mathscr{X}_c(P), \quad \chi(f): = X,$

with $X(u) = \tau_u^{-1}(f(u))$.

As we will show in the subsequent Lemma 2.2, there exists a local addition τ on P such that

(i) $\tau \circ TA^a = A^a \circ \tau$;

and (ii) the fibers of P are additively closed, i.e.,

$$\tau(\text{Ver}_u\, P) \subset P_x, \quad x \in p(u).$$

For such a local addition we have

$$\chi(U_{\mathbf{e}}^\tau \cap \mathscr{G}^\#) = \mathscr{X}_c^\#(P).$$

Actually, if $X = \chi(f)$ with $f \in U_{\mathbf{e}}^\tau \cap \mathscr{G}^\#$, then $X(u) \in \text{Ver}_u P$ since $f(u) \in P_x$ and fibers are additively closed and X is G-invariant since τ is equivariant; vice versa, if $X \in \mathscr{X}_c^\#(P)$, the map $f = \tau \circ X$ is a diffeomorphism of P since χ is surjective and satisfies

$$f(ua) = (\tau \cdot X)(ua) = \tau((TA^a \circ X)(u))$$
$$= (\tau \circ X)(u)a = f(u)a,$$
$$(p \circ f)(u) = p(\tau(X(u))) = p(u),$$

hence $f \in \mathscr{G}^\#$. Thus $\mathscr{G}^\#$ is a splitting submanifold of Diff P, hence a Lie subgroup of Diff P and its Lie algebra is the splitting Lie subalgebra $\mathscr{X}_c^\#(P)$ of $\mathscr{X}_c(P)$. □

Lemma 2.2: Let $P(M,G)$ be a principal fiber bundle with principal action A. There exists a local addition τ on P satisfying conditions (i) and (ii) above.

Proof: Take a G-invariant partition of unity $\{f_\alpha\}$ of P subordinated to a local trivializing system $\{(U_\alpha, \varphi_\alpha)\}$. If ξ_G is the (right) invariant spray on G and ξ_α any spray on U_α, then

$$\xi = \sum_\alpha f_\alpha \xi_\alpha \oplus \xi_G$$

is a G-invariant spray on P. The corresponding exponential map $\exp^{\xi}$ is equivariant and defined on an open G-invariant neighborhood V of the zero section of TP. Using a G-invariant metric on P, a contracting diffeomorphism h: $TP \to h(TP) \subset V$ with $h(0_u) = 0_u$, $\pi_P \circ h = \pi_P$, where π_P: $TP \to P$, can be constructed as in 10.2 of Michor[5], which, moreover, is equivariant. Therefore $\tau = \exp^{\xi} \circ h$ satisfies (i). As to (ii), it is enough to remark that the spray ξ, when restricted to the fiber P_x over x, gives a spray on P_x and the diffeomorphism h preserves Ver P.

Theorem 2.3: If G is compact, $\hat{\mathcal{G}}$ is a splitting Lie subgroup of $C^\infty(P,G)$ and its Lie algebra is the splitting subalgebra $C^\infty_{cG}(P,\mathfrak{g})$ of the Lie algebra $C^\infty_c(P,\mathfrak{g})$ of $C^\infty(P,G)$ consisting of those maps $\hat{\varphi}$: $P \to \mathfrak{g}$ with compact support such that $\hat{\varphi}(ua) = \mathrm{Ad}_{a^{-1}}(\hat{\varphi}(u))$.

Proof: The linear operator

$$\int_G : C^\infty_c(P,\mathfrak{g}) \to C^\infty_c(P,\mathfrak{g}),$$

$$\left(\int_G \varphi\right)(u) := \int_G \mathrm{Ad}_a(\varphi(ua))d\mu(a)$$

is clearly a continuous projection onto $C^\infty_{cG}(P,\mathfrak{g})$ and this shows that $C^\infty_{cG}(P,\mathfrak{g})$ is a splitting subspace of $C^\infty_c(P,\mathfrak{g})$; in fact $C^\infty_{cG}(P,\mathfrak{g})$ is a Lie subalgebra of $C^\infty_c(P,\mathfrak{g})$ since the Lie bracket is pointwise defined. A chart at the identity of the NLF–Lie group $C^\infty(P,G)$ is given by $(U_e, \chi, C^\infty_c(P,V) \subset C^\infty_c(P,\mathfrak{g}))$, where, if $\exp_G: \mathfrak{g} \to G$ is the exponential map of G, V is a zero neighborhood in $\mathfrak{g}$ such that $\exp_G \restriction_V$: $V \to \exp_G(V) \equiv W \subset G$ is a diffeomorphism,

$$U_e = \{f \in C^\infty(P,G) \mid f \sim e,\ f(P) \subset W\}$$

and

$$\chi: U_e \to C^\infty_c(P,V)$$

is given by $\chi(f) = \log_G \circ f$, $\log_G = (\exp_G \restriction_V)^{-1}$: $W \to V$. We may assume that V is invariant under the adjoint action (e.g., an open ball with respect to a G-invariant metric on $\mathfrak{g}$) so W is invariant under conjugation. Then

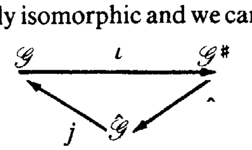
$$\hat{\mathcal{G}} \cap U_e = \hat{U}_e$$
$$:= \{f \in C^\infty(P,G) \mid f \sim e,$$
$$f(P) \subset V,\ f(ua) = a^{-1} f(u) a,\ \forall a \in G\}.$$

Clearly

$$\chi(\hat{U}_e) = \{\varphi \in C^\infty(P,\mathfrak{g}) \mid \varphi \sim 0,$$
$$\varphi(ua) = \mathrm{Ad}_{a^{-1}} \varphi(u),\ \varphi(U) \subset V\}$$
$$= \chi(U_e) \cap C^\infty_{cG}(P,\mathfrak{g}).$$

Then $\hat{\mathcal{G}}$ is a splitting submanifold of $C^\infty(P,G)$, hence a Lie subgroup with Lie algebra the subalgebra $C^\infty_{cG}(P,\mathfrak{g})$ of $C^\infty_c(P,\mathfrak{g})$. □

As remarked above the groups $\mathcal{G}$, $\mathcal{G}^\#$, $\hat{\mathcal{G}}$ are algebraically isomorphic and we can consider the following diagram:



where the isomorphism j is given by

$$h(\hat{f})(x) := [(u,\hat{f}(u))]_G,\quad u \in p^{-1}(x).$$

Now we know that in the case of compact G the three groups are NLF–Lie groups. The following result is expected.

Theorem 2.4: If G is compact, the $\mathcal{G}$, $\mathcal{G}^\#$, and $\hat{\mathcal{G}}$ are isomorphic as NLF–Lie groups.

Proof: We must just prove that the maps in the above diagram are smooth.

(a) The map ι is smooth. Introduce the smooth map

$$r: (P \times_G G) \times_M P \to P,\quad r([(u,a)]_G, u)) := ua,$$

where $(P \times_G G) \times_M P$ is the total space of the fiber product of the bundles $P[G]$ and $P(M,G)$, and consider the following maps:

$$\iota_0: \mathcal{G} \to C^\infty(M, P \times_G G), \text{ the canonical embedding,}$$

$$\iota_1: C^\infty(M, P \times_G G) \to C^\infty(P, P \times_G G),\quad f \rightsquigarrow f \circ p,$$

$$\iota_2: C^\infty(P, P \times_G G) \to C^\infty(P, (P \times_G G) \times P),$$

$$(\iota_2(f))(u) := (f(u), u),$$

$$\iota_3: C^\infty(P, (P \times_G G) \times_M P) \to C^\infty(P,P),\quad \psi \times \eta \rightsquigarrow r \circ (\psi \circ \eta).$$

The maps ι_1 and ι_3 are smooth by Theorem 11.4 of Ref. 5; the map ι_2 is smooth by Proposition 10.5 of Refs. 5; finally, ι_0 is smooth by Proposition 10.10 of Ref. 5. Note that $\iota_2 \circ \iota_1 \circ \iota_0$ takes the values in the submanifold $C^\infty(P, (P \times_G G) \times_M P)$ of $C^\infty(P, (P \times_G G) \times P)$ and that $\iota = \iota_3 \circ \iota_2 \circ \iota_1 \circ \iota_0$.

(b) The map ˆ is smooth. Introduce the smooth map

$$\nu: P \times_M P \to G,\quad \nu(u,v) := a,\quad \text{where } ua = v,$$

and consider the following maps:

$$\kappa_0: \mathcal{G}^\# \to C^\infty(P,P), \quad \text{the canonical embedding,}$$

$$\kappa_1: C^\infty(P,P) \to C^\infty(P, P \times P),\quad (\kappa_1(f))(u) := (f(u),u),$$

$$\kappa_2: C^\infty(P, P \times_M P) \to C^\infty(P,G),\quad \kappa_2(f) = \nu \circ f.$$

Note that $\hat{} = \kappa_2 \circ \kappa_1 \circ \kappa_0$; its smoothness follows by the same arguments as at the end of (a).

(c) The map j is smooth. As is shown in Ref. 3 we can use as charts at the identities of the groups $\hat{\mathcal{G}}$ and $\mathcal{G}$ the canonical charts using the exponential mappings

$$\mathrm{Exp}: L_{\mathcal{G}} \equiv \mathrm{Sec}_c\, P[\mathfrak{g}] \to \mathcal{G},\quad (\mathrm{Exp}\,\lambda)(x) = \exp_x(\lambda(x)),$$

where exp: $P \times_G \mathfrak{g} \to P \times_G G$ is the pointwise defined exponential map, and

$$\widehat{\mathrm{Exp}}: L_{\hat{\mathcal{G}}} \equiv C^\infty_{cG}(P,\mathfrak{g}) \to \hat{\mathcal{G}},\quad (\widehat{\mathrm{Exp}}\,\hat{\sigma})(x) = \exp_G(\hat{\sigma}(u)).$$

The two charts are clearly j-correlated and the local expression of j is the continuous linear operator

$$C^\infty_{cG}(P,\mathfrak{g}) \ni \hat{\lambda} \rightsquigarrow \lambda \in \mathrm{Sec}_c\, P[\mathfrak{g}]$$

with

$$\lambda(x) = [(u, \hat{\lambda}(u))]_G,\quad u \in p^{-1}(x).$$

Hence the isomorphism j is smooth. □

We remark that, taking into account the properties of the exponential map of $\mathscr{G}$,[3] Theorem 2.4 shows that the exponential map of the group Diff P restricted to $\mathscr{G}^{\#}$ is a local diffeomorphism in the case of compact G.

We conclude this section calling attention to two interesting properties of the Schwartz–Lie group $\mathscr{G}$.

(1) The group $\mathscr{G}$ has no small subgroups; this can be easily seen and essentially stems from the fact that the group G, as every ordinary Lie group, has the same property.[7]

(2) The group $\mathscr{G}$ is analytic and the Baker–Campbell–Hausdorff formula holds

$$\operatorname{Exp}\sigma \operatorname{Exp}\sigma' = \operatorname{Exp}\{(\sigma+\sigma') + \tfrac{1}{2}[\sigma,\sigma'] + \tfrac{1}{12}([\sigma,[\sigma,\sigma']] - [\sigma',[\sigma,\sigma']] + \cdots\},$$

for every σ,σ' in a suitable neighborhood of 0 in $L_{\mathscr{G}}$.

This can be seen rather easily using the canonical atlas defined by the exponential map and again remembering that the same property holds for the group G.

Obviously also the group $\mathscr{G}^{\#}$ and $\hat{\mathscr{G}}$ are these two properties in the case of compact G.

III. THE MANIFOLD OF PRINCIPAL CONNECTIONS

In gauge theories an important step is the study of the action of $\mathscr{G}$ on the space $\mathscr{C}$ of principal connections. Usually this action is introduced essentially as a pullback via the gauge transformations considered as diffeomorphisms of P.

As pointed out in Sec. II, the gauge transformations are in fact bundle automorphisms and this point of view is perhaps the only suitable way, in the general case, to treat smoothness properties of the group of gauge transformations.

Accordingly it might be convenient to look at connections too as sections of a suitable bundle over the base space M. This is just the aim of this section.

We need some preliminaries.

As is well known the tangent space TG of the Lie group G with multiplication $\mu: G\times G\to G$ can be given a Lie group structure with multiplication $T\mu$.

The group TG can be made to act on the Lie algebra of G by introducing the affine action

$$B: TG\times\mathfrak{g}\to\mathfrak{g},\quad B(\alpha_a,\delta):\mathrm{Ad}_a\delta-\alpha,$$

where $\alpha_a\in T_aG$, $\alpha\in\mathfrak{g}$, and $\alpha_a=(T_eR_a)\alpha$.

Moreover the group TG can be considered as the structure group of the tangent principal bundle $TP(TM,TG) \equiv (TP,Tp,TM;TG)$. Actually if $(U_\alpha,\varphi_\alpha)$ is a trivializing system for $P(M,G)$ with transition functions $\varphi_{\alpha\beta}$, then $(TU_\alpha,T\varphi_\alpha)$ is a trivializing system for $TP(TM,TG)$ with transition functions $T\varphi_{\alpha\beta}$.

We recall that a connection one-form ω on the principal bundle $P(M,G)$ is a $\mathfrak{g}$-valued one-form on P such that

(a) $\omega(\alpha_u^*)=\alpha,\quad \forall u\in P,$

for every fundamental vector field α^* on P, i.e.,

$$\alpha_u^*=(T_eA_u)\alpha,\quad \alpha\in\mathfrak{g};$$

and

(b) $\omega\circ TA^a=\mathrm{Ad}_a-1\circ\omega,\quad \forall a\in G.$

Looking at ω as a map from TP into $\mathfrak{g}$ we can investigate its equivariant properties with respect to the actions of TG on TP and $\mathfrak{g}$.

We have, with $\xi_u\in T_uP$, $\alpha_a\in T_aG$, and $\alpha\in\mathfrak{g}$ such that $\alpha_a=(T_e,R_a)\alpha$,

$$\begin{aligned}
\omega(T_{(u,a)}A(\xi_u,\alpha_a))
&=\omega((T_uA^a)(\xi_u)+(T_aA_u)(\alpha_a))\\
&=\mathrm{Ad}_{a^{-1}}\omega(\xi_u)+\omega((T_aA_u)((T_eR_a)\alpha))\\
&=\mathrm{Ad}_{a^{-1}}\omega(\xi_u)+\omega((T_uA^a\circ T_eA_u)\alpha)\\
&=\mathrm{Ad}_{a^{-1}}\omega(\xi_u)+\mathrm{Ad}_{a^{-1}}\omega((T_eA_u)\alpha)\\
&=\mathrm{Ad}_{a^{-1}}\omega(\xi_u)+\mathrm{Ad}_{a^{-1}}\alpha\\
&=B((\alpha_a)^{-1},\omega(\xi_u)).
\end{aligned}$$

Thus connection one-forms can be considered as (particular) B-type $\mathfrak{g}$-valued maps on TP. It is well known that the B-type $\mathfrak{g}$-valued maps on TP correspond bijectively to the sections of the bundle $TP[\mathfrak{g}]$ associated to the principal bundle $TP(TM,TG)$. To get a precise characterization of connection one-forms we must investigate this associated bundle.

First of all we remark that $TP[\mathfrak{g}]=(TP\times_{TG}\mathfrak{g},Tp_{\mathfrak{g}},TM)$ is an affine bundle, that is, a bundle of affine spaces; actually the action B of TG on $\mathfrak{g}$ is affine. The transition functions $\psi_{\alpha\beta}$ take values in the group of affine transformations of $\mathfrak{g}$ and are given by

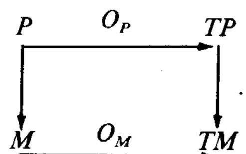
$$\begin{aligned}
\psi_{\alpha\beta}(\xi_x)\delta&=B((T_x\varphi_{\alpha\beta})\xi_x,\delta)\\
&=\mathrm{Ad}\varphi_{\alpha\beta}(x)\delta-(d_x\varphi_{\alpha\beta})\xi_x,\quad \xi_x\in TU_{\alpha\beta},
\end{aligned}$$

where $d_x\varphi_{\alpha\beta}=(T_eR\varphi_{\alpha\beta(x)})^{-1}\circ T_x\varphi_{\alpha\beta}$ is the (right) logarithmic derivative of the transition function $\varphi_{\alpha\beta}$ of the principal bundle $P(M,G)$.

We now introduce the fiber bundle $TP_M[\mathfrak{g}]\equiv(TP\times_{TG}\mathfrak{g},\pi_M\circ Tp,M)$, where $\pi_M: TM\to M$ is the projection of the tangent bundle of M.

By standard arguments it can be seen that $TP_M[\mathfrak{g}]$ is a *vector* bundle for which the fiber over x is $T_xM\times\mathfrak{g}$.

If $O_P: P\to TP$, $O_M: M\to TM$, and $O_G: G\to TG$ are the zero sections of the corresponding tangent bundles, the following diagram commutes:



The pair (O_P,O_G) is an injection of principal bundles over O_M, so it induces uniquely a map

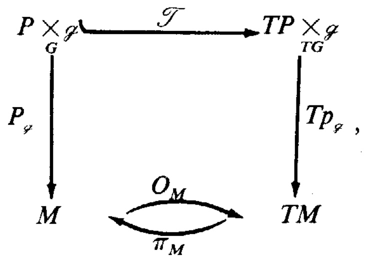
$$\mathscr{T}: P\times_G\mathfrak{g}\to TP\times_{TG}\mathfrak{g},$$

which is a bundle injection over O_M of the associated bundles $P[\mathfrak{g}]$ and $TP[\mathfrak{g}]$. Moreover $\mathscr{T}$ is a vector bundle injection of $P[\mathfrak{g}]$ and $TP_M[\mathfrak{g}]$ over id_M.

Now we can prove the following decomposition theorem, which will be very important later on.

Theorem 3.1: $TP_M[\mathfrak{g}]\cong P[\mathfrak{g}]\oplus TM$, that is, the vector bundle $TP_M[\mathfrak{g}]$ is the Whitney sum of $P[\mathfrak{g}]$ and the tangent bundle of M.

Proof: The image $\mathscr{T}(P[\mathscr{g}])$ of $P[\mathscr{g}]$ is a subbundle of $TP_M[\mathscr{g}]$. Hence there exists a Whitney complement of $\mathscr{T}(P[\mathscr{g}])$, i.e., a subbundle W of $TP_M[\mathscr{g}]$ such that $TP_M[\mathscr{g}] = \mathscr{T}(P[\mathscr{g}]) \oplus W$. Looking at the diagram

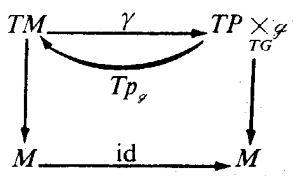


we get $Tp_\varphi \circ \mathscr{T} = O_M \circ p_\varphi$, so that $\operatorname{Im} \mathscr{T} \subset \operatorname{Ker} Tp_\varphi$, Tp_φ being a vector bundle morphism over M. But $\dim \operatorname{Im} \mathscr{T} = \dim \mathscr{g} = \dim \operatorname{Ker} Tp_\varphi$, so $\operatorname{Im} \mathscr{T} = \operatorname{Ker} Tp_\varphi$. Hence Tp_φ induces a vector bundle isomorphism of W and TM. □

Now we can give a precise characterization of the connections on $P(M,G)$ among the sections of $TP[\mathscr{g}]$. To every connection one-form ω there corresponds a section γ of $TP[\mathscr{g}]$ with

$$\gamma(\xi_x) = [(\xi_u, \omega(\xi_u))]_{TG},$$

where $Tp(\xi_u) = \xi_x$. Clearly the connections are exactly those sections γ of $TP[\mathscr{g}]$ that satisfy the following diagram:



and are linear on the fibers, that is on those sections of $TP[\mathscr{g}]$ which are also vector bundle morphisms over the identity of TM and $TP_M[\mathscr{g}]$.

Thus we have, denoting by $\mathscr{C}$ the set of the principal connections,

$$\mathscr{C} \equiv \{\gamma \in \operatorname{Sec} L(TM, TP_M[\mathscr{g}]), \quad Tp_\varphi \circ \gamma(x) = \mathbb{1}_x, \ \forall x \in M\},$$

where $L(TM, TP_M[\mathscr{g}])$ is the vector bundle over M whose fiber at x consists of the linear maps from $T_x M$ into $(\pi_M \circ Tp_\varphi)^{-1}(x)$ and $\mathbb{1}_x$ is the identity operator on $T_x M$.

On the basis of the above identification we can give $\mathscr{C}$ a suitable differentiable structure.

It is shown in Ref. 5, Proposition 10.10, that the vector space $\operatorname{Sec} E$ of the sections of an ordinary vector bundle (E, π, X) is a splitting submanifold of $C^\infty(X,E)$ modeled on the NLF-space $\operatorname{Sec}_c E$. For any $s \in \operatorname{Sec} E$, the set $s + \operatorname{Sec}_c E$ is an open neighborhood of s in FD-topology and an affine subspace, which is isomorphic to $\operatorname{Sec}_c E$; for this reason $\operatorname{Sec} E$ is called a local topological affine space. We now prove that $\mathscr{C}$ is an affine subspace and a splitting submanifold (shortly a local topological affine splitting subspace) of the local topological affine space $\operatorname{Sec} L(TM, TP_M[\mathscr{g}])$.

Theorem 3.2: $\mathscr{C}$ is a topological affine splitting subspace of $\operatorname{Sec} L(TM, TP_M[\mathscr{g}])$ isomorphic to $\operatorname{Sec} L(TM, P[\mathscr{g}])$ as topological affine space.

Proof: Fix $\gamma_0 \in \mathscr{C}$; since $Tp_\varphi \circ (\gamma(x) - \gamma_0(x)) = 0_x$, $\forall x \in M$, implies $\gamma - \gamma_0 \in \operatorname{Ker} Tp_\varphi = \operatorname{Im} \mathscr{T}$, then $\gamma - \gamma_0 \in \operatorname{Sec} L(TM, \mathscr{T}(P[\mathscr{g}]))$. Conversely, let $\sigma \in \operatorname{Sec} L(TM, \mathscr{T}(P[\mathscr{g}]))$. One can easily prove that $\gamma_0 + \sigma \in \mathscr{C}$; moreover, by Theorem 3.1, $\operatorname{Sec} L(TM, TP_M[\mathscr{g}]) \cong \operatorname{Sec} L(TM, TM) \oplus \operatorname{Sec} L(TM, P[\mathscr{g}])$ as locally convex vector spaces, hence $\operatorname{Sec} L(TM, P[\mathscr{g}])$ is a splitting submanifold of $\operatorname{Sec} L(TM, TP_M[\mathscr{g}])$ modeled on $\operatorname{Sec} L(TM, P[\mathscr{g}])$. □

Remark: There is another interesting way to look at connections of $P(M,G)$; they can be considered as reductions of the principal bundle $TP(TM,TG)$ to the subgroup G of the structure group TG. This follows from the fact that the bundles $TP[\mathscr{g}]$ and TP/G are isomorphic and by Proposition 5.6 (Chap. I of Ref. 8).

Here we will not exploit further this point of view on connections.

IV. SMOOTHNESS OF THE ACTION OF $\mathscr{G}$ ON $\mathscr{C}$

In order to prove the smoothness of the action of $\mathscr{G}$ on $\mathscr{C}$ we need some preliminary results of geometric nature.

Given the principal bundle $P(M,G)$, let $Z: G \times F \to F$ be the left action of G on a manifold F which defines the associated bundle $P[F]$; then $TZ: TG \times TF \to TF$ is again a left action and defines the associated bundle $TP[TF]$.

The following theorem can be proved by standard arguments.

Theorem 4.1: (a) The associated bundle $TP[TF]$ is isomorphic to the tangent bundle $(T(P \times_G F), Tp_F, TM)$ of the bundle $P[F]$.

(b) The triple $\Theta(P \times_G F) = (TP \times_{TG} TF, \vartheta_F, P \times_G F)$, where $\vartheta_F([(\xi_u, \xi_f)]_{TG}) = [(u,f)]_G$, is a vector bundle isomorphic to the tangent bundle of the manifold $P \times_G F$.

Given two vector bundles $\xi_1 = (E, \pi_1, X)$ and $\xi_2 = (F, \pi_2, Y)$, we recall that $L(\xi_1, \xi_2)$ stands for the vector bundle $(L(E,F), \alpha \times \omega, X \times Y)$, where $L(E,F)_{(x,y)}$ consists of the linear maps $L_{(x,y)}: E_x \to F_y$ and $(\alpha \times \omega)(L_{(x,y)}) = (x,y)$. We denote by $L_X(\xi_1, \xi_2)$ the bundle over X obtained by the composition of $\alpha \times \omega$ with the canonical projection on X.

Moreover, if X and Y are smooth manifolds, the one-jet map $j^1: C^\infty(X,Y) \to C^\infty(X, J^1(X,Y))$ is a smooth map by Proposition 11.1 of Ref. 5. Now, identifying $J^1(X,Y)$ with $L(TX,TY)$, we remark that the map j^1 takes values in the splitting submanifold $\operatorname{Sec} L_X(TX,TY)$ of $C^\infty(X, L(TX,TY))$. Therefore the map

$$j^1: \mathscr{G} = \operatorname{Sec} P[G] \to \operatorname{Sec} L_M(TM, T(P \times_G G))$$

is a smooth map and by Theorem 4.1 we can consider it as a map

$$j^1: \mathscr{G} \to \operatorname{Sec} L_M(TM, \Theta(P \times_G G)).$$

Coming to the action of $\mathscr{G}$ on $\mathscr{C}$, first we recall that in the definition of $TP \times_{TG} TG$ we use the action of TG on itself by inner automorphisms $\alpha_a \rightsquigarrow \beta_b \alpha_a \beta_b^{-1}$ and in $TP \times_{TG} \mathscr{g}$ the above defined action B: $TG\times\mathscr{g}\to\mathscr{g}$. One can easily check that

$$B(\beta_b\alpha_a\beta_b^{-1},B(\beta_b,\delta))=B(\beta_b,B(\alpha_a,\delta)).$$

Then the following "fibered action" is well defined:

$$\widetilde{B}: (TP\underset{TG}{\times}TG)\underset{TM}{\times}(TP\underset{TG}{\times}\mathscr{g})\to TP\underset{TG}{\times}\mathscr{g},$$

$$\widetilde{B}([(\xi_u,\alpha_a)]_{TG},[(\xi_u,\delta)]_{TG}):=[(\xi_u,\mathrm{Ad}_a\delta-\alpha)]_{TG},$$

where $\alpha_a=T_eR_a\cdot\alpha$.

Using $\widetilde{B}$ we define the left action of $\mathscr{G}$ on $\mathscr{C}$

$$\widetilde{A}: \mathscr{G}\times\mathscr{C}\to\mathscr{C},\quad \widetilde{A}(s,\gamma)=\tilde{\gamma},$$

where

$$\tilde{\gamma}(\xi_x):=\widetilde{B}((j^1s)(\xi_x),\gamma(\xi_x)),\quad \text{for } \xi_x\in T_xM.$$

Theorem 4.2: The action $\widetilde{A}$ is smooth.

Proof: We can decompose $\widetilde{A}$ as follows:

$$\mathscr{G}\times\mathscr{C}\xrightarrow{j^1\times i}\mathrm{Sec}\,L_M(TM,\Theta(P\underset{G}{\times}G))\times\mathrm{Sec}\,L(TM,TP_M[\mathscr{g}])$$

$$\xrightarrow{\mathrm{Comp}_{\widetilde{B}}}\mathrm{Sec}\,L(TM,TP_M[\mathscr{g}])\xrightarrow{i^{-1}}\mathscr{C},$$

where i: $\mathscr{C}\to\mathrm{Sec}\,L(TM,TP_M[\mathscr{g}])$ is the canonical inclusion and use of the fact that $\mathrm{Comp}_{\widetilde{B}}(j^1\times i)(\mathscr{G}\times\mathscr{C})\subseteq\mathrm{Im}\,i$ is made. We have just recalled that j^1 is a smooth map; the inclusion i is an embedding by Theorem 3.2 and $\mathrm{Comp}_{\widetilde{B}}$ is smooth by Proposition 11.4 of Ref. 5. □

Remark: We recall that, for every $\gamma\in\mathscr{C}$, $\gamma(\xi_x)=[(\xi_u,\omega(\xi_u))]_{TG}$, where $Tp(\xi_u)=\xi_x$ and ω is the connection one-form corresponding to γ. Analogously if $s\in\mathscr{G}$ there exists an $\hat{f}\in\hat{\mathscr{G}}$ such that $s(x)=[(u,\hat{f}(u))]_G$ with $u\in p^{-1}(x)$. Moreover

$$(j^1s)(\xi_x)=(Ts)(\xi_x)=[(\xi_u,(T\hat{f})(\xi_u))]_{TG}$$

so that we have

$$\widetilde{B}([(\xi_u,(T_\cdot\hat{f})(\xi_u))]_{TG},[(\xi_u,\omega,\xi_u))]_{TG})$$
$$=[(\xi_u,\mathrm{Ad}_{\hat{f}(u)}\omega(\xi_u)-(d\hat{f})(\xi_u))]_{TG},$$

where $d\hat{f}=(T_eR_{\hat{f}(u)})^{-1}\circ T_u\hat{f}$ is the (right) logarithmic derivative of $\hat{f}$ at u.

If we change the left action into a right action

$$\overline{A}: \mathscr{C}\times\mathscr{G}\to\mathscr{C},\quad \overline{A}(\gamma,s)=\bar{\gamma}:=\widetilde{A}(s^{-1},\gamma),$$

we have

$$\bar{\gamma}(\xi_x)=[(\xi_u,\mathrm{Ad}_{(\hat{f}(u))^{-1}}\omega(\xi_u)-(d\hat{f})^{-1}(\xi_u)]_{TG}$$
$$=[(\xi_u,\mathrm{Ad}_{\hat{f}(u))^{-1}}\omega(\xi_u)$$
$$+(T_{\hat{f}(u)}L_{(\hat{f}(u))^{-1}}\circ T_u\hat{f})(\xi_u)]_{TG},$$

since

$$(df)^{-1}(\xi_u)=((T_eR_{(\hat{f}(u))^{-1}})^{-1}\circ T_u\hat{f}^{-1}(\xi_u)$$
$$=(T_{\hat{f}(u)^{-1}}R_{\hat{f}(u)}\circ T_u\hat{f}^{-1})(\xi_u)$$
$$=-(T_{\hat{f}(u)}L_{(\hat{f}(u))^{-1}}\circ T_u\hat{f})(\xi_u).$$

In the expression

$$\mathrm{Ad}_{(\hat{f}(u))^{-1}}\omega(\xi_u)+(T_{\hat{f}(u)}L_{(\hat{f}(u))^{-1}}\circ T_u\hat{f})(\xi_u),$$

one can easily recognize the usual transformation $f^*\omega$ of the one-form ω via pullback with the automorphism f defined by $f(u)=u\hat{f}(u)$ (i.e., the corresponding element of $\mathscr{G}^\#$).

Once the smoothness of the action $\widetilde{A}$ has been proved, a natural development is the investigation of the properties of the orbits and the structure of the orbit space. In this context the main difficulties one is faced with arise from the lack of inverse map theorems for manifolds modeled on locally convex vector spaces more general than Banach spaces. Perhaps for this reason it is common in physical applications to retire to Banach manifolds or to chains of Banach manifolds. However, a workable version of the inverse map theorem (the Nash–Moser theorem) is now available for a significant subcategory of Fréchet manifolds called "tame Fréchet manifolds" by Hamilton.[9]

Now, if the base manifold M is assumed to be compact, the group $\mathscr{G}$ clearly becomes a nuclear Fréchet–Lie group and $\mathscr{C}$ a splitting affine subspace of a nuclear Fréchet space. Actually we can show that $\mathscr{G}$ is a tame Fréchet–Lie group, $\mathscr{G}$ a tame Fréchet manifold, and the action a tame smooth action.

To some extent, moreover, even the case of noncompact M can be handled: the connected component of the unit of $\mathscr{G}$ can be shown to be a strict inductive limit (in the category of topological groups) of tame Fréchet–Lie groups.

As a consequence of the tameness properties we can prove, in general, that every locally compact subgroup of $\mathscr{G}$ is a splitting Lie subgroup. This result appears as a generalization to $\mathscr{G}$ of a classical Cartan theorem and will be useful in the study of stability subgroups of the action $\mathscr{G}$ on $\mathscr{C}$.

[1]W. Kondracki and J. Rogulski, "On the stratification of the orbit space for the action of automorphisms on connections," Polish Academy of Sciences, Warsawa preprint 281 IMPAN 62/83/160, 1983.

[2]O. Kobayashi, A. Yoshika, and H. Omori, "The theory of infinite dimensional Lie groups and its applications," Acta Appl. Math. **3**, 71 (1985).

[3]R. Cirelli and A. Manià, "The group of gauge transformations as a Schwartz–Lie group," J. Math. Phys. **26**, 3036 (1985).

[4]V. Guillemin and S. Sternberg, *Symplectic Techniques in Physics* (Cambridge U.P., Cambridge, 1984).

[5]P. W. Michor, "Manifolds of differentiable mappings," *Shiva Mathematics Series*, Vol. 3 (Shiva, Orpington, Kent, 1980).

[6]W. Greub, S. Halperin, and R. Vanstone, *Connections, Curvature and Cohomology* (Academic, New York, 1972), Vols. I and II.

[7]D. Montgomery and L. Zippin, "Topological transformation groups," in *Interscience Tracts in Pure and Applied Mathematics*, Vol. 15 (Interscience, New York, 1963).

[8]S. Kobayashi and K. Nomizu, "Foundations of differential geometry. I,II," *Interscience Tracts in Pure and Applied Mathematics*, Vol. 15 (Interscience, New York, 1963).

[9]R. S. Hamilton, "The inverse function theorem of Nash and Moser," Bull. Am. Math. Soc. **7**, 65 (1982).