Differential Geometry and its Applications 1 (1991) 391–401
North-Holland

THE ACTION OF THE DIFFEOMORPHISM GROUP ON THE SPACE OF IMMERSIONS

Vicente Cervera
Francisca Mascaró
Peter W. Michor

Departamento de Geometría y Topología
Facultad de Matemáticas
Universidad de Valencia

Institut für Mathematik, Universität Wien,
Strudlhofgasse 4, A-1090 Wien, Austria.

1989

Abstract. We study the action of the diffeomorphism group $\mathrm{Diff}(M)$ on the space of proper immersions $\mathrm{Imm}_{\mathrm{prop}}(M, N)$ by composition from the right. We show that smooth transversal slices exist through each orbit, that the quotient space is Hausdorff and is stratified into smooth manifolds, one for each conjugacy class of isotropy groups.

Table of contents

1991 *Mathematics Subject Classification.* 58D05, 58D10.
Key words and phrases. immersions, diffeomorphisms.
This paper was prepared during a stay of the third author in Valencia, by a grant given by Cosellería de Cultura, Educación y Ciencia, Generalidad Valenciana. The two first two authors were partially supported by the CICYT grant n. PS87-0115-G03-01.

Introduction

Let M and N be smooth finite dimensional manifolds, connected and second countable without boundary such that $\dim M \le \dim N$. Let $\operatorname{Imm}(M,N)$ be the set of all immersions from M into N. It is an open subset of the smooth manifold $C^\infty(M,N)$, see our main reference [Michor, 1980c], so it is itself a smooth manifold. We also consider the smooth Lie group $\operatorname{Diff}(M)$ of all diffeomorphisms of M. We have the canonical right action of $\operatorname{Diff}(M)$ on $\operatorname{Imm}(M,N)$ by composition.

The space $\operatorname{Emb}(M,N)$ of embeddings from M into N is an open submanifold of $\operatorname{Imm}(M,N)$ which is stable under the right action of the diffeomorphism group. Then $\operatorname{Emb}(M,N)$ is the total space of a smooth principal fiber bundle with structure group the diffeomorphism group; the base is called $B(M,N)$, it is a Hausdorff smooth manifold modeled on nuclear (LF)-spaces. It can be thought of as the "nonlinear Grassmannian" of all submanifolds of N which are of type M. This result is based on an idea implicitly contained in [Weinstein, 1971], it was fully proved by [Binz-Fischer, 1981] for compact M and for general M by [Michor, 1980b]. The clearest presentation is in [Michor, 1980c, section 13]. If we take a Hilbert space H instead of N, then $B(M,H)$ is the classifying space for $\operatorname{Diff}(M)$ if M is compact, and the classifying bundle $\operatorname{Emb}(M,H)$ carries also a universal connection. This is shown in [Michor, 1988].

The purpose of this note is to present a generalization of this result to the space of immersions. It fails in general, since the action of the diffeomorphism group is not free. Also we were not able to show that the orbit space $\operatorname{Imm}(M,N)/\operatorname{Diff}(M)$ is Hausdorff. Let $\operatorname{Imm}_{\text{prop}}(M,N)$ be the space of all proper immersions. Then $\operatorname{Imm}_{\text{prop}}(M,N)/\operatorname{Diff}(M)$ turns out to be Hausdorff, and the space of those immersions, on which the diffeomorphism group acts free, is open and is the total space of a smooth principal bundle with structure group $\operatorname{Diff}(M)$ and a smooth manifold as base space. For the immersions on which $\operatorname{Diff}(M)$ does not act free we give a slice theorem which is explicit enough to describe the stratification of the orbit space in detail. The results are new and interesting even in the special case of the loop space $C^\infty(S^1,N) \supset \operatorname{Imm}(S^1,N)$.

The main reference for manifolds of mappings is [Michor, 1980c]. But the differential calculus used there is a little old fashioned now, so it should be supplemented by the convenient setting for differential calculus presented in [Frölicher-Kriegl, 1988].

If we assume that M and N are real analytic manifolds with M compact, then all infinite dimensional spaces become real analytic manifolds and all results of this paper remain true, by applying the setting of [Kriegl-Michor, 1990].

1. Regular orbits

1.1. Setup. Let M and N be smooth finite dimensional manifolds, connected and second countable without boundary, and suppose that $\dim M \le \dim N$. Let $\operatorname{Imm}(M,N)$ be the manifold of all immersions from M into N and let $\operatorname{Imm}_{\text{prop}}(M,N)$ be the open submanifold of all proper immersions.

Fix an immersion i. We will now describe some data for i which we will use throughout the paper. If we need these data for several immersions, we will distinguish them by appropriate superscripts.

First there are sets $W_\alpha \subset \overline{W}_\alpha \subset U_\alpha \subset M$ such that (W_α) is an open cover of M, $\overline{W}_\alpha$ is compact, and U_α is an open locally finite cover of M, each W_α and U_α is connected, and such that $i|U_\alpha : U_\alpha \to N$ is an embedding for each α.

Let g be a fixed Riemannian metric on N and let $\exp^N$ be its exponential mapping. Then let $p : \mathcal{N}(i) \to M$ be the *normal bundle* of i, defined in the following way: For $x \in M$ let $\mathcal{N}(i)_x := (T_x i(T_x M))^\perp \subset T_{i(x)}N$ be the g-orthogonal complement in $T_{i(x)}N$. Then

$$\begin{array}{ccc} \mathcal{N}(i) & \xrightarrow{\bar{i}} & TN \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow[i]{} & N \end{array}$$

is a vector bundle homomorphism over i, which is fiberwise injective.

Now let $U^i = U$ be an open neighborhood of the zero section which is so small that $(\exp^N \circ \bar{i})|(U|U_\alpha) : U|U_\alpha \to N$ is a diffeomorphism onto its image which describes a tubular neighborhood of the submanifold $i(U_\alpha)$ for each α. Let

$$\tau = \tau^i := (\exp^N \circ \bar{i})|U : \mathcal{N}(i) \supset U \to N.$$

It will serve us as a substitute for a tubular neighborhood of $i(M)$.

1.2. Definition. An immersion $i \in \operatorname{Imm}(M, N)$ is called *free* if $\operatorname{Diff}(M)$ acts freely on it, i.e. if $i \circ f = i$ for $f \in \operatorname{Diff}(M)$ implies $f = Id_M$. Let $\operatorname{Imm}_{\text{free}}(M, N)$ denote the set of all free immersions.

1.3. Lemma. *Let $i \in \operatorname{Imm}(M, N)$ and let $f \in \operatorname{Diff}(M)$ have a fixed point $x_0 \in M$ and satisfy $i \circ f = i$. Then $f = Id_M$.*

Proof. We consider the sets (U_α) for the immersion i of 1.1. Let us investigate $f(U_\alpha) \cap U_\alpha$. If there is an $x \in U_\alpha$ with $y = f(x) \in U_\alpha$, we have $(i|U_\alpha)(x) = ((i \circ f)|U_\alpha)(x) = (i|U_\alpha)(f(x)) = (i|U_\alpha)(y)$. Since $i|U_\alpha$ is injective we have $x = y$, and

$$f(U_\alpha) \cap U_\alpha = \{x \in U_\alpha : f(x) = x\}.$$

Thus $f(U_\alpha) \cap U_\alpha$ is closed in U_α. Since it is also open and since U_α is connected, we have $f(U_\alpha) \cap U_\alpha = \emptyset$ or $= U_\alpha$.

Now we consider the set $\{x \in M : f(x) = x\}$. We have just shown that it is open in M. Since it is also closed and contains the fixed point x_0, it coincides with M. $\square$

1.4. Lemma. *If for an immersion $i \in \operatorname{Imm}(M, N)$ there is a point in $i(M)$ with only one preimage, then i is a free immersion.*

Proof. Let $x_0 \in M$ be such that $i(x_0)$ has only one preimage. If $i \circ f = i$ for $f \in \operatorname{Diff}(M)$ then $f(x_0) = x_0$ and $f = Id_M$ by lemma 1.3. $\square$

Note that there are free immersions without a point in $i(M)$ with only one preimage: Consider a figure eight which consists of two touching circles. Now we may map the circle to the figure eight by going first three times around the upper circle, then twice around the lower one. This immersion $S^1 \to \mathbb{R}^2$ is free.

1.5. Theorem. *Let i be a free immersion $M \to N$. Then there is an open neighborhood $\mathcal{W}(i)$ in $\operatorname{Imm}(M,N)$ which is saturated for the $\operatorname{Diff}(M)$-action and which splits smoothly as*

$$\mathcal{W}(i) = \mathcal{Q}(i) \times \operatorname{Diff}(M).$$

Here $\mathcal{Q}(i)$ is a smooth splitting submanifold of $\operatorname{Imm}(M,N)$, diffeomorphic to an open neighborhood of 0 in $C^\infty(\mathcal{N}(i))$. In particular the space $\operatorname{Imm}_{\text{free}}(M,N)$ is open in $C^\infty(M,N)$.

Let $\pi : \operatorname{Imm}(M,N) \to \operatorname{Imm}(M,N)/\operatorname{Diff}(M) = B(M,N)$ be the projection onto the orbit space, which we equip with the quotient topology. Then $\pi|\mathcal{Q}(i) : \mathcal{Q}(i) \to \pi(\mathcal{Q}(i))$ is bijective onto an open subset of the quotient. If i runs through $\operatorname{Imm}_{\text{free,prop}}(M,N)$ of all free and proper immersions these mappings define a smooth atlas for the quotient space, so that

$$(\operatorname{Imm}_{\text{free,prop}}(M,N), \pi, \operatorname{Imm}_{\text{free,prop}}(M,N)/\operatorname{Diff}(M), \operatorname{Diff}(M))$$

is a smooth principal fiber bundle with structure group $\operatorname{Diff}(M)$.

The restriction to proper immersions is necessary because we are only able to show that $\operatorname{Imm}_{\text{prop}}(M,N)/\operatorname{Diff}(M)$ is Hausdorff in section 2 below.

Proof. We consider the setup 1.1 for the free immersion i. Let

$$\mathcal{U}(i) := \{j \in \operatorname{Imm}(M,N) : j(\overline{W}^i_\alpha) \subseteq \tau^i(U^i|U^i_\alpha) \text{ for all } \alpha, j \sim i\},$$

where $j \sim i$ means that $j = i$ off some compact set in M. Then by [Michor, 1980c, section 4] the set $\mathcal{U}(i)$ is an open neighborhood of i in $\operatorname{Imm}(M,N)$. For each $j \in \mathcal{U}(i)$ we define

$$\varphi_i(j) : M \to U^i \subseteq \mathcal{N}(i),$$
$$\varphi_i(j)(x) := (\tau^i|(U^i|U^i_\alpha))^{-1}(j(x)) \text{ if } x \in W^i_\alpha.$$

Then $\varphi_i : \mathcal{U}(i) \to C^\infty(M, \mathcal{N}(i))$ is a mapping which is bijective onto the open set

$$\mathcal{V}(i) := \{h \in C^\infty(M, \mathcal{N}(i)) : h(\overline{W}^i_\alpha) \subseteq U^i|U^i_\alpha \text{ for all } \alpha, h \sim 0\}$$

in $C^\infty(M, \mathcal{N}(i))$. Its inverse is given by the smooth mapping $\tau^i_* : h \mapsto \tau^i \circ h$, see [Michor, 1980c, 10.14]. We claim that φ_i is itself a smooth mapping: recall the fixed Riemannian metric g on N; τ^i is a local diffeomorphism $U^i \to N$, so we choose the exponential mapping with respect to $(\tau^i)^* g$ on U^i and that with respect to g on N; then in the canonical chart of $C^\infty(M, U^i)$ centered at 0 and of $C^\infty(M,N)$ centered at i as described in [Michor, 1980c, 10.4], the mapping φ_i is just the identity.

We have $\tau^i_*(h \circ f) = \tau^i_*(h) \circ f$ for those $f \in \operatorname{Diff}(M)$ which are near enough to the identity so that $h \circ f \in \mathcal{V}(i)$. We consider now the open set

$$\{h \circ f : h \in \mathcal{V}(i), f \in \operatorname{Diff}(M)\} \subseteq C^\infty((M, U^i)).$$

Obviously we have a smooth mapping from it into $C^\infty_c(U^i) \times \operatorname{Diff}(M)$ given by $h \mapsto (h \circ (p \circ h)^{-1}, p \circ h)$, where $C^\infty_c(U^i)$ is the space of sections with compact support of $U^i \to M$. So if we let $\mathcal{Q}(i) := \tau^i_*(C^\infty_c(U^i) \cap \mathcal{V}(i)) \subset \operatorname{Imm}(M,N)$ we have

$$\mathcal{W}(i) := \mathcal{U}(i) \circ \operatorname{Diff}(M) \cong \mathcal{Q}(i) \times \operatorname{Diff}(M) \cong (C^\infty_c(U^i) \cap \mathcal{V}(i)) \times \operatorname{Diff}(M),$$

since the action of $\operatorname{Diff}(M)$ on i is free. Consequently $\operatorname{Diff}(M)$ acts freely on each immersion in $\mathcal{W}(i)$, so $\operatorname{Imm}_{\text{free}}(M,N)$ is open in $C^\infty(M,N)$. Furthermore

$$\pi|\mathcal{Q}(i) : \mathcal{Q}(i) \to \operatorname{Imm}_{\text{free}}(M,N)/\operatorname{Diff}(M)$$

is bijective onto an open set in the quotient.

We now consider $\varphi_i \circ (\pi|\mathcal{Q}(i))^{-1} : \pi(\mathcal{Q}(i)) \to C^\infty(U^i)$ as a chart for the quotient space. In order to investigate the chart change let $j \in \operatorname{Imm}_{\text{free}}(M,N)$ be such that $\pi(\mathcal{Q}(i)) \cap \pi(\mathcal{Q}(j)) \neq \emptyset$. Then there is an immersion $h \in \mathcal{W}(i) \cap \mathcal{Q}(j)$, so there exists a unique $f_0 \in \operatorname{Diff}(M)$ (given by $f_0 = p \circ \varphi_i(h)$) such that $h \circ f_0^{-1} \in \mathcal{Q}(i)$. If we consider $j \circ f_0^{-1}$ instead of j and call it again j, we have $\mathcal{Q}(i) \cap \mathcal{Q}(j) \neq \emptyset$ and consequently $\mathcal{U}(i) \cap \mathcal{U}(j) \neq \emptyset$. Then the chart change is given as follows:

$$\varphi_i \circ (\pi|\mathcal{Q}(i))^{-1} \circ \pi \circ (\tau^j)_* : C_c^\infty(U^j) \to C_c^\infty(U^i)$$
$$s \mapsto \tau^j \circ s \mapsto \varphi_i(\tau^j \circ s) \circ (p^i \circ \varphi_i(\tau^j \circ s))^{-1}.$$

This is of the form $s \mapsto \beta \circ s$ for a locally defined diffeomorphism $\beta : \mathcal{N}(j) \to \mathcal{N}(i)$ which is not fiber respecting, followed by $h \mapsto h \circ (p^i \circ h)^{-1}$. Both composants are smooth by the general properties of manifolds of mappings. So the chart change is smooth.

We have to show that the quotient space $\operatorname{Imm}_{\text{prop,free}}(M,N)/\operatorname{Diff}(M)$ is Hausdorff. This will be done in section 2 below. □

2. Some orbit spaces are Hausdorff

2.1. Theorem. *The orbit space* $\operatorname{Imm}_{\text{prop}}(M,N)/\operatorname{Diff}(M)$ *of the space of all proper immersions under the action of the diffeomorphism group is Hausdorff in the quotient topology.*

The proof will occupy the rest of this section. We want to point out that we believe that the whole orbit space $\operatorname{Imm}(M,N)/\operatorname{Diff}(M)$ is Hausdorff, but that we were unable to prove this.

2.2. Lemma. *Let i and $j \in \operatorname{Imm}_{\text{prop}}(M,N)$ with $i(M) \neq j(M)$ in N. Then their projections $\pi(i)$ and $\pi(j)$ are different and can be separated by open subsets in* $\operatorname{Imm}_{\text{prop}}(M,N)/\operatorname{Diff}(M)$.

Proof. We suppose that $i(M) \nsubseteq \overline{j(M)} = j(M)$ (since proper immersions have closed images). Let $y_0 \in i(M) \setminus \overline{j(M)}$, then we choose open neighborhoods V of y_0 in N and W of $j(M)$ in N such that $V \cap W = \emptyset$. We consider the sets

$$\mathcal{V} := \{k \in \operatorname{Imm}_{\text{prop}}(M,N) : k(M) \cap V \neq \emptyset\} \quad \text{and}$$
$$\mathcal{W} := \{k \in \operatorname{Imm}_{\text{prop}}(M,N) : k(M) \subseteq W\}.$$

Then $\mathcal{V}$ and $\mathcal{W}$ are $\operatorname{Diff}(M)$-saturated disjoint open neighborhoods of i and j, respectively, so $\pi(\mathcal{V})$ and $\pi(\mathcal{W})$ separate $\pi(i)$ and $\pi(j)$ in $\operatorname{Imm}_{\text{prop}}(M,N)/\operatorname{Diff}(M)$. □

2.3. For a proper immersion $i : M \to N$ and $x \in i(M)$ let $\delta(x) \in \mathbb{N}$ be the number of points in $i^{-1}(x)$. Then $\delta : i(M) \to \mathbb{N}$ is a mapping.

Lemma. *The mapping $\delta : i(M) \to \mathbb{N}$ is upper semicontinuous, i.e. $\{x \in i(M) : \delta(x) \le k\}$ is open in $i(M)$ for each k.*

Proof. Let $x \in i(M)$ with $\delta(x) = k$ and let $i^{-1}(x) = \{y_1, \dots, y_k\})$. Then there are pairwise disjoint open neighborhoods W_n of y_n in M such that $i|W_n$ is an embedding for each n. The set $M \setminus (\bigcup_n W_n)$ is closed in M, and since i is proper the set $i(M \setminus (\bigcup_n W_n))$ is also closed in $i(M)$ and does not contain x. So there is an open neighborhood U of x in $i(M)$ which does not meet $i(M \setminus (\bigcup_n W_n))$. Then obviously $\delta(z) \le k$ for all $z \in U$. $\square$

2.4. We consider two proper immersions i_1 and $i_2 \in \operatorname{Imm}_{\text{prop}}(M, N)$ such that $i_1(M) = i_2(M) =: L \subseteq N$. Then we have mappings $\delta_1, \delta_2 : L \to \mathbb{N}$ as in 2.3.

2.5. Lemma. *In the situation of 2.4, if $\delta_1 \ne \delta_2$ then the projections $\pi(i_1)$ and $\pi(i_2)$ are different and can be separated by disjoint open neighborhoods in* $\operatorname{Imm}_{\text{prop}}(M,N)/\operatorname{Diff}(M)$.

Proof. Let us suppose that $m_1 = \delta_1(y_0) \ne \delta_2(y_0) = m_2$. There is a small connected open neighborhood V of y_0 in N such that $i_1^{-1}(V)$ has m_1 connected components and $i_2^{-1}(V)$ has m_2 connected components. This assertions describe Whitney C^0-open neighborhoods in $\operatorname{Imm}_{\text{prop}}(M,N)$ of i_1 and i_2 which are closed under the action of $\operatorname{Diff}(M)$, respectively. Obviously these two neighborhoods are disjoint. $\square$

2.6. We assume now for the rest of this section that we are given two immersions i_1 and $i_2 \in \operatorname{Imm}_{\text{prop}}(M, N)$ with $i_1(M) = i_2(M) =: L$ such that the functions from 2.4 are equal: $\delta_1 = \delta_2 =: \delta$.

Let $(L_\beta)_{\beta \in B}$ be the partition of L consisting of all pathwise connected components of level sets $\{x \in L : \delta(x) = c\}$, c some constant.

Let B_0 denote the set of all $\beta \in B$ such that the interior of L_β in L is not empty. Since M is second countable, B_0 is countable.

Claim. $\bigcup_{\beta \in B_0} L_\beta$ is dense in L.

Let k_1 be the smallest number in $\delta(L)$ and let B_1 be the set of all $\beta \in B$ such that $\delta(L_\beta) = k_1$. Then by lemma 2.3 each L_β for $\beta \in B_1$ is open. Let L^1 be the closure of $\bigcup_{\beta \in B_1} L_\beta$. Let k_2 be the smallest number in $\delta(L \setminus L^1)$ and let B_2 be the set of all $\beta \in B$ with $\beta(L_\beta) = k_2$ and $L_\beta \cap (L \setminus L^1) \ne \emptyset$. Then by lemma 2.3 again $L_\beta \cap (L \setminus L^1) \ne \emptyset$ is open in L so L_β has non empty interior for each $\beta \in B_2$. Then let L^2 denote the closure of $\bigcup_{\beta \in B_1 \cup B_2} L_\beta$ and continue the process. Since by lemma 2.3 we always find new L_β with non empty interior, we finally exhaust L and the claim follows.

Let $(M^1_\lambda)_{\lambda \in C^1}$ be a suitably chosen cover of M by subsets of the sets $i_1^{-1}(L_\beta)$ such that each $i_2|\operatorname{int} M^1_\lambda$ is an embedding for each λ. Let C^1_0 be the set of all λ such that M^1_λ has non empty interior. Let similarly $(M^2_\mu)_{\mu \in C^2}$ be a cover for i_2. Then there are at most countably many sets M^1_λ with $\lambda \in C^1_0$, the union $\bigcup_{\lambda \in C^1_0} \operatorname{int} M^1_\lambda$ is dense and consequently $\bigcup_{\lambda \in C^1_0} \overline{M^1_\lambda} = M$; similarly for the M^2_μ.

2.7. Procedure. Given immersions i_1 and i_2 as in 2.6 we will try to construct a diffeomorphism $f : M \to M$ with $i_2 \circ f = i_1$. If we meet an obstacle to the construction this will give us enough control on the situation to separate i_1 and i_2.

Choose $\lambda_0 \in C^1_0$ so that $\operatorname{int} M^1_{\lambda_0} \ne \emptyset$. Then $i_1 : \operatorname{int} M^1_{\lambda_0} \to L_{\beta_1(\lambda_0)}$ is an embedding, where $\beta_1 : C^1 \to B$ is the mapping satisfying $i_1(M^1_\lambda) \subseteq L_{\beta_1(\lambda)}$ for all $\lambda \in C^1$.

Now we choose $\mu_0 \in \beta_2^{-1}\beta_1(\lambda_0) \subset C_0^2$ such that $f := (i_2|\operatorname{int} M^2_{\mu_0})^{-1} \circ i_1|\operatorname{int} M^1_{\lambda_0}$ is a diffeomorphism $\operatorname{int} M^1_{\lambda_0} \to \operatorname{int} M^2_{\mu_0}$. Note that f is uniquely determined by the choice of μ_0, if it exists, by lemma 1.3. So we will repeat the following construction for every $\mu_0 \in \beta_2^{-1}\beta_1(\lambda_0) \subset C_0^2$.

Now we try to extend f. We choose $\lambda_1 \in C_0^1$ such that $\overline{M}^1_{\lambda_0} \cap \overline{M}^1_{\lambda_1} \neq \emptyset$.

Case a. Only $\lambda_1 = \lambda_0$ is possible, so $M^1_{\lambda_0}$ is dense in M since M is connected and we may extend f by continuity to a diffeomorphism $f : M \to M$ with $i_2 \circ f = i_1$.

Case b. We can find $\lambda_1 \neq \lambda_0$. We choose $x \in \overline{M}^1_{\lambda_0} \cap \overline{M}^1_{\lambda_1}$ and a sequence (x_n) in $M^1_{\lambda_0}$ with $x_n \to x$. Then we have a sequence $(f(x_n))$ in B.

Case ba. $y := \lim f(x_n)$ exists in M. Then there is $\mu_1 \in C_0^2$ such that $y \in \overline{M}^2_{\mu_0} \cap \overline{M}^2_{\mu_1}$.

Let $U^1_{\alpha_1}$ be an open neighborhood of x in M such that $i_1|U^1_{\alpha_1}$ is an embedding and let similarly $U^2_{\alpha_2}$ be an open neighborhood of y in M such that $i_2|U^2_{\alpha_2}$ is an embedding. We consider now the set $i_2^{-1}i_1(U^1_{\alpha_1})$. There are two cases possible.

Case baa. The set $i_2^{-1}i_1(U^1_{\alpha_1})$ is a neighborhood of y. Then we extend f to $i_1^{-1}(i_1(U^1_{\alpha_1}) \cap i_2(U^2_{\alpha_2}))$ by $i_2^{-1} \circ i_1$. Then f is defined on some open subset of $\operatorname{int} M^1_{\lambda_1}$ and by the situation chosen in 2.6 f extends to the whole of $\operatorname{int} M^1_{\lambda_1}$.

Case bab. The set $i_2^{-1}i_1(U^1_{\alpha_1})$ is not a neighborhood of y. This is a definite obstruction to the extension of f.

Case bb. The sequence (x_n) has no limit in M. This is a definite obstruction to the extension of f.

If we meet an obstruction we stop and try another μ_0. If for all admissible μ_0 we meet obstructions we stop and remember the data. If we do not meet an obstruction we repeat the construction with some obvious changes.

2.8. Lemma. *The construction of 2.7 in the setting of 2.6 either produces a diffeomorphism $f : M \to M$ with $i_2 \circ f = i_1$ or we may separate i_1 and i_2 by open sets in $\operatorname{Imm}_{\text{prop}}(M, N)$ which are saturated with respect to the action of $\operatorname{Diff}(M)$*

Proof. If for some μ_0 we do not meet any obstruction in the construction 2.7, the resulting f is defined on the whole of M and it is a continuous mapping $M \to M$ with $i_2 \circ f = i_1$. Since i_1 and i_2 are locally embeddings, f is smooth and of maximal rank. Since i_1 and i_2 are proper, f is proper. So the image of f is open and closed and since M is connected, f is a surjective local diffeomorphism, thus a covering mapping $M \to M$. But since $\delta_1 = \delta_2$ the mapping f must be a 1-fold covering, so a diffeomorphism.

If for all $\mu_0 \in \beta_2^{-1}\beta_1(\lambda_0) \subset C_0^2$ we meet obstructions we choose small mutually distinct open neighborhoods V^1_λ of the sets $i_1(M^1_\lambda)$. We consider the Whitney C^0-open neighborhood $\mathcal{V}_1$ of i_1 consisting of all immersions j_1 with $j_1(M^1_\lambda) \subset V^1_\lambda$ for all λ. Let $\mathcal{V}_2$ be a similar neighborhood of i_2.

We claim that $\mathcal{V}_1 \circ \operatorname{Diff}(M)$ and $\mathcal{V}_2 \circ \operatorname{Diff}(M)$ are disjoint. For that it suffices to show that for any $j_1 \in \mathcal{V}_1$ and $j_2 \in \mathcal{V}_2$ there does not exist a diffeomorphism $f \in \operatorname{Diff}(M)$ with $j_2 \circ f = j_1$. For that to be possible the immersions j_1 and j_2 must have the same image L and the same functions $\delta(j_1)$, $\delta(j_2) : L \to \mathbb{N}$. But now the combinatorial relations of the slightly distinct new sets M^1_λ, L_β, and M^2_μ are contained in the old ones, so any try to construct such a diffeomorphism f starting from the same λ_0 meets the same obstructions. $\square$

3. Singular orbits

3.1. Let $i \in \operatorname{Imm}(M,N)$ be an immersion which is not free. Then we have a nontrivial isotropy subgroup $\operatorname{Diff}_i(M) \subset \operatorname{Diff}(M)$ consisting of all $f \in \operatorname{Diff}(M)$ with $i \circ f = i$.

Lemma. *Then the isotropy subgroup $\operatorname{Diff}_i(M)$ acts properly discontinuously on M, so the projection $q_1 : M \to M_1 := M/\operatorname{Diff}_i(M)$ is a covering map and a submersion for a unique structure of a smooth manifold on M_1. There is an immersion $i_1 : M_1 \to N$ with $i = i_1 \circ q_1$. In particular $\operatorname{Diff}_i(M)$ is countable, and finite if M is compact.*

Proof. We have to show that for each $x \in M$ there is an open neighborhood U such that $f(U) \cap U = \emptyset$ for $f \in \operatorname{Diff}_i(M) \setminus \{Id\}$. We consider the setup 1.1 for i. By the proof of 1.3 we have $f(U^i_\alpha) \cap U^i_\alpha = \{x \in U^i_\alpha : f(x) = x\}$ for any $f \in \operatorname{Diff}_i(M)$. If f has a fixed point then by 1.3 $f = Id$, so $f(U^i_\alpha) \cap U^i_\alpha = \emptyset$ for all $f \in \operatorname{Diff}_i(M) \setminus \{Id\}$. The rest is clear. □

The factorized immersion i_1 is in general not a free immersion. The following is an example for that: Let

$$M_0 \xrightarrow[\alpha]{} M_1 \xrightarrow[\beta]{} M_2 \xrightarrow[\gamma]{} M_3$$

be a sequence of covering maps with fundamental groups $1 \to G_1 \to G_2 \to G_3$. Then the group of deck transformations of γ is given by $\mathcal{N}_{G_3}(G_2)/G_2$, the normalizer of G_2 in G_3, and the group of deck transformations of $\gamma \circ \beta$ is $\mathcal{N}_{G_3}(G_1)/G_1$. We can easily arrange that $\mathcal{N}_{G_3}(G_2) \not\subseteq \mathcal{N}_{G_3}(G_1)$, then γ admits deck transformations which do not lift to M_1. Then we thicken all spaces to manifolds, so that $\gamma \circ \beta$ plays the role of the immersion i.

3.2. Theorem. *Let $i \in \operatorname{Imm}(M,N)$ be an immersion which is not free. Then there is a covering map $q_2 : M \to M_2$ which is also a submersion such that i factors to an immersion $i_2 : M_2 \to N$ which is free.*

Proof. Let $q_0 : M_0 \to M$ be the universal covering of M and consider the immersion $i_0 = i \circ q_0 : M_0 \to N$ and its isotropy group $\operatorname{Diff}_{i_0}(M_0)$. By 3.1 it acts properly discontinuously on M_0 and we have a submersive covering $q_{02} : M_0 \to M_2$ and an immersion $i_2 : M_2 \to N$ with $i_2 \circ q_{02} = i_0 = i \circ q_0$. By comparing the respective groups of deck transformations it is easily seen that $q_{02} : M_0 \to M_2$ factors over $q_1 \circ q_0 : M_0 \to M \to M_1$ to a covering $q_{12} : M_1 \to M_2$. The mapping $q_2 := q_{12} \circ q_1 : M \to M_2$ is the looked for covering: If $f \in \operatorname{Diff}(M_2)$ fixes i_2, it lifts to a diffeomorphism $f_0 \in \operatorname{Diff}(M_0)$ which fixes i_0, so is in $\operatorname{Diff}_{i_0}(M_0)$, so $f = Id$. □

3.3. Convention. In order to avoid complications we assume that from now on M is such a manifold that

(1) For any covering $M \to M_1$, any diffeomorphism $M_1 \to M_1$ admits a lift $M \to M$.

If M is simply connected, condition (1) is satisfied. Also for $M = S^1$ condition (1) is easily seen to be valid. So what follows is applicable to loop spaces.

Condition (1) implies that in the proof of 3.2 we have $M_1 = M_2$.

3.4. Description of a neighborhood of a singular orbit. Let M be a manifold satisfying 3.3.(1). In the situation of 3.1 we consider the normal bundles $p_i : \mathcal{N}(i) \to M$ and $p_{i_1} : \mathcal{N}(i_1) \to M_1$. Then the covering map $q_1 : M \to M_1$ lifts uniquely to a vector bundle homomorphism $\mathcal{N}(q_1) : \mathcal{N}(i) \to \mathcal{N}(i_1)$ which is also a covering map, such that $\tau^{i_1} \circ \mathcal{N}(q_1) = \tau^i$.

We have $M_1 = M/\operatorname{Diff}_i(M)$ and the group $\operatorname{Diff}_i(M)$ acts also as the group of deck transformations of the covering $\mathcal{N}(q_1) : \mathcal{N}(i) \to \mathcal{N}(i_1)$ by $\operatorname{Diff}_i(M) \ni f \mapsto \mathcal{N}(f)$, where

$$\begin{array}{ccc} \mathcal{N}(i) & \xrightarrow[\mathcal{N}(f)]{} & \mathcal{N}(i) \\ \downarrow & & \downarrow \\ M & \xrightarrow[f]{} & M \end{array}$$

is a vector bundle isomorphism for each $f \in \operatorname{Diff}_i(M)$. If we equip $\mathcal{N}(i)$ and $\mathcal{N}(i_1)$ with the fiber Riemann metrics induced from the fixed Riemannian metric g on N, the mappings $\mathcal{N}(q_1)$ and all $\mathcal{N}(f)$ are fiberwise linear isometries.

Let us now consider the right action of $\operatorname{Diff}_i(M)$ on the space of sections $C_c^\infty(\mathcal{N}(i))$ given by $f^*s := \mathcal{N}(f)^{-1} \circ s \circ f$.

From the proof of theorem 1.5 we recall now the sets

$$\begin{array}{ccc} C^\infty(M, \mathcal{N}(i)) \supset \mathcal{V}(i) & \xleftarrow[\varphi_i]{} & \mathcal{U}(i) \\ \uparrow & & \uparrow \\ C_c^\infty(\mathcal{N}(i)) \supset C_c^\infty(U^i) & \xleftarrow{\varphi_i} & \mathcal{Q}(i). \end{array}$$

All horizontal mappings are again diffeomorphisms and the vertical mappings are inclusions. But since the action of $\operatorname{Diff}(M)$ on i is not free we cannot extend the splitting submanifold $\mathcal{Q}(i)$ to an orbit cylinder as we did in the proof on theorem 1.5. $\mathcal{Q}(i)$ is again a smooth transversal for the orbit though i.

For any $f \in \operatorname{Diff}(M)$ and $s \in C_c^\infty(U^i) \subset C_c^\infty(\mathcal{N}(i))$ we have

$$\varphi_i^{-1}(f^*s) = \tau_*^i(f^*s) = \tau_*^i(s) \circ f.$$

So the space $q_1^* C_c^\infty(\mathcal{N}(i_1))$ of all sections of $\mathcal{N}(i) \to M$ which factor to sections of $\mathcal{N}(i_1) \to M_1$, is exactly the space of all fixed points of the action of $\operatorname{Diff}_i(M)$ on $C_c^\infty(\mathcal{N}(i))$; and they are mapped by $\tau_*^i = \varphi_i^{-1}$ to immersions in $\mathcal{Q}(i)$ which have again $\operatorname{Diff}_i(M)$ as isotropy group.

If $s \in C_c^\infty(U^i) \subset C_c^\infty(\mathcal{N}(i))$ is an arbitrary section, the orbit through $\tau_*^i(s) \in \mathcal{Q}(i)$ hits the transversal $\mathcal{Q}(i)$ again in the points $\tau_*^i(f^*s)$ for $f \in \operatorname{Diff}_i(M)$.

We summarize all this in the following theorem:

3.5. Theorem. *Let M be a manifold satisfying condition* (1) *of 3.3. Let $i \in Imm(M, N)$ be an immersion which is not free, i.e. has non trivial isotropy group* $\operatorname{Diff}_i(M)$.

Then in the setting and notation of 3.4 in the following commutative diagram the bottom mapping

$$\begin{array}{ccc} \operatorname{Imm}_{\text{free}}(M_1, N) & \xrightarrow{(q_1)^*} & \operatorname{Imm}(M, N) \\ \pi \downarrow & & \downarrow \pi \\ \operatorname{Imm}_{\text{free}}(M_1, N)/\operatorname{Diff}(M_1) & \longrightarrow & \operatorname{Imm}(M, N)/\operatorname{Diff}(M) \end{array}$$

is the inclusion of a (possibly non Hausdorff) manifold, the stratum of $\pi(i)$ in the stratification of the orbit space. This stratum consists of the orbits of all immersions which have $\operatorname{Diff}_i(M)$ *as isotropy group.*

3.6. The orbit structure. We have the following description of the orbit structure near i in $\operatorname{Imm}(M,N)$: For fixed $f\in \operatorname{Diff}_i(M)$ the set of fixed points $\operatorname{Fix}(f):=\{j\in \mathcal{Q}(i): j\circ f=j\}$ is called a *generalized wall*. The union of all generalized walls is called the *diagram* $\mathcal{D}(i)$ of i. A connected component of the complement $\mathcal{Q}(i)\setminus \mathcal{D}(i)$ is called a *generalized Weyl chamber*. The group $\operatorname{Diff}_i(M)$ maps walls to walls and chambers to chambers. The immersion i lies in every wall.

We shall see shortly that there is only one chamber and that the situation is rather distinct from that of reflection groups.

If we view the diagram in the space $C^\infty_c(U^i)\subset C^\infty_c(\mathcal{N}(i))$ which is diffeomorphic to $\mathcal{Q}(i)$, then it consist of traces of closed linear subspaces, because the action of $\operatorname{Diff}_i(M)$ on $C^\infty_c(\mathcal{N}(i))$ consists of linear isometries in the following way. Let us tensor the vector bundle $\mathcal{N}(i)\to M$ with the natural line bundle of half densities on M, and let us remember one positive half density to fix an isomorphism with the original bundle. Then $\operatorname{Diff}_i(M)$ still acts on this new bundle $\mathcal{N}_{1/2}(i)\to M$ and the pullback action on sections with compact support is isometric for the inner product

$$\langle s_1,s_2\rangle := \int_M g(s_1,s_2).$$

We consider the walls and chambers now extended to the whole space in the obvious manner.

3.7. Lemma. *Each wall in $C^\infty_c(\mathcal{N}_{1/2}(i))$ is a closed linear subspace of infinite codimension. Since there are at most countably many walls, there is only one chamber.*

Proof. From the proof of lemma 3.1 we know that $f(U^i_\alpha)\cap U^i_\alpha=\emptyset$ for all $f\in\operatorname{Diff}_i(M)$ and all sets U^i_α from the setup 1.1. Take a section s in the wall of fixed points of f. Choose a section s_α with support in some U^i_α and let the section s be defined by $s|U^i_\alpha = s_\alpha|U^i_\alpha$, $s|f^{-1}(U^i_\alpha)=-f^*s_\alpha$, 0 elsewhere. Then obviously $\langle s,s'\rangle=0$ for all s' in the wall of f. But this construction furnishes an infinite dimensional space contained in the orthogonal complement of the wall of f. $\square$

References

Binz, Ernst; Fischer, Hans R., *The manifold of embeddings of a closed manifold*, Proc. Differential geometric methods in theoretical physics, Clausthal 1978, Springer Lecture Notes in Physics 139, 1981.

Frölicher, A.; Kriegl, A., *Linear spaces and differentiation theory*, Pure and Applied Mathematics, J. Wiley, Chichester, 1988.

Kriegl, A.; Michor, P. W., *A convenient setting for real analytic mappings*, 52 p., to appear, Acta Mathematica (1990).

Michor, P. W., *Manifolds of smooth maps*, Cahiers Topol. Geo. Diff. **19** (1978), 47–78.

Michor, P. W., *Manifolds of smooth maps II: The Lie group of diffeomorphisms of a non compact smooth manifold*, Cahiers Topol. Geo. Diff. **21** (1980a), 63–86.

Michor, P. W., *Manifolds of smooth maps III: The principal bundle of embeddings of a non compact smooth manifold*, Cahiers Topol. Geo. Diff. **21** (1980b), 325–337.

Michor, P. W., *Manifolds of differentiable mappings*, Shiva, Orpington, 1980c.

Michor, P. W., *Manifolds of smooth mappings IV: Theorem of De Rham*, Cahiers Top. Geo. Diff. **24** (1983), 57–86.

Michor, P. W., *Gauge theory for diffeomorphism groups*, Proceedings of the Conference on Differential Geometric Methods in Theoretical Physics, Como 1987, K. Bleuler and M. Werner (eds.), Kluwer, Dordrecht, 1988, pp. 345–371.

Weinstein, Alan, *Symplectic manifolds and their Lagrangian manifolds*, Advances in Math. **6** (1971), 329–345.

Departamento de Geometría y Topología, Facultad de Matemáticas, Universidad de Valencia, E-46100 BURJASSOT, VALENCIA, SPAIN

E-mail address: mascaro@evalun11.bitnet

Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria