

## THE JACOBI FLOW

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*For Wlodek Tulczyjew, on the occasion of his 65th birthday.*

It is well known that the geodesic flow on the tangent bundle is the flow of a certain vector field which is called the spray  $S : TM \rightarrow TTM$ . It is maybe less well known that the flow lines of the vector field  $\kappa_{TM} \circ TS : TTM \rightarrow TTTM$  project to Jacobi fields on  $TM$ . This could be called the ‘Jacobi flow’. This result was developed for the lecture course [5], and it is the main result of this paper. I was motivated by the paper [6] of Urbanski in these proceedings to publish it, as an explanation of some of the uses of iterated tangent bundles in differential geometry.

**1. The tangent bundle of a vector bundle.** Let  $(E, p, M)$  be a vector bundle with fiber addition  $+_E : E \times_M E \rightarrow E$  and fiber scalar multiplication  $m_t^E : E \rightarrow E$ . Then  $(TE, \pi_E, E)$ , the tangent bundle of the manifold  $E$ , is itself a vector bundle, with fiber addition denoted by  $+_{TE}$  and scalar multiplication denoted by  $m_t^{TE}$ .

If  $(U_\alpha, \psi_\alpha : E \upharpoonright U_\alpha \rightarrow U_\alpha \times V)_{\alpha \in A}$  is a vector bundle atlas for  $E$ , such that  $(U_\alpha, u_\alpha)$  is a manifold atlas for  $M$ , then  $(E \upharpoonright U_\alpha, \psi'_\alpha)_{\alpha \in A}$  is an atlas for the manifold  $E$ , where

$$\psi'_\alpha := (u_\alpha \times \text{Id}_V) \circ \psi_\alpha : E \upharpoonright U_\alpha \rightarrow U_\alpha \times V \rightarrow u_\alpha(U_\alpha) \times V \subset \mathbb{R}^m \times V.$$

Hence the family  $(T(E \upharpoonright U_\alpha), T\psi'_\alpha : T(E \upharpoonright U_\alpha) \rightarrow T(u_\alpha(U_\alpha) \times V) = u_\alpha(U_\alpha) \times V \times \mathbb{R}^m \times V)_{\alpha \in A}$  is the atlas describing the canonical vector bundle structure of  $(TE, \pi_E, E)$ . The transition functions are in turn:

$$\begin{aligned} (\psi_\alpha \circ \psi_\beta^{-1})(x, v) &= (x, \psi_{\alpha\beta}(x)v) \quad \text{for } x \in U_{\alpha\beta} \\ (u_\alpha \circ u_\beta^{-1})(y) &= u_{\alpha\beta}(y) \quad \text{for } y \in u_\beta(U_{\alpha\beta}) \\ (\psi'_\alpha \circ (\psi'_\beta)^{-1})(y, v) &= (u_{\alpha\beta}(y), \psi_{\alpha\beta}(u_\beta^{-1}(y))v) \\ (T\psi'_\alpha \circ T(\psi'_\beta)^{-1})(y, v; \xi, w) &= (u_{\alpha\beta}(y), \psi_{\alpha\beta}(u_\beta^{-1}(y))v; d(u_{\alpha\beta})(y)\xi, \\ &\quad (d(\psi_{\alpha\beta} \circ u_\beta^{-1})(y))\xi)v + \psi_{\alpha\beta}(u_\beta^{-1}(y))w). \end{aligned}$$

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So we see that for fixed  $(y, v)$  the transition functions are linear in  $(\xi, w) \in \mathbb{R}^m \times V$ . This describes the vector bundle structure of the tangent bundle  $(TE, \pi_E, E)$ .

For fixed  $(y, \xi)$  the transition functions of  $TE$  are also linear in  $(v, w) \in V \times V$ . This gives a vector bundle structure on  $(TE, Tp, TM)$ . Its fiber addition will be denoted by  $T(+_E) : T(E \times_M E) = TE \times_{TM} TE \rightarrow TE$ , since it is the tangent mapping of  $+_E$ . Likewise its scalar multiplication will be denoted by  $T(m_t^E)$ . One may say that the second vector bundle structure on  $TE$ , that one over  $TM$ , is the derivative of the original one on  $E$ .

The space  $\{\Xi \in TE : Tp.\Xi = 0 \text{ in } TM\} = (Tp)^{-1}(0)$  is denoted by  $VE$  and is called the *vertical bundle* over  $E$ . The local form of a vertical vector  $\Xi$  is  $T\psi'_\alpha.\Xi = (y, v; 0, w)$ , so the transition functions are  $(T\psi'_\alpha \circ T(\psi'_\beta)^{-1})(y, v; 0, w) = (u_{\alpha\beta}(y), \psi_{\alpha\beta}(u_\beta^{-1}(y))v; 0, \psi_{\alpha\beta}(u_\beta^{-1}(y))w)$ . They are linear in  $(v, w) \in V \times V$  for fixed  $y$ , so  $VE$  is a vector bundle over  $M$ . It coincides with  $0_M^*(TE, Tp, TM)$ , the pullback of the bundle  $TE \rightarrow TM$  over the zero section. We have a canonical isomorphism  $Vl_E : E \times_M E \rightarrow VE$ , called the *big vertical lift*, given by  $Vl_E(u_x, v_x) := \partial_t|_0(u_x + tv_x)$ , which is fiber linear over  $M$ . We will mainly use the *small vertical lift*  $vl_E : E \rightarrow TE$ , given by  $vl_E(v_x) = \partial_t|_0 t.v_x = Vl_E(0_x, v_x)$ . The local representation of the vertical lift is  $(T\psi'_\alpha \circ vl_E \circ (\psi'_\alpha)^{-1})(y, v) = (y, 0; 0, v)$ .

If  $\varphi : (E, p, M) \rightarrow (F, q, N)$  is a vector bundle homomorphism, then we have  $vl_F \circ \varphi = T\varphi \circ vl_E : E \rightarrow VF \subset TF$ . So  $vl$  is a natural transformation between certain functors on the category of vector bundles and their homomorphisms. The mapping  $vrp_E := pr_2 \circ Vl_E^{-1} : VE \rightarrow E$  is called the *vertical projection*.

**2. The second tangent bundle of a manifold.** All of 1 is valid for the second tangent bundle  $TTM$  of a manifold, but here we have one more natural structure at our disposal. The *canonical flip* or *involution*  $\kappa_M : TTM \rightarrow TTM$  is defined locally by

$$(TTu \circ \kappa_M \circ TTu^{-1})(x, \xi; \eta, \zeta) = (x, \eta; \xi, \zeta),$$

where  $(U, u)$  is a chart on  $M$ . Clearly this definition is invariant under changes of charts ( $Tu_\alpha$  equals  $\psi'_\alpha$  from 1).

The flip  $\kappa_M$  has the following properties:

- (1)  $\kappa_N \circ TTf = TTf \circ \kappa_M$  for each  $f \in C^\infty(M, N)$ .
- (2)  $T(\pi_M) \circ \kappa_M = \pi_{TM}$  and  $\pi_{TM} \circ \kappa_M = T(\pi_M)$ .
- (3)  $\kappa_M^{-1} = \kappa_M$ .
- (4)  $\kappa_M$  is a linear isomorphism from the vector bundle  $(TTM, T(\pi_M), TM)$  to the bundle  $(TTM, \pi_{TM}, TM)$ , so it interchanges the two vector bundle structures on  $TTM$ .
- (5) It is the unique smooth mapping  $TTM \rightarrow TTM$  which satisfies

$$\partial_t \partial_s c(t, s) = \kappa_M \partial_s \partial_t c(t, s)$$

for each  $c : \mathbb{R}^2 \rightarrow M$ .

All this follows from the local formula given above. A quite early use of  $\kappa_M$  is in [4].

**3. Lemma.** For vector fields  $X, Y \in \mathfrak{X}(M)$  we have

$$\begin{aligned} [X, Y] &= \text{vrp}_{TM} \circ (TY \circ X - \kappa_M \circ TX \circ Y), \\ TY \circ X - {}_{TM} \kappa_T \circ TX \circ Y &= \text{Vl}_{TM}(Y, [X, Y]) \\ &= (\text{vl}_{TM} \circ [X, Y]) T(+_{TM}) (0_{TM} \circ Y). \end{aligned}$$

See [3] 6.13, 6.19, or 37.13 for different proofs of this well known result.

**4. Linear connections and their curvatures.** Let  $(E, p, M)$  be a vector bundle. Recall that a linear connection on the vector bundle  $E$  can be described by specifying its *connector*  $K : TE \rightarrow E$ . This notions seems to be due to [2]. Any smooth mapping  $K : TE \rightarrow E$  which is a (fiber linear) homomorphism for both vector bundle structures on  $TE$ ,

$$\begin{array}{ccc} TE & \xrightarrow{K} & E \\ \pi_E \downarrow & & \downarrow p \\ E & \xrightarrow{p} & M \end{array} \qquad \begin{array}{ccc} TE & \xrightarrow{K} & E \\ Tp \downarrow & & \downarrow p \\ TM & \xrightarrow{\pi_M} & M \end{array}$$

and which is a left inverse to the vertical lift,  $K \circ \text{vl}_E = \text{Id}_E : E \rightarrow TE \rightarrow E$ , specifies a linear connection. Namely: The inverse image  $H := K^{-1}(0_E)$  of the zero section  $0_E \subset E$ , it is a subvector bundle for both vector bundle structures, and for the vector bundle structure  $\pi_E : TE \rightarrow E$  the subbundle  $H$  turns out to be a complementary bundle for the vertical bundle  $VE \rightarrow E$ . We get then the associated *horizontal lift mapping*

$$C : TM \times_M E \rightarrow TE, \quad C(\cdot, u) = \left( Tp|_{\ker(K : T_u E \rightarrow E_{p(u)})} \right)^{-1}$$

which has the following properties

$$\begin{aligned} (Tp, \pi_E) \circ C &= \text{Id}_{TM \times_M E}, \\ C(\cdot, u) : T_{p(u)}M &\rightarrow T_u E \text{ is linear for each } u \in E, \\ C(X_x, \cdot) : E_x &\rightarrow (Tp)^{-1}(X_x) \text{ is linear for each } X_x \in T_x M. \end{aligned}$$

Conversely given a smooth horizontal lift mapping  $C$  with these properties one can reconstruct a connector  $K$ .

For any manifold  $N$ , smooth mapping  $s : N \rightarrow E$  along  $f = p \circ s : N \rightarrow M$ , and vector field  $X \in \mathfrak{X}(N)$  a connector  $K : TE \rightarrow E$  defines the *covariant derivative* of  $s$  along  $X$  by

$$(1) \quad \nabla_X s := K \circ Ts \circ X : N \rightarrow TN \rightarrow TE \rightarrow E.$$

See the following diagram for all the mappings.

$$(2) \quad \begin{array}{ccccc} & & TE & & \\ & Ts \nearrow & \downarrow \pi_E & \searrow K & \\ TN & & E & & E \\ X \uparrow & s \nearrow & \nearrow \nabla_X s & & \\ N & \xrightarrow{f} & M & & \end{array}$$

In canonical coordinates as in 1 we have then

$$\begin{aligned} C((y, \xi), (y, v)) &= (y, v; \xi, \Gamma_y(v, \xi)), \\ K(y, v; \xi, w) &= (y, w - \Gamma_y(v, \xi)), \\ \nabla_{(y, \xi)}(\text{Id}, s) &= (\text{Id}, ds(y)\xi - \Gamma_y(s(y), \xi)), \end{aligned}$$

where the *Christoffel symbol*  $\Gamma_y(v, \xi)$  is smooth in  $y$  and bilinear in  $(v, \xi)$ . Here the sign is the negative of the one in many more traditional approaches, since  $\Gamma$  parametrizes the horizontal bundle.

Let  $C_f^\infty(N, E)$  denote the space of all sections along  $f$  of  $E$ , isomorphic to the space  $C^\infty(f^*E)$  of sections of the pullback bundle. The covariant derivative may then be viewed as a bilinear mapping  $\nabla : \mathfrak{X}(N) \times C_f^\infty(N, E) \rightarrow C_f^\infty(N, E)$ . It has the following properties which follow directly from the definitions:

- (3)  $\nabla_X s$  is  $C^\infty(N, \mathbb{R})$ -linear in  $X \in \mathfrak{X}(N)$ . For  $x \in N$  also we have  $\nabla_{X(x)} s = K.Ts.X(x) = (\nabla_X s)(x) \in E$ .
- (4)  $\nabla_X(h.s) = dh(X).s + h.\nabla_X s$  for  $h \in C^\infty(N, \mathbb{R})$ .
- (5) For any manifold  $Q$ , smooth mapping  $g : Q \rightarrow N$ , and  $Y_y \in T_y Q$  we have  $\nabla_{Tg.Y_y} s = \nabla_{Y_y}(s \circ g)$ . If  $Y \in \mathfrak{X}(Q)$  and  $X \in \mathfrak{X}(N)$  are  $g$ -related, then we have  $\nabla_Y(s \circ g) = (\nabla_X s) \circ g$ .

For vector fields  $X, Y \in \mathfrak{X}(M)$  and a section  $s \in C^\infty(E)$  the curvature  $R \in \Omega^2(M, L(E, E))$  of the connection is given by

$$(6) \quad R(X, Y)s = ([\nabla_X, \nabla_Y] - \nabla_{[X, Y]})s$$

**Theorem.** *Let  $K : TE \rightarrow E$  be the connector of a linear connection on a vector bundle  $(E, p, M)$ . If  $s : N \rightarrow E$  is a section along  $f := p \circ s : N \rightarrow M$  then we have for vector fields  $X, Y \in \mathfrak{X}(N)$*

$$(7) \quad \begin{aligned} \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X, Y]} s &= \\ &= (K \circ TK \circ \kappa_E - K \circ TK) \circ TTs \circ TX \circ Y = \\ &= R \circ (Tf \circ X, Tf \circ Y)s : N \rightarrow E, \end{aligned}$$

where  $R \in \Omega^2(M; L(E, E))$  is the curvature.

*Proof.* Let first  $m_t^E : E \rightarrow E$  denote the scalar multiplication. Then we have  $\partial_t|_0 m_t^E = \text{vl}_E$  where  $\text{vl}_E : E \rightarrow TE$  is the vertical lift. We use then lemma 3 and some obvious commutation relations to get in turn:

$$\begin{aligned} \text{vl}_E \circ K &= \partial_t|_0 m_t^E \circ K = \partial_t|_0 K \circ m_t^{TE} = TK \circ \partial_t|_0 m_t^{TE} = TK \circ \text{vl}_{(TE, \pi_E, E)}. \\ \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X, Y]} s &= \\ &= K \circ T(K \circ Ts \circ Y) \circ X - K \circ T(K \circ Ts \circ X) \circ Y - K \circ Ts \circ [X, Y] \\ K \circ Ts \circ [X, Y] &= K \circ \text{vl}_E \circ K \circ Ts \circ [X, Y] \\ &= K \circ TK \circ \text{vl}_{TE} \circ Ts \circ [X, Y] = K \circ TK \circ TTs \circ \text{vl}_{TN} \circ [X, Y] \\ &= K \circ TK \circ TTs \circ ((TY \circ X - \kappa_N \circ TX \circ Y) (T-) 0_{TN} \circ Y) \\ &= K \circ TK \circ TTs \circ TY \circ X - K \circ TK \circ TTs \circ \kappa_N \circ TX \circ Y - 0. \end{aligned}$$

Now we sum up and use  $TTs \circ \kappa_N = \kappa_E \circ TTs$  to get the first result. If in particular we choose  $f = \text{Id}_M$  so that  $s$  is a section of  $E \rightarrow M$  and  $X, Y$  are vector fields on  $M$ , then we get the curvature  $R$ .

To see that in the general case  $(K \circ TK \circ \kappa_E - K \circ TK) \circ TTs \circ TX \circ Y$  coincides with  $R(Tf \circ X, Tf \circ Y)s$  one has to write out (1) and  $(TTs \circ TX \circ Y)(x) \in TTE$  in canonical charts induced from vector bundle charts of  $E$ .  $\square$

**5. Torsion.** Let  $K : TTM \rightarrow M$  be a linear connector on the tangent bundle, let  $X, Y \in \mathfrak{X}(M)$ . Then the torsion is given by

$$\text{Tor}(X, Y) = (K \circ \kappa_M - K) \circ TX \circ Y.$$

If moreover  $f : N \rightarrow M$  is smooth and  $U, V \in \mathfrak{X}(N)$  then we get also

$$\begin{aligned} \text{Tor}(Tf.U, Tf.V) &= \nabla_U(Tf \circ V) - \nabla_V(Tf \circ U) - Tf \circ [U, V] \\ &= (K \circ \kappa_M - K) \circ TTf \circ TU \circ V. \end{aligned}$$

*Proof.* (9) We have in turn

$$\begin{aligned} \text{Tor}(X, Y) &= \nabla_X Y - \nabla_Y X - [X, Y] \\ &= K \circ TY \circ X - K \circ TX \circ Y - K \circ \text{vl}_{TM} \circ [X, Y] \\ K \circ \text{vl}_{TM} \circ [X, Y] &= K \circ ((TY \circ X - \kappa_M \circ TX \circ Y) (T-) \circ 0_{TM} \circ Y) \\ &= K \circ TY \circ X - K \circ \kappa_M \circ TX \circ Y - 0. \end{aligned}$$

An analogous computation works in the second case, and that  $(K \circ \kappa_M - K) \circ TTf \circ TU \circ V = \text{Tor}(Tf.U, Tf.V)$  can again be checked in local coordinates.  $\square$

**6. Sprays.** Given a linear connector  $K : TTM \rightarrow M$  on the tangent bundle with its horizontal lift mapping  $C : TM \times_M TM \rightarrow TTM$ , then  $S := C \circ \text{diag} : TM \rightarrow TM \times_M TM \rightarrow TTM$  is called the *spray*. This notion is due to [1]. The spray has the following properties:

$$\begin{aligned} \pi_{TM} \circ S &= \text{Id}_{TM} && \text{a vector field on } TM, \\ T(\pi_M) \circ S &= \text{Id}_{TM} && \text{a second order differential equation,} \\ S \circ m_t^{TM} &= T(m_t^{TM}) \circ m_t^{TTM} \circ S && \text{'quadratic',} \end{aligned}$$

where  $m_t^E$  is the scalar multiplication by  $t$  on a vector bundle  $E$ . From  $S$  one can reconstruct the torsion free part of  $C$ . The following result is well known:

**Lemma.** For a spray  $S : TM \rightarrow TTM$  on  $M$ , for  $X \in TM$

$$\text{geo}^S(X)(t) := \pi_M(\text{Fl}_t^S(X))$$

defines a geodesic structure on  $M$ , where  $\text{Fl}^S$  is the flow of the vector field  $S$ .

The abstract properties of a geodesic structure are obvious:

$$\begin{aligned} \text{geo} &: TM \times \mathbb{R} \supset U \rightarrow M \\ \text{geo}(X)(0) &= \pi_M(X), \quad \partial_t|_0 \text{geo}(X)(t) = X \\ \text{geo}(tX)(s) &= \text{geo}(X)(ts) \\ \text{geo}(\text{geo}(X)'(t))(s) &= \text{geo}(X)(t+s) \end{aligned}$$

From a geodesic structure one can reconstruct the spray by differentiation.

**7. Theorem.** *Let  $S : TM \rightarrow TTM$  be a spray on a manifold  $M$ . Then  $\kappa_{TM} \circ TS : TTM \rightarrow TTTM$  is a vector field. Consider a flow line*

$$Y(t) = \text{Fl}_t^{\kappa_{TM} \circ TS}(Y(0))$$

of this field. Then we have:

$c := \pi_M \circ \pi_{TM} \circ Y$  is a geodesic on  $M$ .

$\dot{c} = \pi_{TM} \circ Y$  is the velocity field of  $c$ .

$J := T(\pi_M) \circ Y$  is a Jacobi field along  $c$ .

$\dot{J} = \kappa_M \circ Y$  is the velocity field of  $J$ .

$\nabla_{\partial_t} J = K \circ \kappa_M \circ Y$  is the covariant derivative of  $J$ .

The Jacobi equation is given by:

$$\begin{aligned} 0 &= \nabla_{\partial_t} \nabla_{\partial_t} J + R(J, \dot{c})\dot{c} + \nabla_{\partial_t} \text{Tor}(J, \dot{c}) \\ &= K \circ TK \circ TS \circ Y. \end{aligned}$$

This implies that in a canonical chart induced from a chart on  $M$  the curve  $Y(t)$  is given by

$$(c(t), c'(t); J(t), J'(t)).$$

*Proof.* Consider a curve  $s \mapsto X(s)$  in  $TM$ . Then each  $t \mapsto \pi_M(\text{Fl}_t^S(X(s)))$  is a geodesic in  $M$ , and in the variable  $s$  it is a variation through geodesics. Thus  $J(t) := \partial_s|_0 \pi_M(\text{Fl}_t^S(X(s)))$  is a Jacobi field along the geodesic  $c(t) := \pi_M(\text{Fl}_t^S(X(0)))$ , and each Jacobi field is of this form, for a suitable curve  $X(s)$ . We consider now the curve  $Y(t) := \partial_s|_0 \text{Fl}_t^S(X(s))$  in  $TTM$ . Then by 2.(6) we have

$$\begin{aligned} \partial_t Y(t) &= \partial_t \partial_s|_0 \text{Fl}_t^S(X(s)) = \kappa_{TM} \partial_s|_0 \partial_t \text{Fl}_t^S(X(s)) = \kappa_{TM} \partial_s|_0 S(\text{Fl}_t^S(X(s))) \\ &= (\kappa_{TM} \circ TS)(\partial_s|_0 \text{Fl}_t^S(X(s))) = (\kappa_{TM} \circ TS)(Y(t)), \end{aligned}$$

so that  $Y(t)$  is a flow line of the vector field  $\kappa_{TM} \circ TS : TTM \rightarrow TTTM$ . Moreover using the properties of  $\kappa$  from section 2 and of  $S$  from section 6 we get

$$\begin{aligned} T\pi_M.Y(t) &= T\pi_M.\partial_s|_0 \text{Fl}_t^S(X(s)) = \partial_s|_0 \pi_M(\text{Fl}_t^S(X(s))) = J(t), \\ \pi_M T\pi_M Y(t) &= c(t), \text{ the geodesic,} \\ \partial_t J(t) &= \partial_t T\pi_M.\partial_s|_0 \text{Fl}_t^S(X(s)) = \partial_t \partial_s|_0 \pi_M(\text{Fl}_t^S(X(s))), \\ &= \kappa_M \partial_s|_0 \partial_t \pi_M(\text{Fl}_t^S(X(s))) = \kappa_M \partial_s|_0 \partial_t \pi_M(\text{Fl}_t^S(X(s))) \\ &= \kappa_M \partial_s|_0 T\pi_M.\partial_t \text{Fl}_t^S(X(s)) = \kappa_M \partial_s|_0 (T\pi_M \circ S)\text{Fl}_t^S(X(s)) \\ &= \kappa_M \partial_s|_0 \text{Fl}_t^S(X(s)) = \kappa_M Y(t), \\ \nabla_{\partial_t} J &= K \circ \partial_t J = K \circ \kappa_M \circ Y. \end{aligned}$$

Finally let us express the well known Jacobi expression, where we put  $\gamma(t, s) := \pi_M(\text{Fl}_t^S(X(s)))$  for short and use most of the expressions from above:

$$\begin{aligned} \nabla_{\partial_t} \nabla_{\partial_t} J + R(J, \dot{c})\dot{c} + \nabla_{\partial_t} \text{Tor}(J, \dot{c}) &= \\ &= \nabla_{\partial_t} \nabla_{\partial_t} .T\gamma.\partial_s + R(T\gamma.\partial_s, T\gamma.\partial_t)T\gamma.\partial_t + \nabla_{\partial_t} \text{Tor}(T\gamma.\partial_s, T\gamma.\partial_t) \\ &= K.T(K.T(T\gamma.\partial_s).\partial_t).\partial_t \\ &\quad + (K.TK.\kappa_{TM} - K.TK).TT(T\gamma.\partial_t).T\partial_s.\partial_t \\ &\quad + K.T((K.\kappa_M - K).TT\gamma.T\partial_s.\partial_t).\partial_t \end{aligned}$$

Note that for example for the term in the second summand we have

$$TTT\gamma.TT\partial_t.T\partial_s.\partial_t = T(T(\partial_t\gamma).\partial_s).\partial_t = \partial_t\partial_s\partial_t\gamma = \partial_t.\kappa_M.\partial_t.\partial_s\gamma = T\kappa_M.\partial_t.\partial_t.\partial_s\gamma$$

which at  $s = 0$  equals  $T\kappa_M\ddot{J}$ . Using this we get for the Jacobi expression at  $s = 0$ :

$$\begin{aligned} \nabla_{\partial_t}\nabla_{\partial_t}J + R(J,\dot{c})\dot{c} + \nabla_{\partial_t}\text{Tor}(J,\dot{c}) &= \\ &= (K.TK + K.TK.\kappa_{TM}.T\kappa_M - K.TK.T\kappa_M + K.TK.T\kappa_M - K.TK).\partial_t\partial_tJ = \\ &= K.TK.\kappa_{TM}.T\kappa_M.\partial_t\partial_tJ = K.TK.\kappa_{TM}.\partial_tY = K.TK.TS.Y, \end{aligned}$$

where we used  $\partial_t\partial_tJ = \partial_t(\kappa_M.Y) = T\kappa_M\partial_tY = T\kappa_M.\kappa_{TM}.TS.Y$ . Finally the validity of the Jacobi equation  $0 = K.TK.TS.Y$  follows trivially from  $K \circ S = 0_{TM}$ .  $\square$

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