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KNIT PRODUCTS OF GRADED LIE ALGEBRAS AND GROUPS

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Abstract. If a graded Lie algebra is the direct sum of two graded sub Lie algebras, its bracket can be written in a form that mimics a "double sided semidirect product". It is called the *knit product* of the two subalgebras then. The integrated version of this is called a *knit product* of groups — it coincides with the *Zappa-Szép product*. The behavior of homomorphisms with respect to knit products is investigated.

Introduction

If a Lie algebra is the direct sum of two sub Lie algebras one can write the bracket in a way that mimics semidirect products on both sides. The two representations do not take values in the respective spaces of derivations; they satisfy equations (see 1.1) which look "derivatively knitted" — so we call them a derivatively knitted pair of representations. These equations are familiar for the Frölicher-Nijenhuis bracket of differential geometry, see [1] or [2, 1.10]. This paper is the outcome of my investigation of what formulas 1.1 mean algebraically. It was a surprise for me that they describe the general situation (Theorem 1.3). Also the behavior of homomorphisms with respect to knit products is investigated (Theorem 1.4).

The integrated version of a knit product of Lie algebras will be called a knit product of groups — but it is well known to algebraists under the name *Zappa-Szép product*, see [3] and the references therein. I present it here with different

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notation in order to describe afterwards again the behavior of homomorphisms with respect to this product. This gives a kind of generalization of the method of induced representations.

1. Knit products of graded Lie algebras

1.1. Definition. Let A and B be graded Lie algebras, whose grading is in $\mathbf{Z}$ or $\mathbf{Z}_2$, but only one of them. A *derivatively knitted pair of representations* (α, β) for (A, B) are graded Lie algebra homomorphisms $\alpha : A \to \operatorname{End}(B)$ and $\beta : B \to \operatorname{End}(A)$ such that:

$$\begin{aligned}\alpha(a)[b_1, b_2] = [\alpha(a)b_1, b_2] + (-1)^{|a||b_1|}[b_1, \alpha(a)b_2]-\\ - \Big((-1)^{|a||b_1|}\alpha(\beta(b_1)a)b_2 - (-1)^{(|a|+|b_1|)|b_2|}\alpha(\beta(b_2)a)b_1\Big)\end{aligned}$$

$$\begin{aligned}\beta(b)[a_1, a_2] = [\beta(b)a_1, a_2] + (-1)^{|b||a_1|}[a_1, \beta(b)a_2]-\\ - \Big((-1)^{|b||a_1|}\beta(\alpha(a_1)b)a_2 - (-1)^{(|b|+|a_1|)|a_2|}\beta(\alpha(a_2)b)a_1\Big)\end{aligned}$$

Here $|a|$ is the degree of a. For (non-graded) Lie algebras just assume that all degrees are zero.

1.2. Theorem. *Let (α, β) be a derivatively knitted pair of representations for graded Lie algebras $A = \bigoplus A_k$ and $B = \bigoplus B_k$. Then $A \oplus B := \bigoplus_{k,l}(A_k \oplus B_l)$ becomes a graded Lie algebra $A \oplus_{(\alpha,\beta)} B$ with the following bracket:*

$$\begin{aligned}[(a_1, b_1), (a_2, b_2)] := \Big([a_1, a_2] + \beta(b_1)a_2 - (-1)^{|b_2||a_1|}\beta(b_2)a_1,\\ [b_1, b_2] + \alpha(a_1)b_2 - (-1)^{|a_2||b_1|}\alpha(a_2)b_1\Big)\end{aligned}$$

The grading is $(A \oplus B)_k := A_k \oplus B_k$.

Proof. Obviously this bracket is graded anticommutative. The graded Jacobi identity is checked by computation. □

We call $A \oplus_{(\alpha,\beta)} B$ the *knit product* of A and B. If $\beta = 0$ then α has values in the space of (graded) derivations of A and $A \oplus 0$ is an ideal in $A \oplus_{(\alpha,0)} B$ and we get a semidirect product of graded Lie algebras. Note also that $[(a, 0), (0, b)] = \big((-1)^{|b||a|}\beta(b)a, \alpha(a)b\big)$. This is the key to the following theorem.

1.3. Theorem. *Let A and B be graded Lie subalgebras of a graded Lie algebra C such that $A + B = C$ and $A \cap B = 0$. Then C as graded Lie algebra is isomorphic to a knit product of A and B.*

Proof. For $a \in A$ and $b \in B$ we write

$$[a,b] =: \alpha(a)b - (-1)^{|a||b|}\beta(b)a$$

for the decomposition of $[a,b]$ into components in $C = B + A$. Then $\beta : B \to \mathrm{End}(A)$ and $\alpha : A \to \mathrm{End}(B)$ are linear. Now decompose both sides of the graded Jacobi identity

$$[a,[b_1,b_2]] = [[a,b_1],b_2] + (-1)^{|a||b_1|}[b_1,[a,b_2]]$$

and compare the A- and B-components respectively. This gives equation 1.1 for α and that β is a graded Lie algebra homomorphism. The rest follows by interchanging A and B. Now we decompose $[a_1+b_1, a_2+b_2]$ and see that $C = A \oplus_{(\alpha,\beta)} B$. □

1.4. Now let $\Phi : A \oplus_{(\alpha,\beta)} B \to A' \oplus_{(\alpha',\beta')} B'$ be a linear mapping between knit products. Then Φ can be decomposed into $\Phi(a,b) =: (f(a)+\psi(b), g(b)+\varphi(a))$ for linear mappings $\varphi : A \to B'$, $\psi : B \to A'$, $f : A \to A'$, and $g : B \to B'$.

Theorem. *In this situation Φ is a graded Lie algebra homomorphism if and only if the following conditions hold:*

$$\begin{aligned}
\varphi([a_1,a_2]) &= [\varphi(a_1),\varphi(a_2)] + \alpha'(f(a_1))\varphi(a_2) \\
&\quad - (-1)^{|a_1||a_2|}\alpha'(f(a_2))\varphi(a_1) \\
\psi([b_1,b_2]) &= [\psi(b_1),\psi(b_2)] + \beta'(g(b_1))\psi(b_2) \\
&\quad - (-1)^{(|b_1||b_2|}\beta'(g(b_2))\psi(b_1) \\
[\psi(b), f(a)] &= f(\beta(b)a) - \beta'(g(b))f(a) \\
&\quad - (-1)^{|a||b|}\big(\psi(\alpha(a)b) - \beta'(\varphi(a))\psi(b)\big) \\
[g(b),\varphi(a)] &= \varphi(\beta(b)a) - \alpha'(\psi(b))\varphi(a) \\
&\quad - (-1)^{|a||b|}\big(g(\alpha(a)b) - \alpha'(f(a))g(b)\big) \\
f([a_1,a_2]) &= [f(a_1),f(a_2)] + \beta'(\varphi(a_1))f(a_2) \\
&\quad - (-1)^{|a_1||a_2|}\beta'(\varphi(a_2))f(a_1) \\
g([b_1,b_2]) &= [g(b_1),g(b_2)] + \alpha'(\psi(b_1))g(b_2) \\
&\quad - (-1)^{|b_1||b_2|}\alpha'(\psi(b_2))g(b_1)
\end{aligned}$$

If f and g are graded Lie algebra homomorphism the last pair of equations obviously simplifies.

Proof. A long but straightforward computation. □

This theorem can be used to build representations of C out of representations of A and B.

2. Knit products of groups

2.1. Definition. Let A and B be groups. An *automorphically knitted pair of actions* (α, β) for (A, B) are mappings $\alpha : B \times A \to A$ and $\beta : B \times A \to B$ such that:

(1) $\check{\alpha} : B \to \{\text{bijections of A}\}$ is a group homomorphism, so $\alpha_{b_1} \circ \alpha_{b_2} = \alpha_{b_1 b_2}$ and $\alpha_e = Id_A$, where $\alpha_b(a) := \alpha(b, a)$.
(2) $\check{\beta} : A \to \{\text{bijections of B}\}$ is a group anti homomorphism, i.e., $\beta^{a_1} \circ \beta^{a_2} = \beta^{a_2 a_1}$ and $\beta^e = Id_B$, where $\beta^a(b) = \beta(b, a)$.
(3) $\alpha_b(a_1 a_2) = \alpha_b(a_1).\alpha_{\beta^{a_1}(b)}(a_2)$.
(4) $\beta^a(b_1 b_2) = \beta^{\alpha_{b_2}(a)}(b_1).\beta^a(b_2)$.

2.2. Theorem. *Let (α, β) be an automorphically knitted pair of actions for (A, B). Then $A \times B$ is a group $A \times_{(\alpha,\beta)} B$ with the following operations:*

$$(a_1, b_1).(a_2, b_2) := (a_1.\alpha_{b_1}(a_2), \beta^{a_2}(b_1).b_2)$$
$$(a, b)^{-1} := (\alpha_{b^{-1}}(a^{-1}), \beta^{a^{-1}}(b^{-1})).$$

Unit is (e, e). $A \times \{e\}$ and $\{e\} \times B$ are subgroups of $A \times_{(\alpha,\beta)} B$ which are isomorphic to A and B, respectively. If $\check{\alpha} \equiv Id_A$ then $\{e\} \times B$ is a normal subgroup of $A \times_{(\alpha,\beta)} B$ and we have a semidirect product; similarly if $\check{\beta} \equiv Id_B$.

If A and B are topological groups or Lie groups and α, β are continuous or smooth, then $A \times_{(\alpha,\beta)} B$ is also a topological group or Lie group, respectively.

The proof is routine.

We will call $A \times_{(\alpha,\beta)} B$ the *knit product* of A and B in analogy with section 1. In algebra, with different notation, this product is well known under the name *Zappa-Szép product*. I owe this remark to G. Kowol.

2.3. Theorem. *Let G be a group, let A and B be subgroups such that $G = A.B$ and $A \cap B = \{e\}$. Then G is isomorphic to a knit product of A and B.*

Proof. Let $b.a = \alpha(b, a).\beta(b, a)$ be the unique decomposition of $b.a$ in $G = A.B$. Then

$$a_1 b_1 a_2 b_2 = a_1 \alpha(b_1, a_2)\beta(b_1, a_2) b_2 = (a_1 \alpha_{b_1}(a_2)).(\beta^{a_2}(b_1) b_2).$$

So it remains to show that (α, β) satisfies the conditions of 2.1. Obviously we have $\alpha(e, a) = a$, $\beta(e, a) = e$, $\alpha(b, e) = e$, $\beta(b, e) = b$. Comparing coefficients in the law of associativity of G gives two equations. Setting suitable elements in these equations to e gives all conditions of 2.1. □

2.4. Let $\Phi = (\Phi_1, \Phi_2) : A \times_{(\alpha,\beta)} B \to A' \times_{(\alpha',\beta')} B'$ be a mapping between knit products of groups. We put

$$f(a) := \Phi_1(a, e), \qquad g(b) := \Phi_2(e, b) \tag{1}$$
$$\varphi(b) := \Phi_1(e, b), \qquad \psi(a) := \Phi_2(a, e) \tag{2}$$

Then we have $f : A \to A'$, $g : B \to B'$, $\varphi : B \to A'$, $\psi : A \to B'$. Φ is a group homomorphism if and only if

$$\text{(3)} \qquad \begin{cases} \Phi_1(a_1\alpha_{b_1}(a_2), \beta^{a_2}(b_1)b_2) = \Phi_1(a_1, b_1).\alpha'_{\Phi_2(a_1,b_1)}(\Phi_1(a_2, b_2)) \\ \Phi_2(a_1\alpha_{b_1}(a_2), \beta^{a_2}(b_1)b_2) = \beta'^{\Phi_1(a_2,b_2)}(\Phi_2(a_1, b_1)).\Phi_2(a_2, b_2). \end{cases}$$

Now we set in (3) suitable elements to e, use (1) and (2) and get in turn

$$\text{(e)} \qquad \begin{cases} \Phi_1(a_1, b_2) = f(a_1).\alpha'_{\psi(a_1)}(\varphi(b_2)) \\ \Phi_2(a_1, b_2) = \beta'^{\varphi(b_2)}(\psi(a_1)).g(b_2) \end{cases}$$

$$\text{(f)} \qquad \begin{cases} \varphi(b_1b_2) = \varphi(b_1).\alpha'_{g(b_1)}(\varphi(b_2)) \\ \psi(a_1a_2) = \beta'^{f(a_2)}(\psi(a_1)).\psi(a_2) \end{cases}$$

$$\text{(4)} \qquad \begin{cases} \Phi_1(\alpha_{b_1}(a_2), \beta^{a_2}(b_1)) = \varphi(b_1).\alpha'_{g(b_1)}(f(a_2)) \\ \Phi_2(\alpha_{b_1}(a_2), \beta^{a_2}(b_1)) = \beta'^{f(a_2)}(g(b_1)).\psi(a_2) \end{cases}$$

$$\text{(g)} \qquad \begin{cases} f(a_1a_2) = f(a_1).\alpha'_{\psi(a_1)}(f(a_2)) \\ g(b_1b_2) = \beta'^{\varphi(b_2)}(g(b_1)).g(b_2) \end{cases}$$

If f and g are homomorphisms of groups then (g) implies:

$$\text{(g')} \qquad \begin{cases} f(a_2) = \alpha'_{\psi(a_1)}(f(a_2)) \\ g(b_1) = \beta'^{\varphi(b_2)}(g(b_1)) \end{cases}$$

Now we decompose the left hand sides of (4) with the help of (e) and get:

$$\text{(h)} \qquad \begin{cases} f(\alpha_{b_1}(a_2)).\alpha'_{\psi(\alpha_{b_1}(a_2))}(\varphi(\beta^{a_2}(b_1))) = \varphi(b_1).\alpha'_{g(b_1)}(f(a_2)) \\ \beta'^{\varphi(\beta^{a_2}(b_1))}(\psi(\alpha_{b_1}(a_2))).g(\beta^{a_2}(b_1))) = \beta'^{f(a_2)}(g(b_1)).\psi(a_2) \end{cases}$$

2.5. Theorem. *Let $A \times_{(\alpha,\beta)} B$ and $A' \times_{(\alpha',\beta')} B'$ be knit products of groups and let $f : A \to A'$, $g : B \to B'$, $\varphi : B \to A'$, $\psi : A \to B'$ be mappings such that* (f)*,* (g)*, and* (h) *from 2.4 hold. We define $\Phi = (\Phi_1, \Phi_2) : A \times_{(\alpha,\beta)} B \to A' \times_{(\alpha',\beta')} B'$ by 2.4.*(e)*, then Φ is a homomorphism of groups. If f and g are homomorphisms, then we may use* (g') *instead of* (g).

Proof. It suffices to check (3) of 2.5. This is a difficult computation using 2.4 (a)-(h). □

For topological groups and Lie groups all the expected assertions about continuity and smoothness are true.

This theorem may be used to construct representations of $A \times_{(\alpha,\beta)} B$ out of representations of A and B — a sort of generalized induced representation procedure.

Starting from the equations 2.1 for a knit product of Lie groups and deriving the equations of 1.1 for their Lie algebras is a very interesting exercise in calculus on Lie groups.

References

1. A. Frölicher, A. Nijenhuis, *Theory of vector valued differential forms. Part I.*, Indagationes Math **18** (1956), 338–359.
2. P. W. Michor, *Remarks on the Frölicher-Nijenhuis bracket*, Proceedings of the Conference on Differential Geometry and its Applications, Brno 1986, D. Reidel, 1987, pp. 198–220.
3. J. Szép, *On the structure of groups which can be represented as the product of two subgroups*, Acta Sci. Math. Szeged **12** (1950), 57–61.

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