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# EXTENSIONS OF LIE ALGEBRAS

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Abstract. We review (non-abelian) extensions of a given Lie algebra, identify a 3-dimensional cohomological obstruction to the existence of extensions. A striking analogy to the setting of covariant exterior derivatives, curvature, and the Bianchi identity in differential geometry is spelled out.

**1. Introduction.** The theory of group extensions and their interpretation in terms of cohomology is well known, see [2], [5], [3], [1], e.g. When we wrote this paper we thought that the counterpart for Lie algebras (for non-abelian extensions) was not spelled out in detail in the literature. We presented it here in this short note, with special emphasis to connections with the (algebraic) theory of covariant exterior derivatives, curvature and the Bianchi identity in differential geometry (see section 3). Kirill Mackenzie pointed out to us, that most of our results are available in [4], [12], [15], and generalizations for Lie algebroids are in [11]. So this paper will not appear in print.

**2. Describing extensions.** Consider any exact sequence of homomorphisms of Lie algebras:

$$0 \to \mathfrak{h} \xrightarrow{i} \mathfrak{e} \xrightarrow{p} \mathfrak{g} \to 0.$$

Consider a linear mapping $s : \mathfrak{g} \to \mathfrak{e}$ with $p \circ s = \mathrm{Id}_{\mathfrak{g}}$. Then $s$ induces mappings

$$\alpha : \mathfrak{g} \to \operatorname{der}(\mathfrak{h}), \qquad \alpha_X(H) = [s(X), H], \tag{2.1}$$

$$\rho : \bigwedge^2 \mathfrak{g} \to \mathfrak{h}, \qquad \rho(X,Y) = [s(X), s(Y)] - s([X,Y]), \tag{2.2}$$

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which are easily seen to satisfy

$$[\alpha_X, \alpha_Y] - \alpha_{[X,Y]} = \mathrm{ad}_{\rho(X,Y)} \tag{2.3}$$

$$\sum_{\mathrm{cyclic}\{X,Y,Z\}} \Big(\alpha_X \rho(Y,Z) - \rho([X,Y],Z)\Big). = 0 \tag{2.4}$$

We can completely describe the Lie algebra structure on $\mathfrak{e} = \mathfrak{h} \oplus s(\mathfrak{g})$ in terms of $\alpha$ and $\rho$ :

$$\begin{aligned}[H_1 + s(X_1), H_2 + s(X_2)] &= \\ &= ([H_1, H_2] + \alpha_{X_1} H_2 - \alpha_{X_2} H_1 + \rho(X_1, X_2)) + s[X_1, X_2]\end{aligned} \tag{2.5}$$

and one can check that formula (2.5) gives a Lie algebra structure on $\mathfrak{h} \oplus s(\mathfrak{g})$, if $\alpha : \mathfrak{g} \to \mathrm{der}(\mathfrak{h})$ and $\rho : \bigwedge^2 \mathfrak{g} \to \mathfrak{h}$ satisfy (2.3) and (2.4).

**3. Motivation: Lie algebra extensions associated to a principal bundle.** Let $\pi : P \to M = P/K$ be a principal bundle with structure group $K$; i.e. $P$ is a manifold with a free right action of a Lie group $K$ and $\pi$ is the projection on the orbit space $M = P/K$. Denote by $\mathfrak{g} = \mathfrak{X}(M)$ the Lie algebra of the vector fields on $M$, by $\mathfrak{e} = \mathfrak{X}(P)^K$ the Lie algebra of $K$-invariant vector fields on $P$ and by $\mathfrak{h} = \mathfrak{X}_{\mathrm{vert}}(P)^K$ the ideal of the $K$-invariant vertical vector fields of $\mathfrak{e}$. Geometrically, $\mathfrak{e}$ is the Lie algebra of infinitesimal automorphisms of the principal bundle $P$ and $\mathfrak{h}$ is the ideal of infinitesimal automorphisms acting trivially on $M$, i.e. the Lie algebra of infinitesimal gauge transformations. We have a natural homomorphism $\pi_* : \mathfrak{e} \to \mathfrak{g}$ with the kernel $\mathfrak{h}$, i.e. $\mathfrak{e}$ is an extension of $\mathfrak{g}$ by means $\mathfrak{h}$.

Note that we have an additional structure of $C^\infty(M)$-module on $\mathfrak{g}, \mathfrak{h}, \mathfrak{e}$, such that $[X, fY] = f[X,Y] + (\pi_* X)fY$, where $X, Y \in \mathfrak{e}$, $f \in C^\infty(M)$. In particular, $\mathfrak{h}$ is a Lie algebra over $C^\infty(M)$. The extension

$$0 \to \mathfrak{h} \to \mathfrak{e} \to \mathfrak{g} \to 0$$

is also an extension of $C^\infty(M)$-modules.

Assume now that the section $s : \mathfrak{g} \to \mathfrak{e}$ is a homomorphism of $C^\infty(M)$-modules. Then it can be considered as a connection in the principal bundle $\pi$, and the $\mathfrak{h}$-valued 2-form $\rho$ as its curvature. In this sense we interpret the constructions from section 1 as follows. See [7], section 11 for more background information. The analogy with differential geometry has also been noticed by [8] and [9].

**4. Geometric interpretation.** Note that (2.2) looks like the Maurer-Cartan formula for the *curvature* on principal bundles of differential geometry

$$\rho = ds + \tfrac{1}{2}[s,s]_\wedge,$$

where for an arbitrary vector space $V$ the usual Chevalley differential is given by

$$d : L^p_{\mathrm{skew}}(\mathfrak{g}; V) \to L^{p+1}_{\mathrm{skew}}(\mathfrak{g}; V)$$

$$d\varphi(X_0, \dots, X_p) = \sum_{i<j} (-1)^{i+j} \varphi([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_p)$$

and where for a vector space $W$ and a Lie algebra $\mathfrak{f}$ the $\mathbb{N}$-graded (super) Lie bracket $[\quad,\quad]_\wedge$ on $L^*_{\text{skew}}(W,\mathfrak{f})$ is given by

$$[\varphi,\psi]_\wedge(X_1,\dots,X_{p+q}) = \frac{1}{p!\,q!}\sum_\sigma \operatorname{sign}(\sigma)\,[\varphi(X_{\sigma 1},\dots,X_{\sigma p}),\psi(X_{\sigma(p+1)},\dots)]_{\mathfrak{f}}.$$

Similarly formula (2.3) reads as

$$\operatorname{ad}_\rho = d\alpha + \tfrac{1}{2}[\alpha,\alpha]_\wedge.$$

Thus we view $s$ as a *connection* in the sense of a *horizontal lift* of vector fields on the base of a bundle, and $\alpha$ as an *induced connection*. Namely, for every $\operatorname{der}(\mathfrak{h})$-module $V$ we put

$$\alpha_\wedge : L^p_{\text{skew}}(\mathfrak{g};V) \to L^{p+1}_{\text{skew}}(\mathfrak{g};V)$$
$$\alpha_\wedge\varphi(X_0,\dots,X_p) = \sum_{i=0}^p (-1)^i \alpha_{X_i}(\varphi(X_0,\dots,\widehat{X_i},\dots,X_p)).$$

Then we have the *covariant exterior differential* (on the sections of an associated vector bundle)

$$\delta_\alpha : L^p_{\text{skew}}(\mathfrak{g};V) \to L^{p+1}_{\text{skew}}(\mathfrak{g};V), \qquad \delta_\alpha\varphi = \alpha_\wedge\varphi + d\varphi, \tag{3.1}$$

for which formula (2.4) looks like the *Bianchi identity* $\delta_\alpha\rho = 0$. Moreover one finds quickly that another well known result from differential geometry holds, namely

$$\delta_\alpha\delta_\alpha(\varphi) = [\rho,\varphi]_\wedge, \quad \varphi \in L^p_{\text{skew}}(\mathfrak{g};\mathfrak{h}). \tag{3.2}$$

If we change the linear section $s$ to $s' = s + b$ for linear $b : \mathfrak{g} \to \mathfrak{h}$, then we get

$$\alpha'_X = \alpha_X + \operatorname{ad}^{\mathfrak{h}}_{b(X)} \tag{3.3}$$
$$\begin{aligned} \rho'(X,Y) &= \rho(X,Y) + \alpha_X b(Y) - \alpha_Y b(X) - b([X,Y]) + [bX,bY] \\ &= \rho(X,Y) + (\delta_\alpha b)(X,Y) + [bX,bY]. \\ \rho' &= \rho + \delta_\alpha b + \tfrac{1}{2}[b,b]_\wedge. \end{aligned} \tag{3.4}$$

**5. Theorem.** *Let $\mathfrak{h}$ and $\mathfrak{g}$ be Lie algebras.*

*Then isomorphism classes of extensions of $\mathfrak{g}$ over $\mathfrak{h}$, i.e. short exact sequences of Lie algebras $0 \to \mathfrak{h} \to \mathfrak{e} \to \mathfrak{g} \to 0$, modulo the equivalence described by the commutative diagram of Lie algebra homomorphisms*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{h} & \longrightarrow & \mathfrak{e} & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\varphi} & & \| & & \\ 0 & \longrightarrow & \mathfrak{h} & \longrightarrow & \mathfrak{e}' & \longrightarrow & \mathfrak{g} & \longrightarrow & 0, \end{array}$$

*correspond bijectively to equivalence classes of data of the following form:*

$$\text{(5.1)} \qquad \textit{A linear mapping } \alpha : \mathfrak{g} \to \operatorname{der}(\mathfrak{h}),$$

$$\text{(5.2)} \qquad \textit{a skew-symmetric bilinear mapping } \rho : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h}$$

*such that*

$$\text{(5.3)} \qquad [\alpha_X, \alpha_Y] - \alpha_{[X,Y]} = \operatorname{ad}_{\rho(X,Y)},$$

$$\text{(5.4)} \qquad \sum_{\text{cyclic}} \Big( \alpha_X \rho(Y,Z) - \rho([X,Y],Z) \Big) = 0 \quad \textit{equivalently, } \delta_\alpha \rho = 0.$$

*On the vector space* $\mathfrak{e} := \mathfrak{h} \oplus \mathfrak{g}$ *a Lie algebra structure is given by*

$$\text{(5.5)} \quad [H_1 + X_1, H_2 + X_2]_{\mathfrak{e}} = [H_1, H_2]_{\mathfrak{h}} + \alpha_{X_1} H_2 - \alpha_{X_2} H_1 + \rho(X_1, X_2) + [X_1, X_2]_{\mathfrak{g}},$$

*the associated exact sequence is*

$$0 \to \mathfrak{h} \xrightarrow{i_1} \mathfrak{h} \oplus \mathfrak{g} = \mathfrak{e} \xrightarrow{\operatorname{pr}_2} \mathfrak{g} \to 0.$$

*Two data* $(\alpha, \rho)$ *and* $(\alpha', \rho')$ *are equivalent if there exists a linear mapping* $b \colon \mathfrak{g} \to \mathfrak{h}$ *such that*

$$\text{(5.6)} \qquad \alpha'_X = \alpha_X + \operatorname{ad}^{\mathfrak{h}}_{b(X)},$$

$$\text{(5.7)} \qquad \begin{aligned} \rho'(X,Y) &= \rho(X,Y) + \alpha_X b(Y) - \alpha_Y b(X) - b([X,Y]) + [b(X), b(Y)] \\ \rho' &= \rho + \delta_\alpha b + \tfrac{1}{2}[b,b]_\wedge, \end{aligned}$$

*the corresponding isomorphism being*

$$\mathfrak{e} = \mathfrak{h} \oplus \mathfrak{g} \to \mathfrak{h} \oplus \mathfrak{g} = \mathfrak{e}', \qquad H + X \mapsto H - b(X) + X.$$

*Moreover, a datum* $(\alpha, \rho)$ *corresponds to a split extension (a semidirect product) if and only if* $(\alpha, \rho)$ *is equivalent to to a datum of the form* $(\alpha', 0)$ *(then* $\alpha'$ *is a homomorphism). This is the case if and only if there exists a mapping* $b : \mathfrak{g} \to \mathfrak{h}$ *such that*

$$\text{(5.8)} \qquad \rho = -\delta_\alpha b - \tfrac{1}{2}[b,b]_\wedge.$$

*Proof.* Straigthforward computations. □

**6. Corollary.** [10] *Let* $\mathfrak{g}$ *and* $\mathfrak{h}$ *be Lie algebras such that* $\mathfrak{h}$ *has no center. Then isomorphism classes of extensions of* $\mathfrak{g}$ *over* $\mathfrak{h}$ *correspond bijectively to Lie homomorphisms*

$$\bar{\alpha} : \mathfrak{g} \to \operatorname{out}(\mathfrak{h}) = \operatorname{der}(\mathfrak{h}) / \operatorname{ad}(\mathfrak{h}).$$

*Proof.* If $(\alpha, \rho)$ is a data, then the map $\bar{\alpha} : \mathfrak{g} \to \operatorname{der}(\mathfrak{h}) / \operatorname{ad}(\mathfrak{h})$ is a Lie algebra homomorphism by (5.3). Conversely, let $\bar{\alpha}$ be given. Choose a linear lift $\alpha : \mathfrak{g} \to$

$\operatorname{der}(\mathfrak{h})$ of $\bar\alpha$. Since $\bar\alpha$ is a Lie algebra homomorphism and $\mathfrak{h}$ has no center, there is a uniquely defined skew symmetric linear mapping $\rho : \mathfrak{g}\times\mathfrak{g}\to\mathfrak{h}$ such that $[\alpha_X,\alpha_Y]-\alpha_{[X,Y]} = \operatorname{ad}_{\rho(X,Y)}$. Condition (5.4) is then automatically satisfied. For later use also, we record the simple proof:

$$\begin{aligned}
&\sum_{\text{cyclic}X,Y,Z}\Big[\alpha_X\rho(Y,Z)-\rho([X,Y],Z),H\Big]\\
&=\sum_{\text{cyclic}X,Y,Z}\Big(\alpha_X[\rho(Y,Z),H]-[\rho(Y,Z),\alpha_XH]-[\rho([X,Y],Z),H]\Big)\\
&=\sum_{\text{cyclic}X,Y,Z}\Big(\alpha_X[\alpha_Y,\alpha_Z]-\alpha_X\alpha_{[Y,Z]}-[\alpha_Y,\alpha_Z]\alpha_X+\alpha_{[Y,Z]}\alpha_X\\
&\qquad\qquad\qquad -[\alpha_{[X,Y]},\alpha_Z]+\alpha_{[[X,Y]Z]}\Big)H\\
&=\sum_{\text{cyclic}X,Y,Z}\Big([\alpha_X,[\alpha_Y,\alpha_Z]]-[\alpha_X,\alpha_{[Y,Z]}]-[\alpha_{[X,Y]},\alpha_Z]+\alpha_{[[X,Y]Z]}\Big)H=0.
\end{aligned}$$

Thus $(\alpha,\rho)$ describes an extension by theorem 5. The rest is clear. $\square$

**7. Remarks.** If $\mathfrak{h}$ has no center and $\bar\alpha:\mathfrak{g}\to\operatorname{out}(\mathfrak{h})=\operatorname{der}(\mathfrak{h})/\operatorname{ad}(\mathfrak{h})$ is a given homomorphism, the extension corresponding to $\bar\alpha$ can be constructed in the following easy way: It is given by the pullback diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{h} & \longrightarrow & \operatorname{der}(\mathfrak{h})\times_{\operatorname{out}(\mathfrak{h})}\mathfrak{g} & \xrightarrow{\operatorname{pr}_2} & \mathfrak{g} & \longrightarrow & 0\\
 & & \| & & \downarrow{\scriptstyle \operatorname{pr}_1} & & \downarrow{\scriptstyle\bar\alpha} & & \\
0 & \longrightarrow & \mathfrak{h} & \longrightarrow & \operatorname{der}(\mathfrak{h}) & \xrightarrow{\pi} & \operatorname{out}(\mathfrak{h}) & \longrightarrow & 0
\end{array}$$

where $\operatorname{der}(\mathfrak{h})\times_{\operatorname{out}(\mathfrak{h})}\mathfrak{g}$ is the Lie subalgebra

$$\operatorname{der}(\mathfrak{h})\times_{\operatorname{out}(\mathfrak{h})}\mathfrak{g}:=\{(D,X)\in\operatorname{der}(\mathfrak{h})\times\mathfrak{g}:\pi(D)=\bar\alpha(X)\}\subset\operatorname{der}(\mathfrak{h})\times\mathfrak{g}.$$

We owe this remark to E. Vinberg.

If $\mathfrak{h}$ has no center and satisfies $\operatorname{der}(\mathfrak{h})=\mathfrak{h}$, and if $\mathfrak{h}$ is normal in a Lie algebra $\mathfrak{e}$, then $\mathfrak{e}\cong\mathfrak{h}\oplus\mathfrak{e}/\mathfrak{h}$, since $\operatorname{Out}(\mathfrak{h})=0$.

**8. Theorem.** *Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras and let*

$$\bar\alpha:\mathfrak{g}\to\operatorname{out}(\mathfrak{h})=\operatorname{der}(\mathfrak{h})/\operatorname{ad}(\mathfrak{h})$$

*be a Lie algebra homomorphism. Then the following are equivalent:*

1. *For one (equivalently: any) linear lift $\alpha:\mathfrak{g}\to\operatorname{der}(\mathfrak{h})$ of $\bar\alpha$ choose $\rho:\bigwedge^2\mathfrak{g}\to\mathfrak{h}$ satisfying $([\alpha_X,\alpha_Y]-\alpha_{[X,Y]})=\operatorname{ad}_{\rho(X,Y)}$. Then the $\delta_{\bar\alpha}$-cohomology class of $\lambda=\lambda(\alpha,\rho):=\delta_\alpha\rho:\bigwedge^3\mathfrak{g}\to Z(\mathfrak{h})$ in $H^3(\mathfrak{g};Z(\mathfrak{h}))$ vanishes.*
2. *There exists an extension $0\to\mathfrak{h}\to\mathfrak{e}\to\mathfrak{g}\to 0$ inducing the homomorphism $\bar\alpha$.*

*If this is the case then all extensions* $0 \to \mathfrak{h} \to \mathfrak{e} \to \mathfrak{g} \to 0$ *inducing the homomorphism* $\bar{\alpha}$ *are parameterized by* $H^2(\mathfrak{g}, (Z(\mathfrak{h}), \bar{\alpha}))$*, the second Chevalley cohomology space of* $\mathfrak{g}$ *with values in the center* $Z(\mathfrak{h})$*, considered as* $\mathfrak{g}$*-module via* $\bar{\alpha}$*.*

*Proof.* Using once more the computation in the proof of corollary 6 we see that $\operatorname{ad}(\lambda(X,Y,Z)) = \operatorname{ad}(\delta_\alpha\rho(X,Y,Z)) = 0$ so that $\lambda(X,Y,Z) \in Z(\mathfrak{h})$. The Lie algebra $\operatorname{out}(\mathfrak{h}) = \operatorname{der}(\mathfrak{h})/\operatorname{ad}(\mathfrak{h})$ acts on the center $Z(\mathfrak{h})$, thus $Z(\mathfrak{h})$ is a $\mathfrak{g}$-module via $\bar{\alpha}$, and $\delta_{\bar{\alpha}}$ is the differential of the Chevalley cohomology. Using (3.2) we see that

$$\delta_{\bar{\alpha}}\lambda = \delta_\alpha\delta_\alpha\rho = [\rho,\rho]_\wedge = -(-1)^{2\cdot 2}[\rho,\rho]_\wedge = 0,$$

so that $[\lambda] \in H^3(\mathfrak{g}; Z(\mathfrak{h}))$.

Let us check next that the cohomology class $[\lambda]$ does not depend on the choices we made. If we are given a pair $(\alpha,\rho)$ as above and we take another linear lift $\alpha' : \mathfrak{g} \to \operatorname{der}(\mathfrak{h})$ then $\alpha'_X = \alpha_X + \operatorname{ad}_{b(X)}$ for some linear $b : \mathfrak{g} \to \mathfrak{h}$. We consider

$$\rho' : \bigwedge^2 \mathfrak{g} \to \mathfrak{h}, \quad \rho'(X,Y) = \rho(X,Y) + (\delta_\alpha b)(X,Y) + [b(X), b(Y)].$$

Easy computations show that

$$\begin{aligned}[]
[\alpha'_X, \alpha'_Y] - \alpha'_{[X,Y]} &= \operatorname{ad}_{\rho'(X,Y)}\\
\lambda(\alpha,\rho) = \delta_\alpha\rho &= \delta_{\alpha'}\rho' = \lambda(\alpha',\rho')
\end{aligned}$$

so that even the cochain did not change. So let us consider for fixed $\alpha$ two linear mappings

$$\rho, \rho' : \bigwedge^2 \mathfrak{g} \to \mathfrak{h}, \quad [\alpha_X, \alpha_Y] - \alpha_{[X,Y]} = \operatorname{ad}_{\rho(X,Y)} = \operatorname{ad}_{\rho'(X,Y)}.$$

Then $\rho - \rho' =: \mu : \bigwedge^2 \mathfrak{g} \to Z(\mathfrak{h})$ and clearly $\lambda(\alpha,\rho) - \lambda(\alpha,\rho') = \delta_\alpha\rho - \delta_\alpha\rho' = \delta_{\bar{\alpha}}\mu$.

If there exists an extension inducing $\bar{\alpha}$ then for any lift $\alpha$ we may find $\rho$ as in 5 such that $\lambda(\alpha,\rho) = 0$. On the other hand, given a pair $(\alpha,\rho)$ as in (1) such that $[\lambda(\alpha,\rho)] = 0 \in H^3(\mathfrak{g}, (Z(\mathfrak{h}), \bar{\alpha}))$, there exists $\mu : \bigwedge^2 \mathfrak{g} \to Z(\mathfrak{h})$ such that $\delta_{\bar{\alpha}}\mu = \lambda$. But then

$$\operatorname{ad}_{(\rho-\mu)(X,Y)} = \operatorname{ad}_{\rho(X,Y)}, \quad \delta_\alpha(\rho - \mu) = 0,$$

so that $(\alpha, \rho - \mu)$ satisfy the conditions of 5 and thus define an extension which induces $\bar{\alpha}$.

Finally, suppose that (1) is satisfied, and let us determine how many extensions there exist which induce $\bar{\alpha}$. By 5 we have to determine all equivalence classes of data $(\alpha,\rho)$ as in 5. We may fix the linear lift $\alpha$ and one mapping $\rho : \bigwedge^2 \mathfrak{g} \to \mathfrak{h}$ which satisfies (5.3) and (5.4), and we have to find all $\rho'$ with this property. But then $\rho - \rho' = \mu : \bigwedge^2 \mathfrak{g} \to Z(\mathfrak{h})$ and

$$\delta_{\bar{\alpha}}\mu = \delta_\alpha\rho - \delta_\alpha\rho' = 0 - 0 = 0$$

so that $\mu$ is a 2-cocycle. Moreover we may still pass to equivalent data in the sense of 5 using some $b : \mathfrak{g} \to \mathfrak{h}$ which does not change $\alpha$, i.e. $b : \mathfrak{g} \to Z(\mathfrak{h})$. The corresponding $\rho'$ is, by (5.7), $\rho' = \rho + \delta_\alpha b + \frac{1}{2}[b,b]_\wedge = \rho + \delta_{\bar{\alpha}} b$. Thus only the cohomology class of $\mu$ matters. □

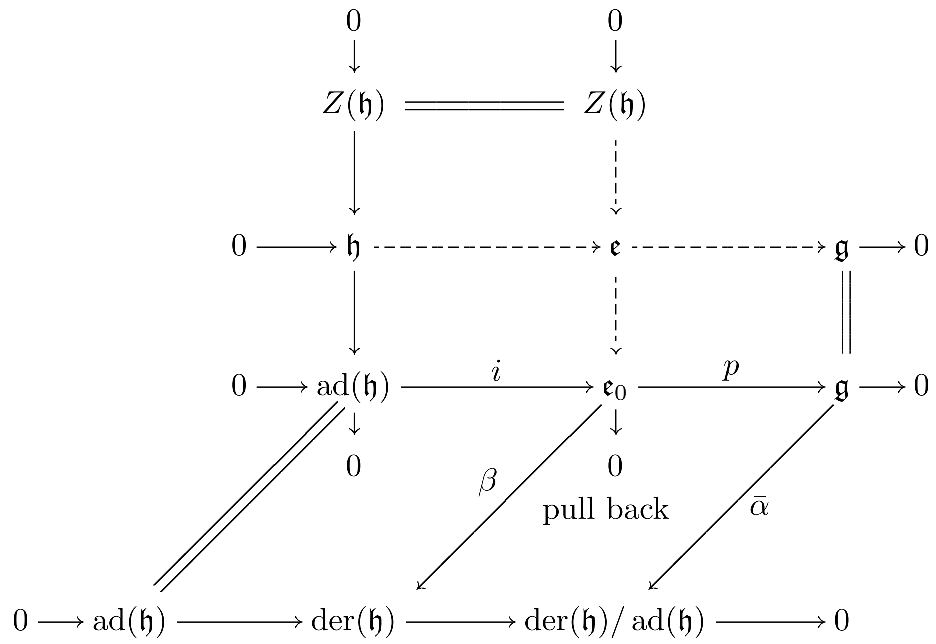
**9. Corollary.** *Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras such that $\mathfrak{h}$ is abelian. Then isomorphism classes of extensions of $\mathfrak{g}$ over $\mathfrak{h}$ correspond bijectively to the set of all pairs $(\alpha, [\rho])$, where $\alpha : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{h}) = \operatorname{der}(\mathfrak{h})$ is a homomorphism of Lie algebras and $[\rho] \in H^2(\mathfrak{g}, \mathfrak{h})$ is a Chevalley cohomology class with coefficients in the $\mathfrak{g}$-module $\mathfrak{h}$.*

*Proof.* This is obvious from theorem 8. □

**10. An interpretation of the class $\lambda$.** Let $\mathfrak{h}$ and $\mathfrak{g}$ be Lie algebras and let a homomorphism $\bar{\alpha} : \mathfrak{g} \to \operatorname{der}(\mathfrak{h})/\operatorname{ad}(\mathfrak{h})$ be given. We consider the extension

$$0 \to \operatorname{ad}(\mathfrak{h}) \to \operatorname{der}(\mathfrak{h}) \to \operatorname{der}(\mathfrak{h})/\operatorname{ad}(\mathfrak{h}) \to 0$$

and the following diagram, where the bottom right hand square is a pullback (compare with remark 7):




The left hand vertical column describes $\mathfrak{h}$ as a central extension of $\operatorname{ad}(\mathfrak{h})$ with abelian kernel $Z(\mathfrak{h})$ which is moreover killed under the action of $\mathfrak{g}$ via $\bar{\alpha}$; it is given by a cohomology class $[\nu] \in H^2(\operatorname{ad}(\mathfrak{h}); Z(\mathfrak{h}))^{\mathfrak{g}}$. In order to get an extension $\mathfrak{e}$ of $\mathfrak{g}$ with kernel $\mathfrak{h}$ as in the third row we have to check that the cohomology class $[\nu]$ is in the image of $i^* : H^2(\mathfrak{e}_0; Z(\mathfrak{h})) \to H^2(\operatorname{ad}(\mathfrak{h}); Z(\mathfrak{h}))^{\mathfrak{g}}$. It would be interesting to express this in terms of of the Hochschild-Serre exact sequence, see [6].

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