COMPLETING LIE ALGEBRA ACTIONS TO LIE GROUP ACTIONS

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Abstract. For a finite dimensional Lie algebra $\mathfrak{g}$ of vector fields on a manifold M we show that M can be completed to a G-space in a unversal way, which however is neither Hausdorff nor T_1 in general. Here G is a connected Lie group with Lie-algebra $\mathfrak{g}$. For a transitive $\mathfrak{g}$-action the completion is of the form G/H for a Lie subgroup H which need not be closed. In general the completion can be constructed by completing each $\mathfrak{g}$-orbit.

1. Introduction. In [7], Palais investigated when one could extend a local Lie group action to a global one. He did this in the realm of non-Hausdorff manifolds, since he showed, that completing a vector field X on a Hausdorff manifold M may already lead to a non-Hausdorff manifold on which the additive group $\mathbb{R}$ acts. We reproved this result in [3], being unaware of Palais' result. In [4] this result was extended to infinite dimensions and applied to partial differential equations like Burgers' equation: Solutions of the PDE were continued beyond the shocks and the universal completion was identified.

Here we give a detailed description of the universal completion of a Hausdorff $\mathfrak{g}$-manifold to a G-manifold. For a homogeneous $\mathfrak{g}$-manifold (where the finite dimensional Lie algebra $\mathfrak{g}$ acts infinitesimally transitive) we show that the G-completion (for a Lie group G with Lie algebra $\mathfrak{g}$) is a homogeneous space G/H for a possibly non-closed Lie subgroup H (theorem 7). In example 8 we show that each such situation can indeed be realized. For general $\mathfrak{g}$-manifolds we show that one can complete each $\mathfrak{g}$-orbit separately and replace the $\mathfrak{g}$-orbits in M by the resulting G-orbits to obtain the universal completion ${}_GM$ (theorem 9). All $\mathfrak{g}$-invariant structures on M 'extend' to G-invariant structures on ${}_GM$. The relation between our results and those of Palais are described in 10.

2. $\mathfrak{g}$-manifolds. Let $\mathfrak{g}$ be a Lie algebra. A $\mathfrak{g}$-manifold is a (finite dimensional Hausdorff) connected manifold M together with a homomorphism of Lie algebras $\zeta = \zeta^M : \mathfrak{g} \to \mathfrak{X}(M)$ into the Lie algebra of vector fields on M. We may assume without loss that it is injective; if not replace $\mathfrak{g}$ by $\mathfrak{g}/\ker(\zeta)$. We shall also say that $\mathfrak{g}$ acts on M.

The image of ζ spans an integrable distribution on M, which need not be of constant rank. So through each point of M there is a unique maximal leaf of that distribution; we also call it the $\mathfrak{g}$-orbit through that point. It is an *initial submanifold* of M in the sense that a mapping from a manifold into the orbit is smooth if and only if it is smooth into M, see [5], 2.14ff.

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Let $\ell : G \times M \to M$ be a left action of a Lie group with Lie algebra $\mathfrak{g}$. Let $\ell_a : M \to M$ and $\ell^x : G \to M$ be given by $\ell_a(x) = \ell^x(a) = \ell(a,x) = a.x$ for $a \in G$ and $x \in M$. For $X \in \mathfrak{g}$ the *fundamental vector field* $\zeta_X = \zeta_X^M \in \mathfrak{X}(M)$ is given by $\zeta_X(x) = -T_e(\ell^x).X = -T_{(e,x)}\ell.(X, 0_x) = -\partial_t|_0 \exp(tX).x$. The minus sign is necessary so that $\zeta : \mathfrak{g} \to \mathfrak{X}(M)$ becomes a Lie algebra homomorphism. For a right action the fundamental vector field mapping without minus would be a Lie algebra homomorphism. Since left actions are more common, we stick to them.

3. The graph of the pseudogroup. Let M be a $\mathfrak{g}$-manifold, effective and connected, so that the action $\zeta = \zeta^M : \mathfrak{g} \to \mathfrak{X}(M)$ is injective. Recall from [1], 2.3 that the pseudogroup $\Gamma(\mathfrak{g})$ consists of all diffeomorphisms of the form

$$\mathrm{Fl}_{t_n}^{\zeta_{X_n}} \circ \ldots \circ \mathrm{Fl}_{t_2}^{\zeta_{X_2}} \circ \mathrm{Fl}_{t_1}^{\zeta_{X_1}} |U$$

where $X_i \in \mathfrak{g}$, $t_i \in \mathbb{R}$, and $U \subset M$ are such that $\mathrm{Fl}_{t_1}^{\zeta_{X_1}}$ is defined on U, $\mathrm{Fl}_{t_2}^{\zeta_{X_2}}$ is defined on $\mathrm{Fl}_{t_1}^{\zeta_{X_1}}(U)$, and so on.

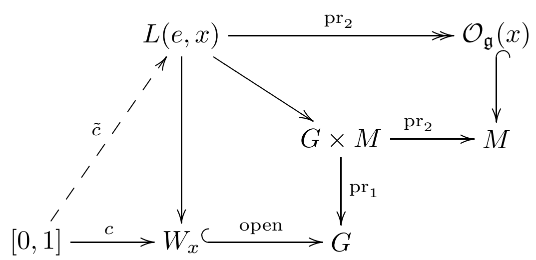
Now we choose a connected Lie group G with Lie algebra $\mathfrak{g}$, and we consider the integrable distribution of constant rank $d = \dim(\mathfrak{g})$ on $G \times M$ which is given by

$$\{(L_X(g), \zeta_X^M(x)) : (g,x) \in G \times M, X \in \mathfrak{g}\} \subset TG \times TM, \tag{3.1}$$

where L_X is the left invariant vector field on G generated by $X \in \mathfrak{g}$. This gives rise to the foliation $\mathcal{F}_\zeta$ on $G \times M$, which we call the *graph foliation* of the $\mathfrak{g}$-manifold M.

Consider the following diagram, where $L(e,x)$ is the leaf through (e,x) in $G \times M$, $\mathcal{O}_\mathfrak{g}(x)$ is the $\mathfrak{g}$-orbit through x in M, and $W_x \subset G$ is the image of the leaf $L(e,x)$ in G. Note that $\mathrm{pr}_1 : L(e,x) \to W_x$ is a local diffeomorphism for the smooth structure of $L(e,x)$.

(3.2)



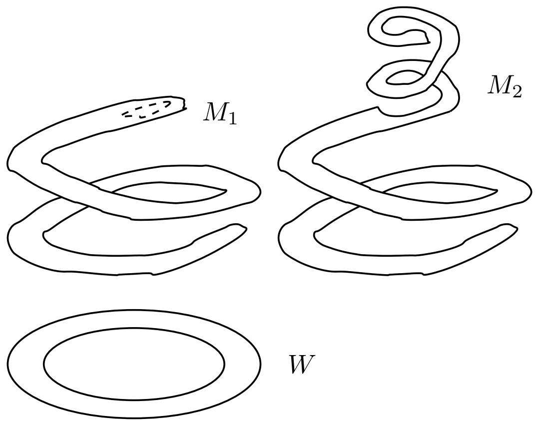
Moreover we consider a piecewise smooth curve $c : [0,1] \to W_x$ with $c(0) = e$ and we assume that it is liftable to a smooth curve $\tilde{c} : [0,1] \to L(e,x)$ with $\tilde{c}(0) = (e,x)$. Its endpoint $\tilde{c}(1) \in L(e,x)$ does not depend on small (i.e. liftable to $L(e,x)$) homotopies of c which respect the ends. This lifting depends smoothly on the choice of the initial point x and gives rise to a local diffeomorphism $\gamma_x(c) : U \to \{e\} \times U \to \{c(1)\} \times U' \to U'$, a typical element of the pseudogroup $\Gamma(\mathfrak{g})$ which is defined near x. See [1], 2.3 for more information and example 4 below. Note, that the leaf $L(g,x)$ through (g,x) is given by

$$L(g,x) = \{(gh, y) : (h,y) \in L(e,x)\} = (\mu_g \times \mathrm{Id})(L(e,x)) \tag{3.3}$$

where $\mu : G \times G \to G$ is the multiplication and $\mu_g(h) = gh = \mu^h(g)$.

4. Examples. It is helpful to keep the following examples in mind, which elaborate upon [1], 5.3. Let $G = \mathfrak{g} = \mathbb{R}^2$, let W be an annulus in $\mathbb{R}^2$ containing 0, and let M_1 be a simply connected piece of finite or infinite length of the universal cover of W. Then the Lie algebra $\mathfrak{g} = \mathbb{R}^2$ acts on M but not the group. Let $p : M_1 \to W$ be the restriction of the covering map, a local diffeomorphism.

Here $G \times_{\mathfrak{g}} M_1 \cong G = \mathbb{R}^2$. Namely, the graph distribution is then also transversal to the fiber of $\mathrm{pr}_2 : G \times M_1 \to M_1$ (since the action is transitive and free on M_1), thus describes a principal G-connection on the bundle $\mathrm{pr}_2 : G \times M_1 \to M_1$. Each leaf is a covering of M_1 and hence diffeomorphic to M_1 since M_1 is simply connected. For $g \in \mathbb{R}^2$ consider $j_g : M_1 \xrightarrow{\mathrm{ins}_g} \{g\} \times M_1 \subset G \times M_1 \xrightarrow{\pi} G \times_{\mathfrak{g}} M_1$ and two points $x \neq y \in M_1$. We may choose a smooth curve γ in M_1 from x to y, lift it into the leaf $L(g,x)$ and project it to a curve c in $g + W$ from g to $c(1) = g + p(y) - p(x) \in g + W$. Then (g,x) and $(c(1),y)$ are on the same leaf. So $j_g(x) = j_g(y)$ if and only if $p(x) = p(y)$. So we see that $j_g(x) = g + p(x)$, and thus $G \times_{\mathfrak{g}} M_1 = \mathbb{R}^2$. This will also follow from 7.



Let us further complicate the situation by now omitting a small disk in M_1 so that it becomes non simply connected but still projects onto W, and let M_2 be a simply connected component of the universal cover of M_1 with the disk omitted. What happens now is that homotopic curves which act equally on M_1 act differently on M_2.

It is easy to see with the methods described below that the completion ${}_GM_i = \mathbb{R}^2$ in both cases.

5. Enlarging to group actions. In the situation of 3 let us denote by ${}_GM = G \times_{\mathfrak{g}} M = G \times M/\mathcal{F}_\zeta$ the space of leaves of the foliation $\mathcal{F}_\zeta$ on $G \times M$, with the quotient topology. For each $g \in G$ we consider the mapping

$$j_g : M \xrightarrow{\mathrm{ins}_g} \{g\} \times M \subset G \times M \xrightarrow{\pi} {}_GM = G \times_{\mathfrak{g}} M. \tag{5.1}$$

Note that the submanifolds $\{g\} \times M \subset G \times M$ are transversal to the graph foliation $\mathcal{F}_\zeta$. The leaf space ${}_GM$ of $G \times M$ of admits a unique smooth structure, possibly singular and non-Hausdorff, such that a mapping $f : {}_GM \to N$ into a smooth manifold N is smooth if and only if the compositions $f \circ j_g : M \to N$ are smooth. For example we may use the structure of a *Frölicher space* or *smooth space* induced by the mappings j_g in the sense of [6], section 23 on ${}_GM = G \times_{\mathfrak{g}} M$. The canonical open maps $j_g : M \to {}_GM$ for $g \in G$ are called the charts of ${}_GM$: By construction, for each $x \in M$ and for $g'g^{-1}$ near enough to e in G there exists a curve $c : [0,1] \to W_x$ with $c(0) = e$ and $c(1) = g'g^{-1}$ and an open neighborhood U of x in M such that for the smooth transformation $\gamma_x(c)$ in the pseudogroup $\Gamma(\mathfrak{g})$ we have

$$j_{g'}|U = j_g \circ \gamma_x(c). \tag{5.2}$$

Thus the mappings j_g may serve as a replacement for charts in the description of the smooth structure on ${}_GM$. Note that the mappings j_g are not injective in general. Even if $g = g'$ there might be liftable smooth loops c in W_x such that (5.2) holds. Note also some similarity of the system of 'charts' j_g with the notion of an *orbifold* where one uses finite groups instead of pseudogroup transformations.

The leaf space ${}_GM = G \times_{\mathfrak{g}} M$ is a smooth G–space where the G-action is induced by $(g', x) \mapsto (gg', x)$ in $G \times M$.

Theorem. *The G–completion ${}_GM$ has the following universal properties:*

(5.3) *Given any Hausdorff G-manifold N and $\mathfrak{g}$-equivariant mapping $f : M \to N$ there exists a unique G-equivariant continuous mapping $\tilde f : {}_GM \to N$ with $\tilde f \circ j_e = f$. Namely, the mapping $\bar f : G \times M \to N$ given by $\bar f(g, x) = g.f(x)$ is smooth and factors to $\tilde f : {}_GM \to N$.*

(5.4) *In the setting of* (5.3)*, the universal property holds also for the T_1-quotient of ${}_GM$, which is given as the quotient $G\times M/\overline{\mathcal{F}_\zeta}$ of $G\times M$ by the equivalence relation generated by the closure of leaves.*

(5.5) *If M carries a symplectic or Poisson structure or a Riemannian metric such that the $\mathfrak{g}$–action preserves this structure or is even a Hamiltonian action then the structure 'can be extended to ${}_GM$ such that the enlarged G-action preserves these structures or is even Hamiltonian'.*

Proof. (5.3) Consider the mapping $\bar f = \ell^N \circ (\mathrm{Id}_G \times f) : G \times M \to N$ which is given by $\bar f(g, x) = g.f(x)$. Then by (3.1) and (3.2) we have for $X \in G$

$$\begin{aligned} T\bar f.(L_X(g), \zeta^M_X(x)) &= T\ell.(L_X(g), T_xf.\zeta^M_X(x)) \\ &= T\ell.(R_{\mathrm{Ad}(g)X}(g), 0_{f(x)}) + T\ell(0_g, \zeta^N_X(f(x))) \\ &= -\zeta_{\mathrm{Ad}(g)X}(g.f(x)) + T\ell_g.\zeta^N_X(f(x)) = 0. \end{aligned}$$

Thus $\bar f$ is constant on the leaves of the graph foliation on $G \times M$ and thus factors to $\tilde f : {}_GM \to N$. Since $\bar f(g.g_1, x) = g.g_1.f(x) = g.\bar f(g, x)$, the mapping $\tilde f$ is G-equivariant. Since N is Hausdorff, $\tilde f$ is even constant on the closure of each leaf, thus (5.4) holds also.

(5.5) Let us treat Poisson structure P on M. For symplectic structures or Riemannian metrics the argument is similar and simpler. Since the Lie derivative along fundamental vector fields of P vanishes, the pseudogroup transformation $\gamma_x(c)$ in (5.2) preserves P. Since ${}_GM$ is the quotient of the disjoint union of all spaces $\{g\} \times M$ for $g \in G$ under the equivalence relation described by (5.2), P 'passes down to this quotient'. Note that we refrain from putting too much meaning on this statement. □

The universal property (5.3) holds also for smooth G-spaces N which need not be Hausdorff, nor T_1, but should have tangent spaces and foliations so that it is meaningful to talk about $\mathfrak{g}$-equivariant mappings. We will not go into this, but see [6], section 23 for some concepts which point in this direction.

As an application of the universal property of the G–completion ${}_GM$, we see that ${}_GM$ depends on the choice of G in the following way. We write $G = \Gamma\backslash\widetilde{G}$, where $\widetilde{G}$ is the simply connected Lie group with Lie algebra $\mathfrak{g}$ and $\Gamma \subset \widetilde{G}$ is the discrete central subgroup such that $\Gamma \cong \pi_1(G)$. Then we have ${}_GM \cong \Gamma\backslash{}_{\widetilde{G}}M$ as G–spaces, so that ${}_{\widetilde{G}}M$ is potentially less singular than ${}_GM$.

6. Example. Let $\mathfrak{g} = \mathbb{R}^2$ with basis X, Y, let $M = \mathbb{R}^3 \setminus \{(0,0,z) : z \in \mathbb{R}\}$, and let $\zeta^\alpha : \mathfrak{g} \to \mathfrak{X}(M)$ be given by

$$(6.1) \qquad \zeta^\alpha_X = \partial_x + \alpha\,\frac{yz}{x^2+y^2}\,\partial_z \quad , \quad \zeta^\alpha_Y = \partial_y - \alpha\,\frac{xz}{x^2+y^2}\,\partial_z \ , \ \alpha > 0$$

which satisfy $[\zeta^\alpha_X, \zeta^\alpha_Y] = 0$. By construction of the graph foliation $\mathcal{F}_{\zeta^\alpha}$ in (3.1) and the procedure summarized in diagram (3.2), the leaves of $\mathcal{F}_{\zeta^\alpha}$ are determined explicitly as follows. For any smooth curve $c(t) = (\xi(t), \eta(t)) \in G$ starting at (ξ_0, η_0) we have $\dot c(t) = \dot\xi(t)\ X + \dot\eta(t)\ Y \in \mathfrak{g}$ and the lifted curve $(c(t), \mathbf{y}(t))$ is in the leaf $L((\xi_0,\eta_0), \mathbf{y}_0)$ if and only if it satisfies the first order ODE

$$(6.2) \qquad (\mathbf{y}(t), \dot{\mathbf{y}}(t)) = \dot\xi(t)\ \zeta^\alpha_X(\mathbf{y}(t)) + \dot\eta(t)\ \zeta^\alpha_Y(\mathbf{y}(t))$$

with initial value $\mathbf{y}(0) = \mathbf{y}_0 = (x_0, y_0, u = z_0) \in M$. Substituting (6.1) into (6.2), we see that this ODE is linear, that is $\dot x = \dot\xi$, $\dot y = \dot\eta$ and $\dot z = -\alpha\ z\ \frac{x\dot\eta - y\dot\xi}{r^2} = -\alpha\ z\ \frac{x\dot y - y\dot x}{r^2}$, where $r^2 = x^2 + y^2$. Thus the projection $\mathbf{x}(t)$ of $\mathbf{y}(t)$ to the (x,y)–plane is given by $\mathbf{x}(t) = c(t) - ((\xi_0,\eta_0) - \mathbf{x}_0) = c(t) - (\xi_0 - x_0, \eta_0 - y_0)$, whereas the third equation leads to

$$(6.3) \qquad z(t) = u\ e^{-\alpha\int_0^t d\theta} = u\ e^{-\alpha(\theta(t)-\theta_0)} = ue^{\alpha\theta_0}\ e^{-\alpha\theta(t)}\ ,$$

where θ is the angle function in the (x,y)–plane. This depends only on the endpoints $\mathbf{x}_0$, $\mathbf{x}(t)$ and the winding number of the curve $\mathbf{x}$ and is otherwise independent of $\mathbf{x}$. Incompleteness occurs whenever the curve $\mathbf{x}$ goes to $(0,0) \in \mathbb{R}^2$ in finite time $\bar t < \infty$, that is $\mathbf{x}(t) \to (0,0)$, $t \uparrow \bar t$ or equivalently $c(t) \to (\xi_0,\eta_0) - \mathbf{x}_0$, $t \uparrow \bar t$. It follows that the leaf $L((\xi_0,\eta_0),\mathbf{y}_0)$ is parametrized by $(r,\theta) \in \mathbb{R}_+ \times \mathbb{R}$ with $z = z(\theta)$ being independent of $r > 0$ and that

$$(6.4) \qquad \mathrm{pr}_1 : L((\xi_0,\eta_0),\mathbf{y}_0) \to W_{(\xi_0,\eta_0),\mathbf{y}_0} = \mathbb{R}^2 \setminus \{(\xi_0,\eta_0) - \mathbf{x}_0\}$$

in (3.2) is a universal covering. This is visibly consistent with (3.3). In order to parametrize the space of leaves ${}_GM$, we observe that the parameter $\mathbf{x}_0$ can be eliminated. In fact, from the previous formulas we see that

$$(6.5) \qquad L((\xi'_0,\eta'_0),(\mathbf{x}'_0,u')) = L((\xi_0,\eta_0),(\mathbf{x}_0,u)),$$

if and only if $(\xi'_0,\eta'_0) - \mathbf{x}'_0 = (\xi_0,\eta_0) - \mathbf{x}_0$ and $u' = ue^{\alpha(\theta_0-\theta'_0)}$, so that we have $z'(\theta) = u'e^{\alpha\theta'_0}\ e^{-\alpha\theta(t)} = ue^{\alpha\theta_0}\ e^{-\alpha\theta(t)} = z(\theta)$. In particular, it follows that

$$(6.6) \qquad L((\xi_0,\eta_0),\mathbf{y}_0) = L((\xi'_0+1,\eta'_0),(1,0,u')),$$

where $(\xi'_0,\eta'_0) = (\xi_0,\eta_0) - \mathbf{x}_0$, $u' = ue^{\alpha\theta_0}$, $\theta'_0 = 0$, projecting to $\mathbb{R}^2 \setminus \{(\xi'_0,\eta'_0)\}$. Therefore the leaves of the form $L((\xi_0+1,\eta_0),(1,0,u))$ are distinct for different values of (ξ_0,η_0) and fixed value of u and from the relation (3.3) we conclude that

$$(6.7) \qquad L((\xi_0+1,\eta_0),(1,0,u)) = (\xi_0,\eta_0) + L((1,0),(1,0,u)),$$

that is $G = \mathbb{R}^2$ acts without isotropy on ${}_GM$. We also need to determine the range for the parameter u. Obviously, we have $L((1,0),(1,0,u')) = L((1,0),(1,0,u))$ if and only if $u' = e^{2\pi\alpha n}u$ for $n \in \mathbb{Z}$. Thus these leaves are parametrized by $[u]$, taking values in the quotient of the additive group $\mathbb{R}$ under the multiplicative group $\{e^{2\pi\alpha n} : n \in \mathbb{Z}\}$, that is

$$\{0\} \cup \mathbb{S}^1_+ \cup \mathbb{S}^1_- \cong \{0\} \cup \mathbb{R}^\times_+/\{e^{2\pi\alpha n} : n \in \mathbb{Z}\} \cup \mathbb{R}^\times_-/\{e^{2\pi\alpha n} : n \in \mathbb{Z}\}. \tag{6.8}$$

The topology on the above space is determined by the leaf closures, respectively the orbit closures. First we have $\overline{L((\xi_0+1,\eta_0),(1,0,u))} = (\xi_0,\eta_0) + \overline{L((1,0),(1,0,u))}$ in $G \times M$ and it is sufficient to determine the closures of $L((1,0),(1,0,u))$. For $(1,0,u) \in M$ with $u \neq 0$ we consider the curve $c(\theta) = e^{i\theta} \in G = \mathbb{R}^2$. It is liftable to $G \times M$ and determines on M the curve $\mathbf{y}(t) = (\cos\theta, \sin\theta, ue^{-\alpha\theta})$. Thus the curve $(c(\theta), \mathbf{y}(\theta))$ in the leaf through $(1,0;1,0,u) \in G \times M \subset \mathbb{R}^5$ has a limit cycle for $\theta \to \infty$ which lies in the different leaf through $(1,0;1,0,0)$ which is closed, given by the (x,y)–plane $(\mathbb{R}^2 \times 0) \setminus 0$ at level $(1,0) \in G$. Thus we have

$$\overline{L((1,0),(1,0,u))} = L((1,0),(1,0,u)) \cup L((1,0),(1,0,0)). \tag{6.9}$$

Hence the leaf $L((1,0),(1,0,u))$ is not closed and the topological space ${}_GM$ is not T_1 and not a manifold. The orbits of the $\mathfrak{g}$-action are determined by the leaf structure via pr_2 in diagram (3.2) and they look here as follows: The (x,y)–plane $(\mathbb{R}^2 \times 0) \setminus 0$ is a closed orbit. Orbits above this plane are helicoidal staircases leading down and accumulating exponentially at the (x,y)–plane. Orbits below this plane are helicoidal staircases leading up and again accumulating exponentially. Thus the orbit space $M/\mathfrak{g}$ of the $\mathfrak{g}$-action is given by (6.8), with the point 0 being closed. By (6.9), the closure of any orbit represented by a point $[u]$ on one of the circles is given by $\{[u], 0\}$. From (6.6) and (6.7), we see that the G-completion ${}_GM$ has a section over the orbit space ${}_GM/G \cong M/\mathfrak{g}$ given by $[u] \mapsto L((1,0),(1,0,u))$. Therefore ${}_GM \cong G \times M/\mathfrak{g} = \mathbb{R}^2 \times \{\ \{0\} \cup \mathbb{S}^1_+ \cup \mathbb{S}^1_-\ \}$.

The structure of the completion and the orbit spaces are independent of the deformation parameter $\alpha > 0$ in (6.1). However for $\alpha \downarrow 0$, the completion just means adding in the z–axis, that is we get ${}_GM \cong \mathbb{R}^3$ with $G = \mathbb{R}^2$ acting by parallel translation on the affine planes $z = c$, and $M/\mathfrak{g} \cong {}_GM/G \cong \mathbb{R}$ as it should be.

It was pointed out to us [2] that one can make this example still more pathological: Consider the above example only in a cylinder over the anulus $0 < x^2 + y^2 < 1$. Add an open handle to the disk and continue the $\mathbb{R}^2$-action on the cylinder over the disk with an open handle added in such a way that there is a shift in the z–direction when one traverses the handle. Then one of the helicoidal staircases is connected to the the disk itself, so it accumulates onto itself. This is called a 'resilient leaf' in foliation theory.

7. Theorem. *Let M be a connected transitive effective $\mathfrak{g}$-manifold. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then we have:*

- (7.1) *Then there exists a subgroup $H \subset G$ such that the G-completion ${}_GM$ is diffeomorphic to G/H.*
- (7.2) *The Hausdorff quotient of ${}_GM$ is the homogeneous manifold $G/\overline{H}$. It has the following universal property: For each smooth $\mathfrak{g}$-equivariant mapping $f : M \to N$ into a Hausdorff G-manifold N there exists a unique smooth G-equivariant mapping $\tilde f : G/\overline{H} \to N$ with $f = \tilde f \circ \pi \circ j_e : M \to G/H \xrightarrow{\pi} G/\overline{H} \to N$.*
- (7.3) *For each leaf $L(g,x_0) \subset G \times M$ the projection $\mathrm{pr}_2 : L(g,x_0) \to M$ is a smooth fiber bundle with typical fiber H.*

Proof. (7.1) We choose a base point $x_0 \in M$. The G-completion is given by ${}_GM = G \times_{\mathfrak{g}} M$, the orbit space of the $\mathfrak{g}$-action on $G \times M$ which is given by $\mathfrak{g} \ni X \mapsto L_X \times \zeta_X^M$, and the G-action on the completion is given by multiplication from the left. The submanifold $G \times \{x_0\}$ meets each $\mathfrak{g}$-orbit in $G \times M$ transversely, since

$$\begin{aligned} T_{(g,x_0)}(G \times \{x_0\}) + T_{(g,x_0)}L(g, x_0) &= \{L_X(g) \times 0_{x_0} + L_Y(g) \times \zeta_Y(x_0) : X, Y \in \mathfrak{g}\} \\ &= T_{(g,x_0)}(G \times M). \end{aligned}$$

By (3.3) we have $L(g,x) = g.L(e,x)$ so that the isotropy Lie algebra $\mathfrak{h} = \mathfrak{g}_{x_0} = \{X \in \mathfrak{g} : \zeta_X(x_0) = 0\}$ is also given by

$$\begin{aligned} X \in \mathfrak{h} \quad &\iff \quad X \times 0_{x_0} \in T_{(e,x_0)}(G \times \{x_0\}) \cap T_{(e,x_0)}L(e, x_0) \\ &\iff \quad L_X(g) \times 0_{x_0} \in T_{(g,x_0)}(G \times \{x_0\}) \cap T_{(g,x_0)}L(g, x_0) \end{aligned}$$

Since $G \times \{x_0\}$ is a leaf of a foliation and the $L(e,x)$ also form a foliation, $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Let H_0 be the connected Lie subgroup of G which corresponds to $\mathfrak{h}$. Then clearly $H_0 \times \{x_0\} \subset G \times \{x_0\} \cap L(e, x_0)$. Let the subgroup $H \subset G$ be given by

$$H = \{g \in G : (g, x_0) \in L(e, x_0)\} = \{g \in G : L(g, x_0) = L(e, x_0)\},$$

then the C^∞-curve component of H containing e is just H_0. So H consists of at most countably many H_0-cosets. Thus H is a Lie subgroup of G (with a finer topology, perhaps). By construction the orbit space $G \times_{\mathfrak{g}} M$ equals the quotient of the transversal $G \times \{x_0\}$ by the relation induced by intersecting with each leaf $L(g, x_0)$ separately, i.e., $G \times_{\mathfrak{g}} M = G/H$.

(7.2) Obviously the T_1-quotient of G/H equals the Hausdorff quotient $G/\overline{H}$ which is a smooth manifold. The universal property is easily seen.

(7.3) Let $x \in M$ and $(g, x) \in L(e, x_0) = L(g, x) = g.L(e, x)$. So it suffices to treat the leaf $L(e, x)$. We choose $X_1, \dots X_n \in \mathfrak{g}$ such that $\zeta_{X_1}(x), \dots, \zeta_{X_n}(x)$ form a basis of the tangent space T_xM. Let $u : U \to \mathbb{R}^n$ be a chart on M centered at x such that $u(U)$ is an open ball in $\mathbb{R}^n$ and such that $\zeta_{X_1}(y), \dots, \zeta_{X_n}(y)$ are still linearly independent for all $y \in U$. For $y \in U$ consider the smooth curve $c_y : [0,1] \to U$ given by $c_y(t) = u^{-1}(t.u(y))$. We consider

$$\begin{aligned} &\partial_t c_y(t) = c_y'(t) = \textstyle\sum_{i=1}^n f_y^i(t)\, \zeta_{X_i}(c_y(t)), \quad f_y^i \in C^\infty([0,1], \mathbb{R}) \\ &X_y(t) = \textstyle\sum_{i=1}^n f_y^i(t)\, X_i \in \mathfrak{g}, \qquad X \in C^\infty([0,1], \mathfrak{g}) \\ &g_y \in C^\infty([0,1], G), \quad T(\mu_{g_y(t)})\partial_t g_y(t) = X_y(t), \quad g_y(0) = e, \end{aligned}$$

and everything is also smooth in $y \in U$. Then for $h \in H$ we have $(h.g_y(t), c_y(t)) \in L(e, x)$ since

$$\partial_t(h.g_y(t), c_y(t)) = (L_{X_y(t)}(h.g_y(t)), \zeta_{X_y(t)}(c_y(t))).$$

Thus $U \times H \ni (y, h) \mapsto \mathrm{pr}_2^{-1}(U) \cap L(e, x)$ is the required fiber bundle parameterization. □

8. Example. Let G be simply connected Lie group and let H be a connected Lie group of G which is not closed. For example, let $G = Spin(5)$ which is compact of rank 2 and let H be a dense 1-parameter subgroup in its 2-dimensional maximal torus. Let $\mathrm{Lie}(G) = \mathfrak{g}$ and $\mathrm{Lie}(H) = \mathfrak{h}$. We consider the foliation of G into right H-cosets gH which is generated by $\{L_X : X \in \mathfrak{h}\}$ and is left invariant under

G. Let U be a chart centered at e on G which is adapted to this foliation, i.e. $u : U \to u(U) = V_1 \times V_2 \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$ such that the sets $u^{-1}(V_1 \times \{x\})$ are the leaves intersected with U. We assume that V_1 and V_2 are open balls, and that U is so small that $\exp : W \to U$ is a diffeomorphism for a suitable convex open set $W \subset \mathfrak{g}$. Of course $\mathfrak{g}$ acts on U and respects the foliation, so this $\mathfrak{g}$-action descends to the leave space M of the foliation on U which is diffeomorphic to V_2.

Lemma. *In this situation, for the G-completion we have $G \times_{\mathfrak{g}} M = G/H$*

Proof. We use the method described in the end of the proof of theorem 7: ${}_GM = G \times_{\mathfrak{g}} M$ is the quotient of the transversal $G \times \{x_0\}$ by the relation induced by intersecting with each leaf $L(g, x_0)$ separately. Thus we have to determine the subgroup $H_1 = \{g \in G : (g, x_0) \in L(e, g)\}$.

Obviously any smooth curve $c_1 : [0, 1] \to H$ starting at e is liftable to $L(e, x_0)$ since it does not move $x_0 \in M$. So $H \subseteq H_1$, and moreover H is the C^∞-path component of the identity in H_1.

Conversely, if $c = (c_1, c_2) : [0, 1] \to L(e, x_0) \subset G \times M$ is a smooth curve from (e, x_0) to (g, x_0) then c_2 is a smooth loop through x_0 in M and there exists a smooth homotopy h in M which contracts c_2 to x_0, fixing the ends. Since $\mathrm{pr}_2 : L(e, x_0) \to M$ is a fiber bundle by (7.3) we can lift the homotopy h from M to $L(e, x_0)$ with starting curve c, fixing the ends, and deforming c to a curve c' in $L(e, x_0) \cap \mathrm{pr}_2^{-1}(x_0)$. Then $\mathrm{pr}_1 \circ c'$ is a smooth curve in H_1 connecting e and g.

Thus $H_1 = H$, and consequently ${}_GM = G/H$. □

9. Theorem. *Let M be a connected $\mathfrak{g}$-manifold. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then the G-completion ${}_GM$ can be described in the following way:*

(9.1) *Form the leaf space $M/\mathfrak{g}$, a quotient of M which may be non-Hausdorff and not T_1 etc.*

(9.2) *For each point $z \in M/\mathfrak{g}$, replace the orbit $\pi^{-1}(z) \subset M$ by the homogeneous space G/H_x described in theorem 7, where x is some point in the orbit $\pi^{-1}(z) \subset M$. One can use transversals to the $\mathfrak{g}$-orbits in M to describe this in more detail.*

(9.3) *For each point $z \in M/\mathfrak{g}$, one can also replace the orbit $\pi^{-1}(z) \subset M$ by the homogeneous space $G/\overline{H_x}$ described in theorem 7, where x is some point in the orbit $\pi^{-1}(z) \subset M$. The resulting G-space has then Hausdorff orbits which are smooth manifolds, but the same orbit space as $M/\mathfrak{g}$.*

See example 6 above.

Proof. Let $\mathcal{O}(x) \subset M$ be the $\mathfrak{g}$-orbit through x, i.e., the leaf through x of the singular foliation (with non-constant leaf dimension) on M which is induced by the $\mathfrak{g}$-action. Then the G-completion of the orbit $\mathcal{O}(x)$ is ${}_G\mathcal{O}(x) = G/H_x$ for the Lie subgroup $H_x \subset G$ described in theorem (7.1). By the universal property of the G-completion we get a G-equivariant mapping ${}_G\mathcal{O}(x) \to {}_GM$ which is injective and a homeomorphism onto its image, since we can repeat the construction of theorem (7.1) on M. Clearly the mapping $j_e : M \to {}_GM$ induces a homeomorphism between the orbit spaces $M/\mathfrak{g} \to {}_GM/G$.

Now let $s : V \to M$ be an embedding of a submanifold which is a transversal to the $\mathfrak{g}$-foliation at $s(v_0)$: We have $Ts \cdot T_{v_0}V \oplus \zeta_{s(v_0)}(\mathfrak{g}) = T_{s(v_0)}M$. Then s induces a mapping $V \to G \times M$ and $V \to {}_GM$ and we may use the point $s(v)$ in replacing $\mathcal{O}(s(v))$ by $G/H_{s(v)}$ for v near v_0. □

The following diagram summarizes the relation between the preceding constructions.

(9.4)
$$\begin{tikzcd}
M \arrow[r] \arrow[d, leftrightarrow, "="] & \bigcup_{[x]\in M/\mathfrak{g}} G/H_x \arrow[r, two heads] \arrow[d, "\cong"] & \bigcup_{[x]\in M/\mathfrak{g}} G/\overline{H_x} \arrow[d, dashed] \arrow[dl, dashed] \arrow[ddl, dashed] \\
M \arrow[r, "j_e"] \arrow[d, "\pi"] & {}_GM = G \times_{\mathfrak{g}} M \arrow[r, two heads] \arrow[d, "\pi_G"] & G \times M/\overline{\mathcal{F}}_\zeta \arrow[d, "\bar\pi_G"] \\
M/\mathfrak{g} \arrow[r, "\cong"] & {}_GM/G \arrow[r, two heads] & (G \times M/\overline{\mathcal{F}}_\zeta)/G
\end{tikzcd}$$

Note that taking the T_1–quotient $G \times M/\overline{\mathcal{F}}_\zeta$ of the leaf space ${}_GM$ may be a very severe reduction. In example 6 the isotropy groups H_x are trivial and we have $G \times M/\overline{\mathcal{F}}_\zeta = \mathbb{R}^2 \times \{0\}$ and $(G \times M/\overline{\mathcal{F}}_\zeta)/G = \{0\}$

10. Palais' treatment of $\mathfrak{g}$-manifolds. In [7], Palais considered $\mathfrak{g}$-actions on finite dimensional manifolds M in the following way. He assumed from the beginning, that M may be a non-Hausdorff manifold, since the completion may be non-Hausdorff. Then he introduces notions which we can express as follows in the terms introduced here:

(10.1) (M, ζ) is called *generating* if it generates a local G–transformation group. See [7], II,2, Def. V and II,7, Thm. XI. This holds if and only if the leaves of the graph foliation on $G \times M$ described in section 3 are Hausdorff. For Hausdorff $\mathfrak{g}$–manifolds this is always the case.

(10.2) (M, ζ) is called *uniform* if $\mathrm{pr}_1 : L(e, x) \to G$ in (3.2) is a covering map for each $x \in M$. See [7], III,6, Def. VIII and III,6, Thm. XVII, Cor., Cor.2. In the Hausdorff case the $\mathfrak{g}$–action is then complete and it may be integrated to a $\widetilde{G}$–action, where $\widetilde{G}$ is a simply connected Lie group with Lie algebra $\mathfrak{g}$, so that ${}_{\widetilde{G}}M \cong M$.

(10.3) (M, ζ) is called *univalent* if $\mathrm{pr}_1 : L(e, x) \to G$ in (3.2) is injective for $\forall x$. See [7], III,2, Def. VI and III,4, Thm. X.

(10.4) (M, ζ) is called *globalizable* if there exists a (non-Hausdorff) G-manifold N which contains M equivariantly as an open submanifold. See [7], III,1, Def. II and III,4, Thm. X. This is a severe condition which is not satisfied in examples 4 and 6 above.

Palais' main result on (non-Hausdorff) manifolds with a vector field says that (10.1), (10.3), and (10.4) are equivalent. See [7], III,7, Thm. XX.

On (non-Hausdorff) $\mathfrak{g}$-manifolds his main result is that (10.3) and (10.4) are equivalent. See [7], III,1, Def. II and III,4, Thm. X, and also III,2, Def. VI and III,4, Thm. X.

11. Concluding remarks. (11.1) A suitable setting for further development might be the class of *discrete* $\mathfrak{g}$–manifolds, that is $\mathfrak{g}$–manifolds for which the $\widetilde{G}$–space ${}_{\widetilde{G}}M$ is T_1, or equivalently the leaves of the graph foliation $\mathcal{F}_\zeta$ on $\widetilde{G} \times M$ are closed. In this case, the charts $j_g : M \to {}_{\widetilde{G}}M$ in (5.1) are local diffeomorphisms with respect to the unique smooth structure on ${}_{\widetilde{G}}M$ and ${}_{\widetilde{G}}M$ is a smooth manifold, albeit not necessarily Hausdorff.

(11.2) In the context of (11.1), there are several definitions of *proper* $\mathfrak{g}$–actions, all of which are equivalent to saying that the $\widetilde{G}$–action on ${}_{\widetilde{G}}M$ is proper. Many properties of proper actions will carry over to this case.

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