Manifolds of Differentiable Mappings

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Preface

Most of the work on this book was done while visiting the University of Mannheim during the academic year 1978/79. I want to thank E. Binz for his invitation and kind hospitality. The project of this book was supported by a research grant from the City of Vienna, 1978. The material included here has been presented in detail in a lecture course at the University of Vienna, 1979. A second volume on hard implicit function theorems on nuclear function spaces, and applications to manifolds of mappings is in the planning stage.

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Peter Michor

Contents

	Introduction	iii
§ 1	Jet bundles	1
§ 2	Manifolds with corners	18
§ 3	Topologies on spaces of continuous mappings	26
§ 4	Topologies on spaces of differentiable mappings	32
§ 5	Open subsets	42
§ 6	Transversality and dense subsets	50
§ 7	Continuity of certain canonical mappings	64
§ 8	Differential calculus on locally convex spaces	73
§ 9	Manifolds modelled on locally convex spaces	83
§ 10	Manifolds of mappings	90
§ 11	Differentiability of certain mappings	108
§ 12	Some tangent mappings	121
§ 13	The principal bundle of embeddings	129
§ 14	The Lie-group of symplectic diffeomorphisms	138
	References	144
	List of symbols	152
	Index	156

Introduction

This book is devoted to the theory of manifolds of differentiable mappings and contains results which can be proved without the help of a hard implicit function theorem on nuclear function spaces. All the necessary background is developed in detail: § 1 (Jet bundles) and § 2 (Manifolds with corners) contain basic material. § 3 - § 7 are devoted to the study of several canonical topologies on spaces of continuous and differentiable mapping and their properties. § 6 is devoted to transversality of mappings between manifolds with corners: the results therein seem to be new. § 8 covers the necessary facts from calculus on locally convex spaces. Here we restrict our attention to the simplest notion of differentiability, called C_c^∞, that admits a chain rule in general. It seems likely that nearly all the main notions of differentiability coincide in the case C^∞ (see H.H. KELLER (1974)), so there is no need to struggle with highly complicated remainder conditions. We prove the so-called Omega lemma and the existence of C_c^∞-partitions of unity on the class of locally convex vector spaces that appears later as the class of manifold model spaces: countable strict inductive limits of separable nuclear Fréchet spaces. § 9 contains general material on C_c^∞-manifolds and a first simple example: $J^\infty(X,Y)$, the C_c^∞-fibre bundle of ∞-jets. § 10 contains the core of the book: $C^\infty(X,Y)$ is made into a C_c^∞-manifold in a natural way, its tangent bundle is identified, certain splitting submanifolds are contructed, and the definite obstacle to a cartesian closed category (i.e. the natural equation $C_c^\infty(A, C_c^\infty(B,C)) = C_c^\infty(A \times B, C)$ in general) in our setting is investigated. § 11 shows that composition and inversion are C_c^∞, so the group Diff(X) of all diffeomorphisms of a finite dimensional manifold (even with corners) is a C_c^∞-Lie-group. It is well known, however, that its exponential mapping is not surjective on any open neighbourhood of the identity in general (see OMORI (1970)).

§ 12 is devoted to the computation of the tangent mappings for several canonical constructions of differential geometry; it is devoted to variational calculus. In § 13 the principal fibre bundle structure of the manifold of embeddings is investigated. § 14 is devoted to the C_c^∞-Lie-group of symplectic diffeomorphisms.

I did not strife for maximal generality in this book, rather for typical results and (hopefully) correct proofs: this field is full of erroneous proofs (e.g. LESLIE) in settings that are too simple.

In the last two decades Global Analysis seemed to become the theory and application of Sobolev spaces: it is convenient to work in a Hilbert space, even if it is an unnatural setting for the problem. The main difficulty there lies in the need for regularity theorems.

There is no Sobolev space in this book. I prefer the natural setting of C^∞, although the methods for solving non-linear partial differential equations are very limited - hopefully a good implicit function theorem will help.

1 Jet bundles

1.1 Jet bundles consist of all possible invariant expressions of Taylor-developments of mappings between manifolds. Their invention goes back to Ehresmann.

1.2 Let X, Y be smooth manifolds without boundary. We define a <u>k-jet from</u> X <u>to</u> Y to be an equivalence class $[f,x]_k$ of pairs (f,x) where $f: X \to Y$ is a smooth mapping, $x \in X$, and where two pairs (f,x), (f',x') are equivalent, $(f,x) \overset{k}{\sim} (f',x')$, iff $x = x'$ and f and f' have the same Taylor development of order k at x in some (hence any) pair of coordinate charts centered at x and $f(x)$ respectively. We write $[f,x]_k =: j^k f(x)$ and call that the <u>k-jet of</u> f <u>at</u> x. x is called the <u>source</u> of the jet, $f(x)$ is called the target.

The set of all k-jets from X to Y is denoted by $J^k(X,Y)$. There are the <u>source mapping</u> $\alpha: J^k(X,Y) \to X$ $\alpha(j^k f(x)) = x$, and the <u>target mapping</u> $\omega: J^k(X,Y) \to Y$, $\omega(j^k f(x)) = f(x)$. We will also use the following notation: $J^k_x(X,Y) := \alpha^{-1}(x)$, $J^k(X,Y) := \omega^{-1}(x)$, $J^k_{x,y}(X,Y) = J^k_x(X,Y) \cap J^k(X,Y)_y$. The last space is the set of all jets with source x and target y.

For formalists there is another definition of the equivalence relation $\overset{k}{\sim}$: $[f,x]_k = [f',x']_k$ iff $x = x'$ and $T^k_x f = T^k_x f'$ (T^k denotes the k-th tangent mapping).

1.3 Now we look at the special case $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$. We write $J^k(\mathbb{R}^n, \mathbb{R}^m) = J^k(n,m)$ too.

Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a smooth mapping. Then the k-jet of f at x has a canonical representative, the Taylor polynomial of order k of f at x:

$$\begin{aligned} f(x+t) &= f(x) + df(x).t + \tfrac{1}{2!}\, d^2f(x)t^2 + \ldots + \tfrac{1}{k!}\, d^kf(x)t^k + \\ &\quad + o(|t|^k) \\ &= f(x) + (T_x^k f)(t) + o(|t|^k) \end{aligned}$$

The "Taylor polynomial of f at x without constant" $T_x^k f\colon t \to T_x^k f(t) = \frac{1}{1!}\, df(x).t + \ldots + \frac{1}{k!}\, d^kf(x).t^k$ is an element of the linear space $P^k(n,m) = \prod_{j=1}^{k} L^j_{sym}(\mathbb{R}^n, \mathbb{R}^m)$, where $L^j_{sym}(\mathbb{R}^n, \mathbb{R}^m)$ is the vector space of j-linear symmetric mappings $\mathbb{R}^n \to \mathbb{R}^m$ (using the total polarisation of polynomials). Conversely each polynomial $p \in P^k(n,m)$ corresponds to the k-jet $[t \to y + p(t-x), x]_k$ with (arbitrary) source x and target y. So we get a canonical identification $J^k_{x,y}(n,m) = P^k(n,m)$, $J^k(n,m) = \mathbb{R}^n \times \mathbb{R}^m \times P^k(n,m)$.

If $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ are open subsets then clearly $J^k(U,V) = U \times V \times P^k(n,m)$ in the same canonical way.

1.4 For later uses we consider the <u>"truncated composition"</u> $\bullet\colon P^k(n,m) \times P^k(1,n) \to P^k(1,m)$, $p \bullet q =$ = (polynomial $p \circ q$ without terms of order $> k$). This is a polynomial mapping of the coefficients of the polynomials, so it is real analytic. Now let $U \subseteq \mathbb{R}^n$, $W \subseteq \mathbb{R}^1$, $V \subseteq \mathbb{R}^m$ be open subsets, let $J^k(U,V) \times_U J^k(W,U) :=$ $= \{(\sigma,\tau) \in J^k(U,V) \times J^k(W,U)\colon \alpha(\sigma) = \omega(\tau) \text{ in } U\} \cong$ $\cong U \times W \times V \times P^k(n,m) \times P^k(1,n)$. Then $\gamma\colon J^k(U,V) \times_U J^k(W,U) \to J^k(W,V)$, given by $\gamma(\sigma,\tau) =$ $= \gamma((\alpha(\sigma),\omega(\sigma),\bar{\sigma}), (\alpha(\tau),\omega(\tau),\bar{\tau})) =$ $= (\alpha(\tau),\omega(\sigma),\bar{\sigma} \bullet \bar{\tau}) \in W \times V \times P^k(1,m)$, is a real analytic mapping, called the <u>fibered composition of jets</u>. It will be used heavily later on.

1.5 Let $U, U' \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be open and let $g: U' \to U$ be a smooth diffeomorphism. Define $J^k(g,V): J^k(U,V) \to J^k(U',V)$ by $J^k(g,V)\ [f,x]_k = [f \circ g,\ g^{-1}(x)]_k$. Using the canonical polynomial representation of jets (1.4) $J^k(g,V)$ has the following form:
$J^k(g,V).\sigma = \gamma(\sigma, j^k g(g^{-1}(x)))$, or $J^k(g,V)(x,y,\bar{\sigma}) = (g^{-1}(x), y, \bar{\sigma} \bullet (T^k_{g^{-1}(x)} g))$. $J^k(g,V)$ is a C^{r-k} diffeomorphism, if g is a C^r diffeomorphism. If $g': U'' \to U'$ is another diffeomorphism, then clearly $J^k(g',V) \circ J^k(g,V) = J^k(g \circ g', V)$, and $J^k(g^{-1},V) = J^k(g,V)^{-1}$. So $J^k(.,V)$ is a contravariant functor acting on diffeomorphisms between open subsets of $\mathbb{R}^n$. Since the truncated composition $\bar{\sigma} \to \bar{\sigma} \bullet (T^k_{g^{-1}(x)} g)$ is a linear mapping, the mapping $J^k_x(g,\mathbb{R}^m) := J^k(g,\mathbb{R}^m) | J^k_x(U,\mathbb{R}^m): J^k_x(U,\mathbb{R}^m) \to J^k_{g^{-1}(x)}(U',\mathbb{R}^m)$ is linear.

1.6 Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$, $W \subseteq \mathbb{R}^l$ be open subsets, let $h: V \to W$ be a smooth mapping. Define $J^k(U,h): J^k(U,V) \to J^k(U,W)$ by $J^k(U,h)[f,x]_k = [h \circ f, x]_k$ or $J^k(U,h)\sigma = \gamma(j^k h(\omega(\sigma)), \sigma)$ or $J^k(U,h)(x,y,\bar{\sigma}) = (x, h(y), (T^k_y k) \bullet \bar{\sigma})$. $J^k(U,h)$ is C^{r-k} if h is C^r. Clearly we have $J^k(U,h) \circ J^k(U,h') = J^k(U, h \circ h')$, $J^k(U, Id_V) = Id_{J^k(U,V)}$; so $J^k(U,.)$ is a covariant functor, acting on smooth mappings between open subsets of finite dimensional vector spaces. The mapping $J^k_{x,y}(U,h): J^k_{x,y}(U,V) \to J^k_{x,h(y)}(U,V')$ is linear if the mapping $\bar{\sigma} \to (T^k_y h) \bullet \bar{\sigma}$ is linear (e.g. if h is affine, or if $k=1$).

1.7 Let $g: X' \to X$ be a diffeomorphism between manifolds. Then the mapping $J^k(g,Y): J^k(X,Y) \to J^k(X',Y)$, defined by $J^k(g,Y)(j^k f(x)) = j^k(f \circ g)(g^{-1}(x))$ or $J^k(g,Y)[f,x]_k = [f \circ g, g^{-1}(x)]_k$, is a bijection. Clearly $J^k(.,Y)$ is a contravariant functor, acting on diffeomorphisms of manifolds, with values in the category of sets and mappings.

If $h: Y \to Y'$ is a smooth mapping between manifolds, then we have the mapping $J^k(X,h): J^k(X,Y) \to J^k(X,Y')$, defined by $J^k(X,h)(j^k f(x)) = j^k(h \circ f)(x)$ or $J^k(X,h)[f,x]_k = [h \circ f, x]_k$. $J^k(X,.)$ is a covariant functor.

1.8 Let X,Y be smooth manifolds of dimensions n,m resp. and let (U,u) be a chart for X (i.e. $U \subseteq X$ is open, $u: U \to u(U) \subseteq \mathbb{R}^n$ is a diffeomorphism) and (V,v) be a chart for Y. We consider the set $J^k_{U,V}(X,Y) = J^k(U,V) = (\alpha \times \omega)^{-1}(U \times V) \subseteq J^k(X,Y)$ and the mapping $J^k(u^{-1},v) = J^k(u^{-1},V) \circ J^k(u(U),v) = J^k(U,v) \circ J^k(u^{-1},v(V)): J^k(U,V) \to J^k(u(U),v(V))$. $J^k(u^{-1},v)$ is bijective and we will use $(J^k(U,V),\ J^k(u^{-1},v): J^k(U,V) \to J^k(u(U),v(V)))$ as a typical chart for $J^k(X,Y)$.

If (U',u'), (V',v') are some other charts for X,Y resp. then the chart change $J^k(u^{-1},v) \circ J^k(u'^{-1},v')^{-1}$: $J^k(u'(U \cap U'),\ v'(V \cap V')) \to J^k(u(U \cap U'),\ v(V \cap V'))$ is the mapping $J^k(u' \circ u^{-1},\ v \circ v'^{-1})$ which is a diffeomorphism by 1.5 and 1.6. It remains to check that $J^k(X,Y)$ is Hausdorff in the topology induced by the atlas of all charts of the form above. This is clear, since $J^k(X,Y)$ is not only a manifold with this atlas, but even a fibre bundle over $X \times Y$ in the sense of the following definition.

1.9 <u>Definition:</u> A C^∞ fibre bundle (E,π,B,F) consist of smooth manifolds E,B,F and a smooth mapping $\pi: E \to B$ which satisfies the following local triviality condition:

There is an open cover (U_i) of B and a family of diffeomorphisms $\psi_i: U_i \times F \to \pi^{-1}(U_i)$ such that $\pi \circ \psi_i(x,y) = x$ for all $x \in U_i$, $y \in F$.

$$\begin{tikzcd} U_i \times F \arrow[r, "\psi_i"] \arrow[dr, "pr_1"'] & \pi^{-1}(E) = :E|U_i \subseteq E \arrow[d, "\pi"] \\ & U_i \end{tikzcd}$$

E is called the <u>total space</u>, B is called the <u>basis</u>, π is the <u>projection</u>, F is the <u>typical fibre</u>. For $x \in B$ the set $E_x := \pi^{-1}(x)$ is the <u>fibre</u> over x. A family (U_i,ψ_i) as above is called a fibre bundle atlas. Let $U_{ij} = U_i \cap U_j$, then the mapping $\psi_j^{-1} \circ \psi_i \colon U_{ij} \times F \to U_{ij} \times F$ is a fibre respecting diffeomorphism, so it is of the following form: $\psi_j^{-1} \circ \psi_i(x,y) = (x, \psi_{ji}(x)(y))$ where $\psi_{ji}(x)\colon F \to F$ is a diffeomorphism for any x. If there is a (finite dimensional) Lie-group G acting smoothly on F (i.e. there is a C^∞ mapping $T\colon G \times F \to F$ such that $T(g_1, T(g_2,y)) = T(g_1 g_2, y)$ and $T(e,.) = Id_F$) and if all $\psi_{ji}(x)$ lie in G (i.e. if all $\psi_{ji}\colon U_{ij} \to G$ are smooth mappings and $\psi_{ji}(x)(y) = T(\psi_{ji}(x), y)$) then G is called a structure group of the fibre bundle.

If $F = \mathbb{R}^1$, and there is a structure group $GL(1,\mathbb{R})$, then $(E, \pi, B, \mathbb{R}^1)$ is said to have the structure of a vector bundle. With these notions we collect the structure of jet bundles in the following theorem.

1.10 Theorem: Let X,Y be smooth manifolds.

1. $J^k(X,Y)$ is a C^{r-k} manifold if X,Y are C^r manifolds; a canonical atlas is given by $\{(J^k(U,V), J^k(u^{-1},v))\colon (U,u)$ chart on X, (V,v) chart on $Y\}$.

2. $(J^k(X,Y), (\alpha,\omega), X \times Y, P^k(n,m))$ is a fibre bundle; the canonical atlas of 1. induces a fibre bundle atlas. $GL^k(n,\mathbb{R}) \times GL^k(m,\mathbb{R})$ is a structure group, where $GL^k(n,\mathbb{R}) = GL(n,\mathbb{R}) \times \prod_{j=2}^{k} P^j_{sym}(n,n)$ with truncated composition, $n = \dim X$, $m = \dim Y$.

3. $(J^k(X,\mathbb{R}^m), \alpha, X, \mathbb{R}^m \times P^k(n,m))$ is a vector bundle; $\{(U, J^k(u,\mathbb{R}^m) \circ (u \times Id)\colon U \times (\mathbb{R}^m \times P^k(n,m)) \to J^k(U,\mathbb{R}^m)\colon (U,u)$ chart on $X\}$ is a vector bundle atlas.

4. If $f\colon X \to Y$ is a C^r-mapping, then $j^k f\colon X \to J^k(X,Y)$ is a C^{r-k}-mapping, sometimes called the k-jet extension of f.

5. If $g\colon X' \to X$ is a (C^r-) diffeomorphism and $h\colon Y \to Y'$ is a (C^r) mapping, then $J^k(g,Y)\colon J^k(X,Y) \to J^k(X',Y)$ and $J^k(X,h)\colon J^k(X,Y) \to J^k(X,Y')$ are C^{r-k} mappings. $J^k(.,..)$ is a contra-covariant bifunctor.

6. For $k' < k$ in $\mathbb{N}_0$ we have canonical projections $\pi^k_{k'}\colon J^k(X,Y) \to J^{k'}(X,Y)$ given by $\pi^k_{k'}[f,x]_k = [f,x]_{k'}$. These

satisfy $\pi^k_{k'} \circ \pi^{k'}_{k''} = \pi^k_{k''}$, $\pi^k_o = (\alpha,\omega)$: $J^k(X,Y) \to X \times Y$.

7. $(J^k(X,Y), \pi^k_{k'}, J^k(X,Y), \prod_{j=k'+1}^{k} L^j_{sym}(\mathbb{R}^n,\mathbb{R}^m))$ are fibre bundles; $(J^k(X,Y), \pi^k_{k-1}, J^{k-1}(X,Y), L^k_{sym}(\mathbb{R}^n,\mathbb{R}^m))$ are affine bundles; $(J^1(X,Y), (\alpha,\omega), X \times Y)$ is a vector bundle and coincides with $(L(TX,TY),(\pi_X,\pi_Y),X\times Y)$. Furthermore we get: $J^1_o(\mathbb{R},Y) = TY$, $J^1(X,\mathbb{R})_o = T^*X$.

Proof:

1. see 1.8.

2. $\{(U\times V, J^k(u,v^{-1}) \circ ((u\times v)\times Id): (U\times V)\times P^k(n,m) \to$ $\to J^k(U,V)$: (U,u) chart on X, (V,v) chart on $Y\}$ is a fibre bundle atlas. The form of the structure group and its action can be read of 1.5, 1.6.

3. Follows from 1.5 where we checked that $J^k(u,\mathbb{R}^m)$: $J^k(u(U),\mathbb{R}^m) \to J^k(U,\mathbb{R}^m)$ is fibrewise linear.

4. If (U,u) is a chart on X and (V,v) is a chart on Y with $f(U) \subseteq V$, then $J^k(u^{-1},v) \circ j^k f \circ u^{-1}$: $u(U) \subseteq \mathbb{R}^n \to$ $\to J^k(u(U),v(V)) = u(U) \times v(V) \times P^k(n,m)$ is just the mapping $x \to (x, v\circ f\circ u^{-1}(x), T^k_x(v\circ f\circ u^{-1}))$ (cf. 1.3) which is visibly of class C^{r-k}.

5. See 1.7. Composing with the chart mappings of 1. and using 1.5, 1.6 it follows that these mappings are C^{r-k}.

6. and 7. In a local chart the mapping $\pi^k_{k'}$ is just truncation of Taylor polynomials to order k'. Let (U,u), (V,v) be charts of X,Y resp. Then we investigate: $J^{k'}(u^{-1},v) \circ \pi^k_{k'} \circ J^k(u^{-1},v)^{-1}$: $J^k(u(U),v(V)) \to J^{k'}(u(U), v(V))$. We have $J^k(u(U),v(V)) = u(U) \times v(V) \times P^k(n,m) =$
$= u(U)\times v(V) \times \prod_{j=1}^{k} L^j_{sym}(\mathbb{R}^n,\mathbb{R}^m) = u(U)\times v(V)\times P^{k'}(n,m) \times$
$\times \prod_{j=k'+1}^{k} L^j_{sym}(\mathbb{R}^n,\mathbb{R}^m) = J^{k'}(u(U),v(V)) \times \prod_{j=k'+1}^{k} L^j_{sym}(\mathbb{R}^n,\mathbb{R}^m)$
and the above mapping is just projection onto the first factor. Therefore $\pi^k_{k'}$: $J^k(X,Y) \to J^{k'}(X,Y)$ is locally trivial with unique fibre type, so is a fibre bundle projection.

To investigate π^k_{k-1} we have $J^k(\varphi(U),\psi(V)) = J^{k-1}(\varphi(U), \psi(V)) \times L^k_{sym}(\mathbb{R}^n,\mathbb{R}^m)$. A chart change in X acts linear on the fibres $P^k(n,m)$ anyhow by 1.5. A chart change h in Y acts on $L^k_{sym}(\mathbb{R}^n,\mathbb{R}^m)$ as follows: $\frac{1}{k!} d^k(h \circ f)(x) = \frac{1}{k!} dh(x) \circ d^k f(x) +$ a sum of terms involving $d^l f(x)$, $l<k$, multilinearly. The first part is linear in $d^k f(x)$, the second part produces the affine shift. So $(J^k(X,Y), \pi^k_{k-1}, J^{k-1}(X,Y), L^k_{sym}(\mathbb{R}^n,\mathbb{R}^m))$ is an affine bundle (i.e. the structure group is the affine group of the model fibre vector space). By the same method one can prove that a structure group of $\pi^k_{k'}$ is $GL^{k-k'}(n,\mathbb{R}) \times GL^{k-k'}(m,\mathbb{R})$, acting polynomially on the model fibre vector space (by truncated composition). $(J^1(X,Y), (\alpha,\omega), X\times Y, L(\mathbb{R}^n,\mathbb{R}^m))$ looks locally like: $J^1(\varphi(U),\psi(V)) = \varphi(U) \times \psi(V) \times L(\mathbb{R}^n,\mathbb{R}^m)$, $J^1(g,h)(x,y,\bar\sigma) = (g^{-1}(x),h(y), dh(y) \circ \bar\sigma \circ dg(g^{-1}(x)))$ is linear in $\bar\sigma$ for fixed x,y, iff g is a local diffeomorphism in $\mathbb{R}^n$ and h any smooth mapping on $\mathbb{R}^m$. But this gives just the same system of transition mappings as $L(TX,TY)$ has.

Finally: $J^1_o(\mathbb{R},Y) = L(T_o\mathbb{R},TY) = TY$, $J^1(X,\mathbb{R})_o = L(TX,T_o\mathbb{R}) = T^*X$. q.e.d.

1.11 Let (E,π,B,F) be a vector bundle, let $(U_i,\psi_i: U_i\times F\to E|U_i)_{i\in I}$ be a vector bundle atlas where the U_i are so small that there is an atlas (U_i,u_i) for B.

<u>Definition:</u> A smooth section of the bundle E (of π) is a smooth mapping $s: B\to E$ with $\pi\circ s = Id_B$. Let $\Gamma(E) = \Gamma(E,\pi,B)$ denote the space of all smooth sections of E with the pointwise linear structure, let $\Gamma_c(E) = \Gamma_c(E,\pi,B)$ denote the sub vector space of all smooth sections of E with compact support.

Our next aim is to prove that the set $\{j^k s(x): s\in\Gamma(E), x\in B\} \subseteq J^k(B,E)$ has the structure of a vector bundle in a natural way.

If $s\in\Gamma(E)$ then $\psi_i^{-1}\circ(s|U_i): U_i\to U_i\times F$ is of the form $x\to(x,s_i(x))$ for suitable $s_i: U_i\to F$.

For simplicity's sake we identify silently U_i with $u_i(U_i)\subseteq\mathbb{R}^n$. We compare $j^k s_i(x)\in J^k(U_i,F)$ with $j^k(\psi_i{}^{-1}\circ(s|U_i))(x)\in J^k(U_i,U_i\times F)$. Let $p=\dim F$.

$j^k s_i(x) = (x, s_i(x), T^k_x(s_i)) \in U_i \times F \times P^k(n,p)$. $J^k(U_i, U_i \times F) = = U_i \times (U_i \times F) \times P^k(n,n+p) \cong U_i \times F \times P^k(n,p) \times U_i \times P^k(n,n)$ and $j^k(\psi_i^{-1} \circ (s|U_i)) = (x, s_i(x), T^k_x(s_i), x, Id_{\mathbb{R}^n})$ in this decomposition.

Therefore the mapping ε: $J^k(U_i,F) \to J^k(U_i, U_i \times F)$, given by $\varepsilon(x,y,\bar{\sigma}) = (x,y,\bar{\sigma},x,Id_{\mathbb{R}^n})$, is an embedding of the vector bundle $(J^k(U_i,F), \alpha, U_i, F \times P^k(n,p))$ onto an "affine subbundle" of the fibre bundle $(J^k(U_i, U_i \times F), (\alpha,\omega), U_i \times U_i \times F, P^k(n,n+p))$. Now let $U_{ij} = U_i \cap U_j$ be silently identified with both $u_i(U_{ij})$ and $u_j(U_{ij})$ in $\mathbb{R}^n$. Then $\psi_i^{-1} \circ \psi_j$: $U_{ij} \times F \to U_{ij} \times F$ is of the form $\psi_i^{-1} \circ \psi_j(x,y) = = (x, \psi_{ij}(x).y)$ for transition mappings ψ_{ij}: $U_{ij} \to GL(F)$ (cf. 1.9).

For $s \in \Gamma(E)$ and $x \in U_{ij}$ we have $(x, s_i(x)) = \psi_i^{-1} s(x) = = \psi_i^{-1} \psi_j \psi_j^{-1} s(x) = \psi_i^{-1} \psi_j(x, s_j(x)) = (x, \psi_{ij}(x).s_j(x))$. So $s_i(x) = \psi_{ij}(x).s_j(x)$. Since $\psi_{ij}(x) \in GL(F)$ we have $(s + \lambda s')_i = s_i + \lambda s'_i$ etc. and $\psi_{ij}(x)(s_j(x) + \lambda s_j'(x)) = = \psi_{ij}(x)s_j(x) + \lambda\psi_{ij}(x)s_j'(x)$, so $T^k_x(\psi_{ij}.(s_j + \lambda s_j')) = = T^k_x(\psi_{ij}.s_j) + \lambda \quad T^k_x(\psi_{ij}.s_j')$. So the mapping ψ_{ij}: $J^k(U_{ij},F) \to J^k(U_{ij},F)$, given by $j^k s_j(x) \to j^k s_i(x)$, is a vector bundle homomorphism of $(J^k(U_{ij},F), \alpha, U_{ij}, F \times P^k(n,p))$ into itself (more exactly: over the mapping $u_{ij} = u_i \circ u_j^{-1}$: $u_j(U_{ij}) \to u_i(U_{ij})$). These transition mappings $\psi_{ij}^{\ k}$ can be used to glue all the trivial vector bundles $(J^k(U_i,F), \alpha, U_i, F \times P^k(n,p))$ into the vector bundle $J^k(E)$, which we define by this process. So we have proved the following result:

1.12 Theorem: Let (E,π,B,F) be a smooth vector bundle. Then we have:

1. $(J^k(E), \pi^k_o, B, F \times P^k(n,p))$ is a vector bundle for each k, where $p = \dim F$, $u = \dim B$. Here $J^k(E) = J^k(E,\pi,B)$ is the set of all k-jets of sections of E. For any $s \in \Gamma(E)$ we denote its k-jet at x by $\bar{j}^k s(x)$ if we consider it to be an element of $J^k(E)$.

2. *The mapping* $\epsilon = \epsilon^k\colon J^k(E) \to J^k(B,E)$, *given by* $\epsilon(\bar{j}^k s(x)) = j^k s(x)$, *is a smooth embedding, fibered over* $(\mathrm{Id}_B, s)\colon B \to B \times E$.

3. *If* $\varphi\colon (E,\pi,B,F) \to (E',\pi',B,F')$ *is a vector bundle homomorphism over* Id_B (*i.e.* $\varphi\colon E \to E'$ *is* C^∞, $\pi' \circ \varphi = \pi$, $\varphi_x = \varphi | E_x\colon E_x \to E_x{}'$ *is linear for all* x), *then* $J^k(\varphi)\colon J^k(E) \to J^k(E')$, *given by* $J^k(\varphi)\bar{j}^k s(x) = \bar{j}^k(\varphi . s)(x)$, *is a vector bundle homomorphism again.* $J^k(.)$ *is a covariant functor, acting on strict vector bundle homomorphisms, and* $\epsilon^k\colon J^k(.) \to J^k(B,.)$ *is a natural transformation.*

1.13 *Remark:* The following result shows that $J^k(TX)$ generate all "natural vector bundles over X" in some sense. The result is due to CHUU LIAN TERNG (1979) and D.B.A. EPSTEIN, W. THURSTON (1979).

A natural vector bundle over n-dimensional manifolds is a functor acting on the category of all n-manifolds with embeddings as morphisms, such that:

(1) $F(X)$ is a vector bundle over X for any n-manifold X.

(2) $F(i)\colon F(X) \to F(Y)$ is a continuous vector bundle map over $i\colon X \to Y$ for any embedding i.

Now the following holds: For any natural vector bundle F there is some k such that $F(i)$ depends only on $j^k i$. k is called the order of F. It follows that there is a one to one correspondence between isomorphism classes of natural vector bundles of order k and representations of the Lie group $GL(n) \times \prod_{j=2}^{n} L^j_{sym}(\mathbb{R}^n, \mathbb{R}^n)$.

1.14 *The vertical bundles of a fibre bundle.*

1. Let (E,p,X,F) be a fibre bundle. A locally trivializing fibre bundle atlas (U_i, ψ_i) of E consists of an open cover (U_i) of X and fibre respecting diffeomorphisms

$$\begin{tikzcd} U_i \times F \arrow[rr, "\psi_i"] \arrow[dr, "pr_1"'] & & E|U_i = p^{-1}(U_i) \arrow[dl, "p"] \\ & U_i & \end{tikzcd}$$

Then we have $\psi_i^{-1} \circ \psi_j(x,y) = (x, \psi_{ij}(x,y))$ for $x \in U_{ij} = U_i \cap U_j$; the $\psi_{ij}: U_{ij} \to \mathrm{Diff}(F)$ are called the transition mappings for E. If (V_α, v_α) is a manifold atlas for F and (U_i, u_i) is a manifold atlas for X, then $(\psi_i(U_i \times V_\alpha), (u_i \times v_\alpha) \circ \psi_i^{-1})_{i,\alpha}$ is a manifold atlas for E.

The coordinate transitions for this atlas look like $(x,y) \to (x',y') = (u_{ji}(x), v_{\beta\alpha}\bar{\psi}_{ji}(x,y)) = (u_j \circ u_i^{-1}(x), v_\beta \circ v_\alpha^{-1} \circ \psi_{ji}(u_i^{-1}(x),y))$.

2. In the setting of 1., the tangent bundle $(TE, \pi_E, E, \mathbb{R}^n \times \mathbb{R}^m)$ of the total space E of the fibre bundle (E,p, X,F) (where $n = \dim X$, $m = \dim F$) is a vector bundle. We have the vector bundle homomorphism of constant rank:

$$\begin{array}{ccc} TE & \xrightarrow{Tp} & TX \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi_X} \\ E & \xrightarrow{p} & X \end{array}$$

The coordinate transformations for TE which are induced by the atlas for E described in 1. look like:

$(x,y;\xi,\eta) \to (x',y';\xi',\eta')$

$x' = u_{ji}(x)$

$y' = v_{\beta\alpha}\bar{\psi}_{ji}(x,y)$

$\xi' = d(u_{ji})(x).\xi$

$\eta' = d_1(v_{\beta\alpha} \circ \bar{\psi}_{ji})(x,y).\xi + d_2(v_{\beta\alpha} \circ \bar{\psi}_{ji})(x,y).\eta$

3. The (fibre wise) kernel of the vector bundle homomorphism $Tp: TE \to TX$ is a vector subbundle of TE, the vertical bundle $(V(E), \pi_E|V(E), E, \mathbb{R}^m)$. In local coordinates (as described in 2.) Tp looks like:

$p(x,y) = x$

$Tp(x,y;\xi,\eta) = (x,\xi)$.

So locally V(E) is given by $\{(x,y;0,\eta)\}$.

1.15 The vertical bundle of a vector bundle

1. Let $(E,p,X,\mathbb{R}^m)$ be a vector bundle. Then the coordinate transformations of 1.14.1: $(x,y) \to (x',y')$ where $x' = u_{ji}(x)$

$y' = \bar{\psi}_{ji}(x).y$,

can be chosen in such a way, that $\bar{\psi}_{ji}(x,y)$ is linear in y for fixed x, and the atlas (V_α, v_α) is just $(\mathbb{R}^m, Id_{\mathbb{R}^m})$.

The coordinate transitions for $(TE, \pi_E, E, \mathbb{R}^n \times \mathbb{R}^m)$ are:

$(x,y;\xi,\eta) \to (x',y',\xi',\eta')$

$x' = \underline{u}_{ji}(x)$

$y' = \overline{\psi}_{ji}(x).y$

$\xi' = d\ \underline{u}_{ji}(x).\xi$

$\eta' = d_1\overline{\psi}_{ji}(x,y).\xi + d_2\overline{\psi}_{ji}(x,y).\eta$

$\quad = (d\ \underline{\psi}_{ji}(x).\xi).y + \overline{\psi}_{ji}(x).\eta$

The vertical bundle $(V(E), \pi_E | V(E), E, \mathbb{R}^m)$ is again locally given by $\{(x,y;0,\eta)\}$.

2. Since $(E,p,X,\mathbb{R}^m)$ is a vector bundle, TE has two vector bundle structures:

a) The tangent bundle structure $(TE, \pi_E, E, \mathbb{R}^n \times \mathbb{R}^m)$, given locally by $(x,y;\xi,\eta) + \lambda(x,y;\xi',\eta') = (x,y;\xi+\lambda\xi',\ \eta+\lambda\eta')$.

b) The derivative of the vector bundle structure $(E,p,X,\mathbb{R}^m)$, i.e. the vector bundle $(TE, Tp, TX, \mathbb{R}^m \times \mathbb{R}^m)$, given locally by $(x,y;\xi,\eta) \oplus \lambda(x,y';\xi,\eta') = (x,y+\lambda y';\xi,\eta+\lambda\eta')$.

3. <u>The vertical lift</u>: let $x \in X$, $E_x = p^{-1}(x) \subseteq E$ be the fibre over x, let $i_x: E_x \to E$ denote the embedding of the fibre.

For $v,w \in E_x$ let $V_x(v,w) = T_v(i_x).w \in TE$, where $T_v(i_x)$: $T_v(E_x) \cong E_x \to T_vE$. $V(v,w) = V_x(v,w)$ is called the <u>vertical lift of</u> w <u>over</u> v. In canonical coordinates V has the following form: $V((x,y),(x,z)) = (x,y;0,z)$.

$V: E \oplus E \to V(E)$ is a C^∞-mapping and even a vector bundle isomorphism $(E\oplus E,\ p, X, \mathbb{R}^m \times \mathbb{R}^m) \to (V(E), p \circ \pi_E = \pi_X \circ Tp, X, \mathbb{R}^m \times \mathbb{R}^m)$.

4. We will use the mapping $\zeta_E = pr_2 \circ V^{-1}$: $V(E) \to E \oplus E \to E$, given locally by $\zeta_E(x,y;0,\eta) = (x,\eta)$. ζ_E is called the <u>vertical projection</u>.

1.16 <u>The fibre derivative.</u> Let $(E_1,p_1,X_1,\mathbb{R}^m)$, $(E_2,p_2,X_2,\mathbb{R}^l)$ be vector bundles and let

$$\begin{array}{ccc} E_1 & \xrightarrow{\varphi} & E_2 \\ p_1 \downarrow & & \downarrow p_2 \\ X_1 & \xrightarrow{\overline{\varphi}} & X_2 \end{array}$$

be a fibre respecting smooth mapping. The <u>fibre derivative</u> of φ is the mapping

$$\begin{array}{ccc} E_1 \oplus E_1 & \xrightarrow{d_F\varphi} & E_2 \\ p_1 \downarrow & & \downarrow p_2 \\ X_1 & \xrightarrow{\overline{\varphi}} & X_2 \end{array}$$

defined like follows. For $x \in X_1$ we have a C^∞-mapping $\varphi_x = \varphi|(E_1)_x \colon (E_1)_x \to (E_2)_{\overline{\varphi}(x)}$ between vector spaces, so its derivative makes sense: $d(\varphi_x)(\eta).\eta' \in (E_2)_{\varphi(x)}$, $\eta,\eta' \in (E_1)_x$. We put $d_F\varphi(\eta,\eta') = d(\varphi_x)(\eta).\eta'$, if $\eta,\eta' \in (E_1)_x$. The local expression for $d_F\varphi$ is: $d_F\varphi\,((x,y),(x,y')) = (\overline{\varphi}(x),\ d_2\varphi(x,y).y')$.

It is clear that the following formula holds: $d_F\varphi = \zeta_{E_2} \circ T_\varphi \circ V_{E_1} \colon E_1 \oplus E_1 \to V(E_1) \to V(E_2) \to E_2$.

Of course the fibre derivative makes sense if φ is only defined on an open subset of E_1.

1.17 <u>Pullbacks of vector bundles</u>

Let $(E,p,X,\mathbb{R}^m)$ be a vector bundle, $f\colon Y \to X$ be a C^∞-mapping. Then the <u>pullback</u> of E by f is the vector bundle $(f^*E, f^*p, Y, \mathbb{R}^m)$, given by

$$\begin{array}{ccc} f^*E = & Y \times_X E & = \{(y,e) \in Y \times E,\ f(y) = p(e)\} \\ f^*p \downarrow & \downarrow pr_1 & \\ Y & = \ Y & \end{array}$$

If (U_i,ψ_i) is a locally trivializing vector bundle atlas for $(E,p,X,\mathbb{R}^m)$ (cf. 1.9, 1.15), i.e.

$$\begin{array}{ccc} U_i \times \mathbb{R}^m & \xrightarrow{\psi_i} & p^{-1}(U_i), \quad U_i \subseteq X \text{ open.} \\ & {\scriptstyle pr_1} \searrow \quad \swarrow {\scriptstyle p} & \\ & U_i & \end{array}$$

Then $(f^{-1}(U_i),\ f^*\psi_i)$ is a locally trivializing vector bundle atlas for $(f^*E, f^*p, Y, \mathbb{R}^m)$, where $(f^*\psi_i)(y,v) = (y, \psi_i(f(y),v))$, i.e.

$$\begin{array}{ccc} f^{-1}(U_i) \times \mathbb{R}^m & \xrightarrow{(Id \times (\psi_i \circ (f \times Id)))} & (f^*p)^{-1}(f^{-1}(U_i)) \\ & {\scriptstyle pr_1} \searrow \quad \swarrow & \\ & f^{-1}(U_i) & \end{array}$$

1.18 Lemma:

1. Let $\mathrm{Vect}(X)$ denote the category of vector bundles over X and strict vector bundle homomorphisms. If $f\colon Y \to X$ is a C^∞-mapping, then $f^* = Y \times_X .\colon \mathrm{Vect}(X) \to \mathrm{Vect}(Y)$ is a functor, mapping strict vector bundle homomorphisms to strict vector bundle homomorphisms.

2. In a natural way we have $T(f^*E) = T(Y \times_X E) = TY \times_{TX} TE$

3. For the vertical bundle we have $V(f^*E) = V(Y \times_X E) = $ $= Y \times_X V(E) = f^*V(E)$ in a natural way. Moreover $V_{f^*E}\colon f^*E \oplus f^*E \to V(f^*E)$ coincides with $Y \times_X (V_E)$: $Y \times_X (E \oplus E) = f^*(E \oplus E) = f^*E \otimes f^*E \to Y \times_X (V(E)) = f^*(V(E))$, and $\zeta_{f^*E}\colon V(f^*E) \to f^*E$ coincides with $Y \times_X (\zeta_E)$: $Y \times_X V(E) \to Y \times_X E$.

Proof: 1. is clear.

2. Since $p\colon E \to X$ is a submersion, $f \times p\colon Y \times E \to X \times X$ is transversal to the diagonal $\Delta_X \subseteq X \times X$, so $Y \times_X E =$ $= (f \times p)^{-1}(\Delta_X)$ is a submanifold of $Y \times E$ and we have
$T(Y \times_X E) = (T(f \times p))^{-1}(T\Delta_X) = \{(v_y, w_e) \in T(Y \times E)\colon$
$(Tf.v_y, Tp.w_e) \in T_{f(y)=p(e)}\Delta_X\}$,
$= \{(v,w) \in TY \times TE\colon Tf.v = Tp.w\}$
$= TY \times_{TX} TE$.

3. $(f^*E)_y = \{y\} \times E_{f(y)} \subseteq Y \times_X E$, so $T_{v_y}((f^*E)_y) = \{0_y\} \times$ $\times T_{v_y} E_{f(x)} \subseteq T(Y \times_X E)$, so $V(f^*E) = \bigcup_{v_y \in f^*E} T_{f_y}((f^*E)_y) =$
$= 0_y \times_X V(E) = Y \times_X V(E)$.
The rest is clear. q.e.d.

1.19 The double tangent bundle

Let X be a manifold. There is the double tangent bundle $T^2X = T(TX)$, the vertical subbundle $V(TX) \subseteq T(TX)$, the vertical lift $V_{TX}\colon TX \oplus TX \to V(TX)$ and the vertical projection $\zeta_{TX}\colon V(TX) \to TX$, cf. 1.15. But there is more structure in this special situation:

There is the canonical flip mapping $\varkappa_X\colon T^2X \to T^2X$, given locally in canonical coordinates by $\varkappa_X(x,y;\xi,\eta) = (x,\xi;y,\eta)$. $\varkappa_X$ is C^∞, idempotent: $\varkappa_X \circ \varkappa_X =$ $= \mathrm{Id}$; $\varkappa_X$ is a vector bundle isomorphism over Id_{TX} between the two vector bundle structures on T^2X:

$$\begin{array}{ccc} T^2X & \xrightarrow{\varkappa_X} & T^2X \\ \pi_{TX}\downarrow & & \downarrow\ T(\pi_X) \\ TX & \xrightarrow{Id_{TX}} & TX \end{array}$$

This is clear from the local expression. The fixed points of $\varkappa_X$ are given locally by $\{(x,y;y,\eta)\}$. These are exactly the elements of $J_o^2(\mathbb{R},X)$. For further use we note again:

$T(\pi_X)\circ\varkappa_X = \pi_{TX}$

$\pi_{TX}\circ\varkappa_X = T(\pi_X)$.

1.20 Vector fields and flows:

Let $\xi\in\mathfrak{X}(X)$ be a vector field on X, let $t\to\alpha_t$ denote its local flow: $\alpha_o(x)=x$, $\frac{d}{dt}\alpha_t(x)=\xi\alpha_t(x)$. Then $t\to T(\alpha_t)$ is a local flow on TX since T is a functor. We compute its time derivative locally: $\frac{d}{dt}T(\alpha_t)(x,y)=\frac{d}{dt}(\alpha_t(x),$ $d\alpha_t(x).y)=(\alpha_t(x),d\alpha_t(x).y;\ \dot\alpha_t(x),d\dot\alpha_t(x).y)=$
$=\varkappa_X(\alpha_t(x),\dot\alpha_t(x);\ d\alpha_t(x).y,d\dot\alpha_t(x).y)=\varkappa_X\ T(\dot\alpha_t)(x,y)$.
So we get: $\frac{d}{dt}T(\alpha_t)=\varkappa_X\circ T(\dot\alpha_t)=\varkappa_X\circ T(\xi\circ\alpha_t)=\varkappa_X\circ T\xi\circ T\alpha_t$.
$T\xi$: $TX\to T^2X$ is not a vector field, but $\varkappa_X\circ T\xi$ is one:
$\pi_{TX}\circ\varkappa_X\circ T\xi=T(\pi_X)\circ T\xi=T(\pi_X\circ\xi)=T(Id_X)=Id_{TX}$ (cf. 1.19).
So we have:

<u>Lemma: Let</u> $\xi\in\mathfrak{X}(X)$ <u>be a vector field on</u> X. <u>Then</u> $\varkappa_X\circ T\xi$ <u>is a vector field on</u> TX. <u>If</u> $t\to\alpha_t$ <u>is the local flow for</u> ξ, <u>then</u> $t\to T\alpha_t$ <u>is the local flow for</u> $\varkappa_X\circ T\xi$.

1.21 Sprays.

Let X be a manifold. We consider the mapping μ: $\mathbb{R}\times TX\to$ $\to TX$, $\mu(t,v)=t.v$. For any $t\neq 0$ the mapping μ_t: $TX\to TX$, $\mu_t(v)=t.v$, is a diffeomorphism.

<u>Definition:</u> A <u>spray</u> ξ on X is a C^∞-mapping ξ: $TX\to T^2X$ subject to the following conditions:

1. $\pi_{TX}\circ\xi=Id_{TX}$ (i.e. ξ is a vector field on TX)
2. $T(\pi_X)\circ\xi=Id_{TX}$
3. $T(\mu_t).\xi(v)=\frac{1}{t}.\xi(tv)$, $t\neq 0$.

Example: Let X be a manifold such that TX is trivial, $TX \cong X \times \mathbb{R}^n$. Let $\gamma\colon X \to L^2_{sym}(\mathbb{R}^n;\mathbb{R}^n)$ be a C^∞-mapping. Then the following mapping is a spray: $\xi(x,v) = (x,v;v,\gamma(x)(v,v))$.

Proof: $\mu_t(x,v) = (x,tv)$, so $T(\mu_t)(x,v;\xi,\eta) = (x,tv;\xi,t\eta)$. So $T(\mu_t).\xi(x,v) = T(\mu_t)(x,v;v,\gamma(x)(v,v)) = (x,tv;v,t\gamma(x)(v,v)) = \frac{1}{t}(x,tv;tv,t^2\gamma(x)(v,v)) = \frac{1}{t}(x,tv;tv,\gamma(x)(tv,tv)) = \frac{1}{t}\xi(x,tv)$.

It can be shown, that any spray on a parallelizable X is of this form. The mapping γ appears in Riemannian geometry as "Christoffel symbols".

Lemma: On any manifold X there is a spray.

Proof: Let (U_i,u_i) be an atlas for X. Then any U_i is a manifold with trivial tangent bundle, so there is a spray ξ_i on U_i by the example above. Let (φ_i) be a smooth partition of unity, subordinate to the open cover (U_i). Then $\xi = \sum_i (\varphi_i \circ \pi_X).\xi_i$ is a spray on X, since condition 2. above is affine, 1. and 3. are linear.

1.22 The local flow of a spray

Let ξ be a spray on the manifold X, let $\varphi\colon D(\xi) \subseteq TX \times \mathbb{R} \to TX$ be the local flow of ξ: $D(\xi) \supseteq TX \times \{0\}$, $D(\xi)$ is open and "radial" ($(v,r) \in D(\xi)$, $|t| \leq 1$ implies $(v,tr) \in D(\xi)$), and $\varphi(v,0) = v$, $\frac{d}{dt}\varphi_t = \xi \circ \varphi_t$, and $D(\xi)$ is maximal.

1. Claim: $\varphi(sv,t) = s.\varphi(v,st)$, $s,t \in \mathbb{R}$, if one side is defined.

Proof: For fixed s and $v \in TX$ let $\alpha(t) = \varphi(t,sv)$, $\beta(t) = s\varphi(v,st)$. Then $\alpha(0) = sv = \beta(0)$; $\dot\alpha(t) = \frac{d}{dt}\varphi(t,sv) = \xi(\varphi(t,sv)) = \xi(\alpha(t))$, $\dot\beta(t) = \frac{d}{dt}s\varphi(v,st) = \frac{d}{dt}(\mu_s\varphi(st,v)) = T(\mu_s)\frac{d}{dt}\varphi(st,v) = T(\mu_s).\xi(\varphi(st,v)).s = \frac{1}{s}\xi(s\varphi(st,v)).s = \xi(\beta(t))$. So α and β are integral curves of the vectorfield ξ with the same initial conditions, so $\alpha = \beta$. q.e.d.

2. Claim: Let $\chi = \pi_X \circ \varphi\colon D(\xi) \to X$. Then the following holds:

2.1. $\chi(0,v) = \pi_X(v)$

2.2. $\dot\chi_v(0) = v$

2.3. $\ddot\chi_v(t) = \xi(\dot\chi_v(t))$, $(v,t) \in D(\xi)$.

2.4. $\chi(t,sv)=\chi(st,v)$, $s,t\in\mathbb{R}$ if one side exists.

Proof: 2.1. $\chi(0,v)=\pi_X\varphi(0,v)=\pi_X(v)$. $\dot{\chi}_v(t)=$
$=\frac{d}{dt}\pi_X\varphi(t,v)=T(\pi_X)\dot{\varphi}_v(t)=T\pi_X\xi(\varphi_v(t))=\varphi_v(t)$ (by 1.21.2).
So $\dot{\chi}_v(0)=\varphi_v(0)=v$, so 2.2. holds. $\ddot{\chi}_v(t)=\dot{\varphi}_v(t)=\xi(\varphi_v(t))=$
$=\xi(\dot{\chi}_v(t))$, so 2.3. holds. 2.4. follows from claim 1:
$\chi(t,sv)=\pi_X\varphi(t,sv)=\pi_X(s.\varphi(st,v))=\pi_X(\varphi(st,v))=\chi(st,v)$.

q.e.d.

3. Claim: There is an open neighbourhood $U\subseteq TX$ of the zero section such that $[-1,1]\times U\subseteq D(\xi)$.

Proof: Let $v\in TX$; there is $\delta>0$ such that $(\delta,v)\in D(\xi)$.
By claim 1: $\delta.\varphi(\delta.1,v)=\varphi(1,\delta v)$, so $(1,\delta v)\in D(\xi)$. q.e.d.

1.23 The exponential mapping of a spray

Definition: In the setting of 1.22, the mapping exp: $U\subseteq TX\to X$, given by $\exp(v)=\chi(1,v)=\pi_X\varphi(1,v)$, is called the exponential mapping of the spray ξ. We write: $\exp_x=\exp|U\cap T_xX$, $x\in X$.

Theorem: If the open neighbourhood $U\subseteq TX$ of the zero section is small enough, then the exponential mapping exp: $U\to X$ has the following properties:

1. $\exp 0_x=x$
2. $\exp_x$: $U\cap T_xX\to X$ is a diffeomorphism onto an open neighbourhood of x in X.
3. $\mathrm{Exp}=(\pi_X,\exp)$: $U\to X\times X$ is a diffeomorphism onto an open neighbourhood of the diagonal in $X\times X$.

Proof: 1. $\exp(0_x)=\chi(1,0_x)=x$ (1.22.2.1).

2. For $v\in U\cap T_xX$ let $\alpha(t)=\exp_x(tv)=\chi(1,tv)=$
$=\chi(t,v)$ (by 1.22.2.4). Then $\dot{\alpha}(0)=\dot{\chi}_v(0)=v$ (by 1.22.2.2).
So $T_{0_x}(\exp_x).v=\frac{d}{dt}\exp_x(tv)|_{t=0}=v$, so $T_{0_x}(\exp_x)=\mathrm{Id}_{T_xX}$;
so by the inverse function theorem $\exp_x$ is a local diffeomorphism. Choose U small enough.

3. By 2. $\mathrm{Exp}=(\pi_X,\exp)$: $U\to X\times X$ is injective, $\mathrm{Exp}(0_x)=(x,x)$. We claim that $T_v\mathrm{Exp}$ is injective for all $v\in U$. Let $x=\pi_X(v)$, choose $w\in T_v(U)=T_v(TX)$. If $0=T_v\mathrm{Exp}.w=(T_v(\pi_X).w,\ T_v(\exp).w)$, then $T_v(\pi_X).w=0$, so w is vertical, $w\in T_v(T_xX)$, so $0=T_v(\exp).w=T_v(\exp_x).w$.

But then $w = 0$ since $\exp_x$ is a diffeomorphism by 2. So (by dimension) $T_v\mathrm{Exp}$ is invertible for all $v \in U$, so Exp is a local diffeomorphism and injective, so 3. follows.

q.e.d.

2 Manifolds with corners

2.1 Definition: A quadrant $Q \subseteq \mathbb{R}^n$ is a subset of the form $Q = \{x \in \mathbb{R}^n : l_1(x) \geq 0, \ldots, l_k(x) \geq 0\}$ where $\{l_1, \ldots, l_k\}$ is a linearly independent subset of $(\mathbb{R}^n)^*$. Here $0 \leq k \leq n$ and k is called the index of Q.

If $x \in Q$ and exactly j of the l_i's satisfy $l_i(x) = 0$, then x is called a corner of index j. The index of a corner depends only on x and Q and not on the special system $\{l_1, \ldots, l_k\}$ describing Q.

2.2 Let $U \subseteq Q$ be an open subset of a quadrant Q. A function $f: U \to \mathbb{R}^p$ is called C^r $(0 \leq r \leq \infty)$ if all partial derivatives of f of order $\leq r$ exist and are continuous on U. By the Whitney extension theorem (cf. H. WHITNEY (1936), J.C. TOUGERON (1972)) this is the case iff f can be extended to a C^r function $\tilde{f}: \tilde{U} \to \mathbb{R}^p$, where $\tilde{U} \subseteq \mathbb{R}^n$ is open and $U = Q \cap \tilde{U}$.

2.3 The border ∂Q of a quadrant Q is $\{x \in \mathbb{R}^n : l_1(x) = 0$ or $l_2(x) = 0$ or ... or $l_k(x) = 0\}$; it is the disjoint union of finitely many (plane) submanifolds of $\mathbb{R}^n$, the faces, edges, corners etc. of Q. ∂Q is "stratified" by this set of submanifolds.

Let $U \subseteq Q$, $U' \subseteq Q'$ be open subsets of quadrants in $\mathbb{R}^n$. A mapping $f: U \to U'$ is a diffeomorphism iff f is bijective and locally of maximal rank. It follows that f maps corners of index j in U to corners of index j in U'. So: $x \in U \subseteq Q$ is of index j iff $f(x) \in U' \subseteq Q'$ is of index j.

2.4 Manifolds with corners are defined in the usual way: they are modelled on open subsets of quadrants in $\mathbb{R}^n$, and are supposed to be of class C^r, $r \geq 1$ (otherwise we just get back topological manifolds with boundary):

A chart (U,u,Q) on a manifold X with corners is a diffeomorphism $u: U \to u(U) \subseteq Q$ of an open set $U \subseteq X$ onto an open subset $u(U)$ of a quadrant in $\mathbb{R}^n$ ($u = \dim X$). (U,u) is called centered at $x \in X$, if $u(x) = 0$.

$x \in X$ is called a corner of index j if there is a chart (U,u,Q) of X with $x \in U$ and $u(x)$ is of index j in $u(U) \subseteq Q$. The index of x is independent of the chosen chart, by the invariance of the index under diffeomorphisms.

The set of all corners of index $j \geq 1$ is called the border ∂X of X; x is called an inner point of X if $\mathrm{ind}_X(x) = 0$.

2.5 A subset $Y \subseteq X$ is called a submanifold with corners of the manifold with corners X, if for any $y \in Y$ there is a chart (U,u,Q) of X centered at y and there is a quadrant $Q' \subseteq \mathbb{R}^k \subseteq \mathbb{R}^n$ such that $Q' \subseteq Q$ and $u(Y \cap U) = u(U) \cap Q'$. A submanifold with corners Y of X is called neat, if the index in Y of each $y \in Y$ coincides with its index in X: any corner of Y is a corner of the same index of X. Only neat submanifolds will be seen to have tubular neighbourhoods. Let us denote $\partial^j X = \{$corners of index j of $X\}$. Then each $\partial^j X$ is a submanifold without boundary of X, and each closure (in X) of a connected component of $\partial^j X$ is a submanifold with corners of X. Neither of them is neat. Let $\partial X = \bigcup_j \partial^j X$.

2.6 The tangent bundle. Let X be a manifold with corners, let $(U_i, u_i, Q_i)_{i \in I}$ be an atlas of X. The tangent bundle TX of X is the quotient space of the set $\{(i,v,x): x \in U_i,\ i \in I,\ v \in \mathbb{R}^n\}$ for the following equivalence relation: $(i,v,x) \sim (j,w,y)$ iff $x = y$ and $d(u_j \circ u_i^{-1})(u_i(x)).v = w$. TX is again a manifold with corners: charts are $(\{[(i,v,x)],\ x \in U_i\},\ \bar{u}_i[(i,v,x)] =$

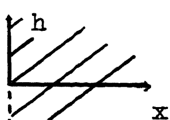
$= (v,u_i(x)))$; so $\bar{u}_i\colon \pi_X^{-1}(U_i) \to \mathbb{R}^n \times u_i(U_i) \subseteq \mathbb{R}^n \times Q_i \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is given by choosing the index i in the equivalence class. $(TX, \pi_X, X, \mathbb{R}^n)$ is a vector bundle.

A tangent vector $\xi \in TX$ is called <u>inner</u> (short for: not outer) if there is a smooth curve $c\colon [0,1) \to X$ with $\dot{c}(0) = \xi$. Let $\xi \in T_xX$, let (U,u,Q) be a chart with $x \in U$, let $Q = \{y \in \mathbb{R}^n\colon l_1(y) \geq 0, \dots, l_k(y) \geq 0\}$. Let $\bar{u}(\xi) = (v,u(x))$ be the coordinate representation of ξ. ξ is inner iff the following holds: if $l_i(u(x)) = 0$, then $l_i(v) \geq 0$. Let us call ξ <u>strictly inner</u> if $l_i(u(x)) = 0$ implies $l_i(v) > 0$.

Let us denote the set of all inner tangent vectors by ${}^iTX$. Then ${}^iTX$ is a subset of TX.

<u>Example:</u> Let $Q = \{x \in \mathbb{R}^n;\ l_1(x) \geq 0, \dots, l_k(x) \geq 0\}$. If x_0 is a corner of index j of Q, i.e. exactly $l_{i_1}(x_0) = 0, \dots, l_{i_j}(x) = 0$, then ${}^iT_{x_0}Q$ is the quadrant $\{v \in \mathbb{R}^n\colon l_{i_1}(v) \geq 0, \dots, l_{i_j}(v) \geq 0\}$ of index j.

<u>Example:</u> ${}^iT\mathbb{R}_+ = {}^iT\ \{x \geq 0\} = \{(x,h) \in \mathbb{R}^2\colon x \geq 0,\ x = 0 \Rightarrow h \geq 0\}$.



This set is a convex cone in $\mathbb{R}^2$, but it is not an open subset of a quadrant, neither is it diffeomorphic to some.

So in general ${}^iTX$ ceases to be a manifold with corners.

<u>Remark:</u> It is possible to enlarge the category of manifolds with corners in such a way, that it contains finite products and inner tangent spaces and there are still inner sprays on all these manifolds (just take the hull of the set of all quadrants under the operation $Q \overset{i}{\to} TQ$ and describe it nicely; then model manifolds on open subsets of these sets). But Whitney's extension theorem is no longer applicable in the form of 2.2. So we prefer to stick to the notion of manifolds with corners.

2.7 <u>Integration of inner vector fields</u>

Let X be a manifold with corners, let ξ be a vector field on X. Let $x \in X$ be an inner point, then there is a unique integral curve of ξ through x. If x is a corner of X, then there is an integral curve $c: [0,\varepsilon)\to X$ with $c(0)=x$ and $c'(t)=\xi(c(t))$ iff all $\xi(y)$ are inner for y near x.

So if ξ is an inner vector field (i.e. ξ has values in ${}^{i}TX$) then there exists a local flow for ξ in the following sense: There is a set $W\subseteq \mathbb{R}\times X$ containing $\{0\}\times X$ and $[0,\varepsilon_x)\times\{x\}$ for some $\varepsilon_x>0$ for each $x\in X$, and a mapping $\alpha: W\to X$ with $\alpha(0,x)=x$ and $\frac{d}{dt}\alpha(t,x)=\xi(\alpha(t,x))$. But α_t is not even a local diffeomorphism (it may map a corner to an inner point).

By a partition of unity argument one may prove that on each manifold with corners X there is a "strictly inner" vector field ξ, i.e. $\xi(x)$ is strictly inner if $x\in\partial X$. Then the local flow α of ξ flows ∂X into the interior of X. By multiplication with a (small) function one may adapt ξ in such a way that $\alpha(t,x)$ is defined for all $0\le t\le\varepsilon$ and all x. Then $\alpha_\varepsilon: X\to X$ maps X diffeomorphically onto its image which is contained in $X\backslash\partial(X)$. So we have proved: (cf. A. DOUADY, L. HERAULT (1973)).

Lemma: Each manifold with corners is a submanifold with corners of a manifold without boundary of the same dimension.

Now let ξ be a vector field which is tangential to the border, i.e. if $x\in\partial^j X$ then $\xi(x)\in T\partial^j X$ $\forall j$. Then there exists on local flow α for ξ for positive and negative time:

There is an open set $W\subseteq\mathbb{R}\times X$ containing $\{0\}\times X$, $\alpha: W\to X$ such that $\alpha(0,x)=x$, $\frac{d}{dt}\alpha(x,t)=\xi(\alpha(x,t))$. α_t is a local diffeomorphism. Corners of index j may flow only along $\partial^j X$.

2.8 The second tangent bundle

Definition: A vector $\Lambda\in T(TX)$ with $\pi_{TX}\Lambda\in{}^{i}TX$ is said to be an inner tangent vector to ${}^{i}TX$ if there is a curve $c: [0,\varepsilon)\to TX$ with $c(0)=\pi_{TX}\cdot\Lambda$, $c([0,\varepsilon))\subseteq{}^{i}TX$ and $c'(0)=\Lambda$.

Example: Let $Q=\{x\in\mathbb{R}^n: l_1(x)\ge 0,\dots,l_k(x)\ge 0\}$ be a quadrant and $(x_o, v) \in {}^iTQ$. Let $(x_o, v\ h, k) \in T^2Q$. This vector is an inner vector to ${}^iTQ$ if the following holds:

1. If x_o is inner (so v arbitrary), then (h,k) is arbitrary.
2. If $l_i(x_o) = 0$, $l_i(v) > 0$, then $l_i(h) \geq 0$. k is arbitrary.
3. If $l_i(x_o) = 0$, $l_i(v) = 0$, then $l_i(h) \geq 0$ and $l_i(k) \geq 0$.

Let us denote by ${}^iT^2X$ the set of all inner vectors to ${}^iTX$.

2.8 Sprays: A spray ξ on a manifold with corners X is a mapping ξ: $TX \to T^2X$ such that

1. $T(\pi_X) \circ \xi = Id_{TX}$
2. $\pi_{TX} \circ \xi = Id_{TX}$ (i.e. ξ is a vector field)
3. $T(\mu_t)\xi(v) = \frac{1}{t}\ \xi(tv)$, $t \in \mathbf{R}$, $t \neq 0$, $v \in TX$, where μ_t: $TX \to TX$ is given by $v \to t.v$.

A spray ξ is called an inner spray, if $\xi({}^iTX) \subseteq {}^iT^2X$.

Example: Let $Q = \{x \in \mathbf{R}^n\colon\ l_1(x) \geq 0, \ldots, l_k(x) \geq 0\}$ be a quadrant of index k. Let $U \subseteq Q$ be open. Then $T^2U = (U \times \mathbf{R}^n) \times (\mathbf{R}^n \times \mathbf{R}^n)$; define ξ: $TU \to T^2U$ by $\xi(x,v) = (x,v;v,0)$. From 1.21 it is seen that ξ is a spray; it is easily checked that ξ is an inner spray.

Lemma: Each manifold with corners X admits an inner spray. Use a partition of unity on X to paste together locally given inner sprays (by the example).

We will need another sort of sprays:

Tangential sprays: A spray ξ on TX is called tangential, if for any submanifold ∂^jX ($= \{x \in X\colon \mathrm{ind}_X(x) = j\}$) we have: $\xi(T\partial^jX) \subseteq T^2(\partial^jX)$. It is easily checked that the spray in the example above is even tangential. By a partition of unity on X we can paste together locally given tangential sprays and get:

Lemma: Any manifold with corners admits tangential sprays.

A word of warning: If ξ is an inner spray on a quadrant Q and if $(x,v) \in TQ$ is not inner, then $(x,v;v,\ldots)$ is no inner vector to the manifold TX either. More so: There does not exist a spray on a manifold X with non trivial corners that is an inner vectorfield on TX.

2.9 The flow of inner and tangential sprays.

Let ξ be an inner spray on a manifold with corners X. Then ξ is a vectorfield on TX and $\xi({}^{i}TX) \subseteq {}^{i}T^{2}X$, but ξ is not an inner vectorfield in general.

For any $v \in {}^{i}TX$ there is an integral curve $c: [0,\epsilon) \to {}^{i}TX$ with $c(0) = v$ and $\dot{c} = \xi \circ c$ since $\xi({}^{i}TX) \subseteq {}^{i}T^{2}X$. So there is a maximal set $W \subseteq \mathbb{R} \times {}^{i}TX$ containing $\{0\} \times {}^{i}TX$ and $[0,\epsilon_x) \times \{x\}$ for any $x \in {}^{i}TX$ and some $\epsilon_x > 0$, and there is a smooth mapping $\alpha: W \longrightarrow {}^{i}TX$ with $\alpha(0,x) = X$ and $\frac{d}{dt}\alpha(t,x) = \xi(\alpha(t,x))$. This mapping α is called the local flow of the spray ξ in ${}^{i}TX$. All the properties 1.22.1 through 1.22.3 hold for this local flow α too, with some more restrictions on the parameter (for not leaving ${}^{i}TX$), by the same proofs.

If the spray ξ is tangential (so $\xi | T\partial^{j}X$ is again a spray on $\partial^{j}X$ for all j) then of course its local flow α leaves invariant all submanifolds ${}^{i}T\partial_{k}^{j}X$, induces a local flow on each ${}^{i}T\bar{\partial}_{k}^{j}X$, where $\bar{\partial}_{k}^{j}X$ is the closure in X of a connected component $\partial_{k}^{j}X$ of $\partial^{j}X$. It is a manifold with corners.

2.10 The exponential mapping.

Let ξ be an inner spray on X. Then there is an open neighbourhood U of the zero section in ${}^{i}TX$ such that $[0,1] \times {}^{i}TX \subseteq W$, by the analogon of 1.22.3 for manifolds with corners. Let $\chi = \pi_X \circ \alpha: W \to X$, where α is the local flow of ξ. The mapping $\exp = \exp^{\xi}: U \to X$, defined by $\exp(v) = \chi(1,v) = \pi_X\alpha(1,v)$, $v \in U$, is called the exponential mapping of the spray ξ.

<u>Theorem: Let</u> ξ <u>be an inner spray on a manifold with corners</u> X <u>and let</u> $\exp: U \to X$, U <u>a suitable open neighbourhood of the zero section in</u> ${}^{i}TX$, <u>be the exponential map of</u> ξ. <u>If</u> U <u>is chosen small enough, the</u> exp <u>has the following properties</u>:

1. $\exp 0_x = x$, $x \in X$

2. $\exp_x \colon U \cap {}^iT_xX \to X$ *is a diffeomorphism of* U_x *onto an open neighbourhood of* x *in* X.

3. *The mapping* $\mathrm{Exp} = (\pi_X, \exp)\colon U \to X \times X$ *is a diffeomorphism of* U *onto an open neighbourhood of the diagonal in* $X \times X$.

If ξ *is moreover tangential* (2.8) *then* $\exp | {}^iT\overline{\partial_k^j X} \cap U\colon {}^iT\overline{\partial_k^j X} \cap U \to \overline{\partial_k^j X}$ *is again the exponential mapping for the spray* $\xi | {}^iT\partial_k^j X$, *where* $\overline{\partial_k^j X}$ *is the closure of a connected component* $\partial_k^j X$ *of* $\partial^j X$.

Proof: As in 1.23. The last part is obvious.

2.11 *Jet bundles:*

Let U be open in a quadrant Q in $\mathbb{R}^n$, let V be open in a quadrant $Q' \subseteq \mathbb{R}^m$. Let $\tilde{U} \subseteq \mathbb{R}^n$, $\tilde{V} \subseteq \mathbb{R}^m$ be open sets with $U = Q \cap \tilde{U}$, $V = Q' \cap \tilde{V}$. We define: $J^k(U,V) := J^k_{U\times V}(\tilde{U},\tilde{V})$

$$= \bigcup_{(x,y)\in U\times V} J^k_{x,y}(\tilde{U},\tilde{V}).$$

(compare 1.3). Then we have again $J^k(U,V) = U \times V \times P^k(n,m)$. If $\varphi\colon U' \to U$, $\psi\colon V \to V'$ are a diffeomorphism resp. smooth mapping between open subsets of quadrants, then we have again a diffeomorphism $J^k(\varphi,V)\colon J^k(U,V) \to J^k(U',V)$ resp. smooth mappings $J^k(U,\psi)\colon J^k(U,V) \to J^k(U,V')$

$$J^k(\varphi,\psi) = J^k(\varphi,V') \circ J^k(U,\psi) = J^k(U',\psi) \circ$$
$$\circ\, J^k(\varphi,V)\colon J^k(U,V) \to J^k(U',V'),$$ defined exactly as in 1.5 and 1.6 by truncated composition.

Let now X, Y be manifolds with corners, let (U_i, u_i, Q_i) be an atlas for X, $(V_\alpha, v_\alpha, Q_\alpha')$ be an atlas for Y, then we have the system of chart change mappings:
$J^k(u_i \circ u_j^{-1}, v_\beta \circ v_\alpha^{-1})\colon J^k(u_i(U_{ij}), v_\alpha(V_{\alpha\beta})) \to J^k(u_j(U_{ij}), v_\beta(V_{\alpha\beta}))$ which can be used to define the fibre bundle: $(J^k(X,Y), (\alpha,\beta), X \times Y, P^k(n,m))$.

2.12 A coordinate free definition of $J^k(X,Y)$ can be given as follows (MATHER (1969)): A k-jet in $J^k(X,Y)$ is an equivalence class $[U,V,f,x]_k$ of quadruples (U,V,f,x),

where U is an open set in a manifold <u>without</u> boundary containing a neighbourhood of x in X, V is an open subset in another manifold without boundary containing a neighbourhood of f(x) in Y, and f: $U \to V$ is a C^k-mapping. The equivalence relation is the following:
$[U,V,f,x]_k = [U',V',f',x']_k$ iff $x = x'$ and in a (hence any) pair of charts around x,f(x) of U,V have f and f' the same Taylor expression at x up to order k.

Another way to define $J^k(X,Y)$ coordinate free is the following: Let $\tilde{X},\tilde{Y}$ be manifolds without boundary containing X,Y as submanifolds with corners respectively and $\dim \tilde{X} = \dim X$, $\dim \tilde{Y} = \dim Y$ (cf. the lemma in 2.7). The $J^k(X,Y) = J^k_{X\times Y}(\tilde{X},\tilde{Y}) = (\alpha,\omega)^{-1}(X \times Y) \subseteq J^k(\tilde{X},\tilde{Y})$.

Then complicated ways of defining J^k for manifolds with corners is chosen in order to get a fibre bundle structure on $J^k(X,Y)$; in other words, to include "outward pointing jets".

2.13 If f: $X \to Y$ is a smooth mapping and $x \in X$, then the <u>k-jet of</u> f <u>at</u> x, in symbols $j^k f(x)$, is defined as follows: Let U,V be (open subsets of) manifolds without boundary containing neighbourhoods of x, f(x) in X,Y resp. with a prolongation $\tilde{f}$: $U \to V$ of f. Then put $j^k f(x) = [U,V,\tilde{f},x]_k$. Another way to define $j^k f(x)$ is via local representatives of f in charts of X,Y, using 2.11.

All k-jets in $J^k(X,Y)$ of the form $j^k f(x)$ for $f \in C^\infty(X,Y)$ and $x \in X$ are called <u>inner</u> k-<u>jets</u>. The set of all these is denoted by ${}^iJ^k(X,Y)$. It is not a manifold with corners any more: We have the same difficulty as with ${}^iTX$. We have in fact ${}^iTX = {}^iJ^k_0([0,\varepsilon),X)$ for $\varepsilon > 0$.

2.14 Theorem 1.10 holds for manifolds with corners in its full content the proof is the same with the obvious changes.

3 Topologies on spaces of continuous mappings

3.1 Let X,Y be Hausdorff topological spaces, let $C(X,Y)$ denote the space of all continuous mappings $X \to Y$. The best known topology on $C(X,Y)$ is the <u>compact-open topology</u> (or CO-<u>topology</u>). A subbasis for this topology consists of sets of the following form: $\{f \in C(X,Y): f(K) \subseteq U\}$ where K is compact in X and U is open in Y. This is a Hausdorff-topology.

<u>Lemma: If</u> X <u>is locally compact with countable basis of open sets and if</u> Y <u>is a complete metric space, then there is a complete metric on</u> $(C(X,Y),CO)$. <u>So</u> $(C(X,Y),CO)$ <u>is a Baire space</u>.

3.2 If $f \in C(X,Y)$ let $\Gamma_f: X \to X \times Y$ be given by $\Gamma_f(x) = (x,f(x))$. We use Γ_f too to denote the image of this mapping $\{(x,f(x)): x \in X\}$, the <u>graph</u> of f.

<u>Definition:</u> The <u>wholly open topology</u> (or WO-<u>topology</u>) on $C(X,Y)$ is given by the basis $\{W(U) = \{f \in C(X,Y): f(X) \subseteq U\}: U \text{ open in } Y\}$. This is not a Hausdorff topology (surjective mappings cannot be separated).

<u>Definition:</u> The <u>graph-topology</u> or WO^o-<u>topology</u> on $C(X,Y)$ is the topology induced by the embedding $\Gamma: C(X,Y) \to C(X,X \times Y)$, $f \to \Gamma_f$, where $C(X,X \times Y)$ bears the WO-topology. The WO^o-topology has the basis of open sets: $\{M(U) = \{f \in C(X,Y): \Gamma_f \subseteq U\}, U \text{ open in } X \times Y\}$. The WO^o-topology is Hausdorff since it can easily be seen to be finer than the compact open topology.

3.3 Lemma: Let X be paracompact and let (Y,d) be a metric space, then for $f \in C(X,Y)$ the following family is a neighbourhood basis for the graph topology:
$N(f,\varepsilon) := \{g \in (X,Y) : d(g(x),f(x)) < \varepsilon(x)$ for all $x \in X\}$,
where $\varepsilon \in C(X,]0,\infty[)$.

Proof: Put $W = \{(x,y) \in X \times Y : d(y,f(x)) < \varepsilon(x)\}$ then W is an open neighbourhood of Γ_f in $X \times Y$ and $N(f,\varepsilon) = M(W)$.

Conversely, let W be any open neighbourhood of Γ_f in $X \times Y$. For each $x \in X$ there is an open neighbourhood U_x of x in X and some $0 < \varepsilon_x \leq 1$ such that $U_x \times B_{\varepsilon_x}(f(x)) = U_x \times \{y \in Y : d(y,f(x)) < \varepsilon_x\} \subseteq W$, by definition of the product topology. Now let $(\varphi_x)_{x \in X}$ be a continuous partition of unity subordinated to the open cover $(U_x)_{x \in X}$ of X, and put $\varepsilon = \sum_{x \in X} \varepsilon_x \cdot \varphi_x$. Then ε is continuous on X, $\varepsilon > 0$, and $\{(x,y) : d(y,f(x)) < \varepsilon(x)\} \subseteq W$, so $N(f,\varepsilon) \subseteq M(U)$.

q.e.d.

3.4 Lemma: Let X be paracompact and let Y be metrizable. Then for any sequence (f_n) in $C(X,Y)$ the following holds: (f_n) converges to $f \in C(X,Y)$ in the graph topology iff there exists a compact set $K \subseteq X$ such that f_n equals f off K for all but finitely many n's and $f_n|K$ converges to $f|K$ uniformly.

Proof: It is clear that the condition above implies convergence. Conversely, let (f_n) and f in $C(X,Y)$ be such that the condition does not hold. Then either f_n does not converge to f in CO or there exists a sequence (x_n) in X without a cluster point and a sequence $\varepsilon_n > 0$ in $\mathbb{R}$ such that $d(f_n(x_n),f(x_n)) \geq \varepsilon_n$ for all n, where d is a metric on Y. Now there is a continuous function ε on X with $0 < \varepsilon(x)$, $\varepsilon(x_n) < \varepsilon_n$ for all n. But then $d(f_n(x_n),f(x_n)) > \varepsilon(x_n)$, i.e. $f_n \notin N(f,\varepsilon)$ for all n. So f_n cannot converge to f in the graph topology. q.e.d.

3.5 Corollary: Let $f: T \to (C(X,Y),WO^o)$ be a continuous mapping, where T is compact connected metrizable, X is paracompact and Y is metrizable. Then there exists a compact set $K \subseteq X$ such that $t \to f(t)(x)$ is constant for any $x \in X \setminus K$.

Proof: For any $t \in T$ there exists $\varepsilon_t > 0$ and a compact set $K_t \subseteq X$ such that $t \to f(t)(x)$ is constant on $B_{\varepsilon_t}(t) = \{t' \in T: d(t,t') < \varepsilon_t\}$ for any $x \in X \setminus K_t$. If not then one may find a sequence $t_n \to t$ in T and a sequence x_n in X without a cluster point such that $f(t_n)(x_n) \neq f(t)(x_n)$ for all n. But then $f(t_n) \not\to f(t)$ by 3.4, so f is not continuous.

Now cover T by finitely many balls $B_{\varepsilon_1}(t_1), \ldots, B_{\varepsilon_k}(t_k)$ and choose $K = K_{t_1} \cup \ldots \cup K_{t_n}$. q.e.d.

3.6 Let (Y,d) be a metric space. A subset $Q \subseteq C(X,Y)$ is called uniformly closed with respect to d, if Q contains all limits of sequences in Q which are uniformly convergent on X with respect to d.

Any set which is closed in the topology of pointwise convergence is uniformly closed, as is a set which is CO-closed.

Proposition: Let X be paracompact, let (Y,d) be a complete metric space. Then any set $Q \subseteq C(X,Y)$ which is uniformly closed with respect to d is a Baire space in the graph topology.

Proof: Let (A_n) be a sequence of open dense subsets of Q (for the trace of the graph topology). Let $U \subseteq Q$ be any non empty open subset.

We have to show that $U \cap \bigcap A_n \neq \emptyset$. $A_o \cap U \neq \emptyset$, open, so there is $f_o \in A_o \cap U$ and $\varepsilon_o \in C(X,]0,1[)$ such that $Q \cap \overline{N}(f,\varepsilon_o) \subseteq A_o \cap U$ (where $\overline{N}(f,\varepsilon_o) = \{g \in C(X,Y): d(g(x), f(x)) \leq \varepsilon(x)$ for all $x\}$). By recursion one gets sequences (f_n) in Q, ε_n in $C(X,]0,1[)$, with $\varepsilon_{n+1} \leq \frac{\varepsilon_n}{2}$ for all n and $Q \cap \overline{N}(f_{n+1}, \varepsilon_{n+1}) \subseteq A_{n+1} \cap N(f_n, \varepsilon_n)$. Then $d(f_{n+1}(x), f_n(x)) \leq \leq 2^{-n}$, so (f_n) is uniformly convergent on X, so its limit f is in Q since Q is uniformly closed. Moreover $f \in \overline{N}(f_n, \varepsilon_n) \cap A_n$ for all n, so $f \in U$ and $f \in \bigcap_n A_n$. q.e.d.

3.7 Definition: Let X be paracompact and let Y be Hausdorff. We define the locally finite open topology or LO-topology on $C(X,Y)$ by the following basis:

$M(L,U) = \{f \in C(X,Y): f(L_\alpha) \subseteq U_\alpha\}$ where $L = (L_\alpha)$ is a locally finite family of closed subsets $L_\alpha \subseteq X$ and $U = (U_\alpha)$ is a family of open subsets of Y with the same index set.

Definition: The LO^o-topology on $C(X,Y)$ is the topology induced by the embedding $\Gamma: C(X,Y) \to C(X, X \times Y)$ where $C(X, X \times Y)$ bears the LO-topology.

Lemma: The LO^o-topology is finer than the graph-topology; so it is Hausdorff. (Since the LO-topology is finer than the WO-topology.)

$(C(X,Y),\ LO^o)$ has the following basis of open sets.

$M(L,U) = \{f \in C(X,Y): \Gamma_f(L_\alpha) \subseteq U_\alpha$ for all $\alpha\}$, where $L = (L_\alpha)$ is a locally finite family of closed sets in X and $U = (U_\alpha)$ is a family of open subsets in $X \times Y$ with the same index set.

3.8 Lemma: If X is paracompact and (Y,d) is a metric space, then the following families are neighbourhood bases of $f \in C(X,Y)$ in the LO^o-topology.

1. $N(f,L,\varepsilon) = \{g \in C(X,Y): d(g(x),f(x)) < \varepsilon_\alpha$ for all $x \in L_\alpha$, for all $\alpha\}$, where $L = (L_\alpha)$ is a locally finite family of closed sets in X and $\varepsilon = (\varepsilon_\alpha)$ is a family of positive real numbers.

2. $N(f,\varphi) = \{g \in C(X,Y): \varphi_\alpha(x) d(g(x),f(x)) < 1$ for all $x \in X$, for all $\alpha\}$, where $\varphi = (\varphi_\alpha)$ is a family of continuous non-negative functions on X such that $(\operatorname{supp} \varphi_\alpha)$ is a locally finite family in X.

Proof: a) Let $N(f,L,\varepsilon)$ be as above. Let $M = (M_\alpha)$ be an locally finite family of open subsets in X with $M_\alpha \supset L_\alpha$. Let $U_\alpha = \{(x,y) \in X \times Y: x \in M_\alpha,\ d(y,f(x)) < \varepsilon_\alpha\}$, then U_α is open in $X \times Y$, $f(L_\alpha) \subset U_\alpha$. So $f \in M(L,U) \subseteq N(f,L,\varepsilon)$, so $N(f,L,\varepsilon)$ is a LO^o-neighbourhood of f.

b) Let $N(f,\varphi)$ be as in 2. Let $L_{\alpha,n} = \{x \in X: \frac{1}{n} \leq \varphi_\alpha(x) \leq \frac{1}{n-1}\}$, then $L = (L_{\alpha,n})$ is locally finite and $f \in N(f, L = (L_{\alpha,n}),\ \varepsilon = (\varepsilon_{\alpha,n} := \frac{1}{n})) \subseteq N(f,\varphi)$, so $N(f,\varphi)$ is a LO^o-neighbourhood of f by a).

c) Let $f \in M(L,U)$ be given, $M(L,U)$ as in 2.7. Then $L = (L_\alpha)$ is locally finite closed, $U = (U_\alpha)$ is open and

$f(L_\alpha) \subseteq U_\alpha$. As in the proof of lemma 3.3 we can find a continuous function $\delta_\alpha\colon L_\alpha \to \mathbf{R}$, $1 > \delta_\alpha > 0$, such that $\{(x,y)\colon d(y,f(x)) < \delta_\alpha(x)$ for all $x \in L_\alpha\} \subseteq U_\alpha$ for any α. Then $\frac{1}{\delta_\alpha}$ is defined on L_α and is >1, so we may find a family (φ_α) of continuous non-negative functions on X such that $(\operatorname{supp} \varphi_\alpha)$ is a locally finite family and $\varphi_\alpha(x) < \frac{1}{\delta_\alpha(x)}$ for all $x \in L_\alpha$. But then we have $f \in N(f,\varphi = (\varphi_\alpha)) \subseteq M(L,U)$.

d) $\{N(f,\varphi)\colon \varphi$ as in the lemma $\}$ is a neighbourhood basis of f in the LO^o-topology by b), c). Then by a), b) $\{N(f,L,\varepsilon)\colon L,\varepsilon\}$ is neighbourhood basis also. q.e.d.

3.9 <u>Lemma:</u> <u>If</u> X <u>is paracompact</u>, (Y,d) <u>is metric</u>, (f_n) <u>is a sequence in</u> $C(X,Y)$ <u>and</u> $f \in C(X,Y)$, <u>then the following holds</u>: $f_n \to f$ <u>in the</u> LO^o-<u>topology iff there exists a compact set</u> $K \subseteq X$ <u>such that</u> $f_n|X\setminus K = f|X\setminus K$ <u>for all but finitely many</u> n's <u>and</u> $f_n|K \to f|K$ <u>uniformly</u>.

<u>Proof:</u> If $f_n \to f$ in LO^o then $f_n \to f$ in WO^o by lemma 3.7, but then (f_n) and f satisfy the condition by lemma 3.4. If $(f_n), f$ satisfy the condition, the clearly $f_n \to f$ in LO^o. q.e.d.

3.10 <u>Corollary:</u> <u>Let</u> $f\colon T \to (C(X,Y), LO^o)$ <u>be a continuous mapping</u>, <u>where</u> T <u>is compact metric connected</u>, X <u>is paracompact and</u> Y <u>is metric</u>. <u>Then there exists a compact set</u> $K \subseteq X$ <u>such that</u> $t \to f(t)(x)$ <u>is constant on</u> T <u>for any</u> $x \in X\setminus K$.

<u>Proof:</u> This follows from 3.8 as 3.5 follows from 3.4.

q.e.d.

3.11 <u>Proposition:</u> <u>Let</u> X <u>be paracompact</u>, <u>let</u> (Y,d) <u>be a complete metric space</u>. <u>Then any subset</u> $Q \subseteq C(X,Y)$ <u>is a Baire space in the</u> LO^o-<u>topology</u>, <u>if it is uniformly closed in</u> $C(X,Y)$ <u>with respect to</u> d (<u>cf</u>. 3.6).

<u>Proof:</u> Let A_n be a sequence of open dense subsets of Q (in the trace of the LO^o-topology). Let $U \subseteq Q$ be any non empty open subset. We have to show that $U \cap \bigcap_n A_n$ is not

empty. Now $A_o \cap U$ is non empty and open, so there are $f_o \in A_o \cap U$ and a locally finite cover $L^{(o)} = (L_\alpha^{(o)})$ of X consisting of closed sets $L_\alpha^{(o)}$, and a family $\epsilon^{(o)} = \epsilon_\alpha^{(o)}$ of real numbers $\epsilon_\alpha^{(o)}$, $0 < \epsilon_\alpha^{(o)} \leq 1$, such that $f_o \in Q \cap \overline{N}(f_o, L^{(o)}, \epsilon^{(o)}) \subseteq A_o \cap U$ (where $\overline{N}(f_o, L^{(o)}, \epsilon^{(o)}) = \{g \in C(X,Y): d(g(x), f(x)) \leq \epsilon_\alpha$ for all $x \in L_\alpha$, for all $\alpha\}$).

By recursion one finds sequences (f_n) in Q, $L^{(n)} = (L_\beta^{(n)})_{\beta \in B_n}$, $\epsilon^{(n)} = (\epsilon_\beta^{(n)})_{\beta \in B_n}$ such that $L^{(n)}$ is a closed locally finite cover of X, $\epsilon^{(n)}$ is a family of real numbers with $0 < \epsilon_\beta^{(n)} \leq \frac{1}{2^n}$, and such that
$f_{n+1} \in Q \cap \overline{N}(f_{n+1}, L^{(n+1)}, \epsilon^{(n+1)}) \subseteq A_{n+1} \cap N(f_n, L^{(n)}, \epsilon^{(n)})$.
Then we have $d(f_{n+1}(x), f_n(x)) \leq \frac{1}{2^{n-1}}$ for all $x \in X$ (since any $L^{(n)}$ is a cover of X and all $\epsilon_\beta^{(n)} \leq \frac{1}{2^n}$), so (f_n) is uniformly convergent on X and its limit f is in Q (since Q is uniformly closed), $f \in \overline{N}(f_n, L^{(n)}, \epsilon^{(n)}) \cap A_n$ for all n, so $f \in U \cap \bigcap_n A_n$. q.e.d.

4 Topologies on spaces of differentiable mappings

4.1 Let X,Y be manifolds with corners. For any $k \in \mathbb{N}$ we have the fibre bundle $J^k(X,Y)$ of k-jets over $X \times Y$. We consider the following:

$$X \times Y = J^0(X,Y) \xleftarrow{\pi_0^1} J^1(X,Y) \xleftarrow{\pi_1^2} J^2(X,Y) \xleftarrow{\pi_2^3} \dots$$

Let us denote by $J^\infty(X,Y)$ the projective limit (in the category of Hausdorff topological spaces) of this system. We have again mappings: $\alpha: J^\infty(X,Y) \to X$, the source map, $\omega: J^\infty(X,Y) \to Y$, the target map, π_k^∞ : $J^\infty(X,Y) \to J^k(X,Y)$ which define the projective limit. For any $f \in C^\infty(X,Y)$ all the k-jets $j^k f(x)$ constitute the element $j^\infty f(x)$ (given by $j^k f(x) = \pi_k^\infty j^\infty f(x)$), and this gives (by the limit property) a continuous mapping $j^\infty f: X \to J^\infty(X,Y)$. $(J^\infty(X,Y), (\alpha,\omega) = \pi_0^\infty, X \times Y, P^\infty(n,m))$ is a topological fibre bundle with typical fibre $P^\infty(n,m)$, the space of all formal power series without constant terms in n variables with values in $\mathbb{R}^m$ ($n = \dim X$, $m = {}$ $= \dim Y$). This fibre bundle is the projective limit of the above system of fibre bundles in the category of topological fibre bundles. We will see that $J^\infty(X,Y)$ is a C_c^∞-fibre bundle later on (in § 9).

$J^\infty(X,Y)$ is a complete metric space, as a countable projective limit of complete metric spaces.

4.2 For C^∞-manifolds with corners X,Y the mappings $j^k: C^\infty(X,Y) \to C^0(X,J^k(X,Y))$, $0 \leq k \leq \infty$, are all injective (since $\omega \circ j^k f = f$).

<u>Lemma: For any</u> $k = 0,1,2,\dots,\infty$ <u>the image of the mapping</u>

$j^k\colon C^k(X,Y)\to C^o(X,J^k(X,Y))$ <u>is closed in the compact open topology</u>.

<u>Proof:</u> We have to show that the image is closed under taking limits of sequences which converge uniformly on compact subsets. If suffices to consider compact convex subsets of chart neighbourhoods. Finally we have to prove the following: Let U be closed and convex in a quadrant Q in $\mathbb{R}^n$, let f_n be a sequence of C^k-mappings $U\to\mathbb{R}^m$ such that for any $r=0,1,\dots,k$ the sequence $(d^r f_n)$ converges uniformly on U to a continuous mapping $g_r\colon U\to L^r_{sym}(\mathbb{R}^n,\mathbb{R}^m)$, then $d^r g_o = g_r$ for all such r.

The proof is by induction on r. The general step is the same as the first step: if $df_n\to g_1$ uniformly on U, $f_n\to g_o$ uniformly on U, then for x,x+y in U (U is convex) we have:

$$\begin{aligned} g_o(x+y) &= \lim_{n\to\infty} f_n(x+y)\\ &= \lim_{n\to\infty}\left(f_n(x)+\int_0^1 df_n(x+ty).y\,dt\right)\\ &= \lim_n f_n(x)+\int_0^1 \lim_n df_n(x+ty).y\,dt\\ &= g_o(x)+\int_0^1 g_1(x+ty).y\,dt. \end{aligned}$$

Therefore $g_1 = dg_o$, also on the boundary of U. q.e.d.

4.3 The <u>compact C^k-topology</u> or <u>CO^k-topology</u> on $C^r(X,Y)$ $(0\le k\le r\le\infty)$ is the topology induced on $C^r(X,Y)$ by the embedding (4.2) $j^k\colon C^r(X,Y)\to C^o(X,J^k(X,Y))$ from the compact open topology. The CO^k-topology has the following properties:

1. <u>The</u> CO^k<u>-topology on</u> $C^k(X,Y)$ <u>is completely metrizable</u> (3.1 <u>and</u> 4.2), <u>thus a Baire-topology</u>.

2. $(C^r(X,Y),CO^k)$ <u>for</u> $k<r$ <u>is metrizable but not complete</u>. <u>Its completion</u> (<u>in the canonical metric</u>) <u>is just</u> $C^k(X,Y)$.

4.4 The <u>Whitney-C^k-topology</u> or <u>WO^k-topology</u> on $C^r(X,Y)$ $(0\le k\le r\le\infty)$ is the topology induced by the embedding $j^k\colon C^r(X,Y)\to C^o(X,J^k(X,Y))$ from the graph topology (or the WO-topology, see below).

The WO^k-topology has the following properties:

1. *A basis for open sets is given by all sets of the form* $W(U) = \{g \in C^r(X,Y),\ j^k g(X) \subseteq U\}$, *where* U *is open in* $J^k(X,Y)$.

2. *If* d_k *is a metric on* $J^k(X,Y)$ $(0 \leq k \leq \infty)$ *generating the topology, and if* $f \in C^r(X,Y)$, *then the following is a neighbourhood basis for* f *in the* WO^k*-topology*:
$N(f,k,\varepsilon) := \{g \in C^r(X,Y):\ d_k(j^k g(x), j^k f(x)) < \varepsilon(x)$ *for all* $x \in X\}$, *where* $\varepsilon \in C(X,]0,\infty[)$.

3. *A sequence* f_n *in* $C^r(X,Y)$ *converges to* $f \in C^r(X,Y)$ *in* WO^k *iff there exists a compact set* $K \subseteq X$ *such that* f_n *equals* f *off* K *for all but finitely many* n's *and* $j^k f_n \to j^k f$ *uniformly on* K.

4. *If* T *is a compact connected metric space and* $f: T \to (C^r(X,Y), WO^k)$ *is a continuous mapping, then there exists a compact set* $K \subseteq X$ *such that* $t \to f(t)(x)$ *is constant for* $x \in X \setminus K$.

5. $(C^r(X,Y), WO^r)$ *is a Baire space. Each* CO^r*-closed subset of* $C^r(X,Y)$ *is a Baire space too in* WO^r $(0 \leq r \leq \infty)$.

6. WO^∞ *is the projective limit topology of all the topologies* WO^k, $0 \leq k < \infty$.

7. *A basis of open sets for* $(C^\infty(X,Y), WO^\infty)$ *consists of all sets* $W(U,k)$ *from* 1., *for* $k = 0,1,2,\ldots$.

Proof: The WO-topology and the WO^0-topology coincide on the image of j^k: $C^r(X,Y) \to C^0(X, J^k(X,Y))$: $J^k(X,Y) \overset{\alpha}{\to} X$ is already a fibration and j^k takes values in the subsets of sections of this fibration. In more detail: the graph of any $j^k f$, $\Gamma_{j^k f}$ lies already in $X \times_X J^k(X,Y) =$
$= \{(x,\sigma) \in X \times J^k(X,Y):\ \alpha(\sigma) = x\} \subseteq X \times J^k(X,Y)$ and $J^k(X,Y) \overset{\sim}{\to}$
$\to X \times_X J^k(X,Y) \subseteq X \times J^k(X,Y)$, $\sigma \to (\alpha(\sigma), \sigma)$, is a topological embedding. So 1. and 2. follow from 3.2, 3. follows from 3.3, 4. is implies by 3.4, 5. by 3.5 and 4.2.

6. is seen as follows: $(\pi_k^\infty)^{-1}(U)$, U open in $J^k(X,Y)$, $k = 0,1,2,\ldots$, is a basis of the topology on $J^\infty(X,Y)$ by construction. This implies 7. and 7. implies 6. q.e.d.

4.5 We now want to compare $(C^\infty(X,Y), WO^\infty)$ with function spaces known from functional analysis. For that we need some subresults.

1. $J^k(X,Y\times Z)\cong J^k(X,Y)\times_X J^k(X,Z)$; this is evident in charts.

2. $(C^k(X,Y\times Z),WO^k)\cong(C^k(X,Y),WO^k)\times(C^k(X,Z),WO^k)$; the decomposition of 1. is compatible with the decomposition $j^k(f,g)=(j^kf,j^kg)$ for $(f,g)\in C^k(X,Y\times Z)$:

$$\begin{tikzcd} X \arrow[r, "(j^kf,j^kg)"] \arrow[dr, "j^k(f,g)"'] & J^k(X,Y)\times_X J^k(X,Z) \subseteq J^k(X,Y)\times J^k(X,Z) \arrow[d, equal] \\ & J^k(X,Y\times Z) \end{tikzcd}$$

3. For $h\in C^k(Y,Z)$ the mapping $h_*=C^k(X,h)\colon C^k(X,Y)\to$ $\to C^k(X,Z)$, given by $h_*(f)=h\circ f$, is WO^k-continuous: $J^k(X,h)\colon J^k(X,Y)\to J^k(X,Z)$ is continuous by 1.10, 2.14, even for $k=\infty$ by 4.1, so $J^k(X,h)_*\colon C^0(X,J^k(X,Y))\to$ $\to C^0(X,J^k(X,Z))$ is trivially WO-continuous, nearly by definition, and the following diagram commutes:

$$\begin{tikzcd} C^k(X,Y) \arrow[r, "j^k"] \arrow[d, "h_*"] & C^0(X,J^k(X,Y)) \arrow[d, "J^k(X,h)_*"] \\ C^k(X,Z) \arrow[r, "j^k"] & C^0(X,J^k(X,Z)). \end{tikzcd}$$

4. $C^\infty(X,\mathbb{R})$ is a topological ring for the WO^∞-topology; i.e. $(f,g)\to f-g$, $(f,g)\to f\cdot g$ are continuous in WO^∞. This follows from 2. and 3. But $(C^\infty(X,\mathbb{R}),WO^\infty)$ is <u>no</u> topological vector space, otherwise $\frac{1}{n}f\to 0$ in WO^∞ for all $f\in C^\infty(X,\mathbb{R})$, but this is only true for f with compact support by 4.4.3.

Therefore the space $\mathfrak{D}$ of all C^∞-functions with compact support on $\mathbb{R}^n$ is the maximal subspace of $C^\infty(\mathbb{R}^n,\mathbb{R})$ which is a topological vector space in WO^∞.

Let $K\subseteq\mathbb{R}^n$ be compact, denote $\mathfrak{D}_K=\{g\in\mathfrak{D}\colon \operatorname{supp} g\subseteq K\}$, then on $\mathfrak{D}_K$ the CO^∞-topology and the WO^∞-topology coincide. If $r<\infty$, then $(\mathfrak{D}^r,WO^r)=\varinjlim_{K\subseteq\mathbb{R}^n}(\mathfrak{D}^r_K,CO^r)$ in the category of topological vector spaces, as can be seen from HORVATH (1966), p. 171. By 4.4.6 we have $(\mathfrak{D},WO^\infty)=$

$= \varprojlim_r (\mathfrak{D}^r, WO^r) = \mathfrak{D}^F$ (as it is denoted by HORVATH), whose dual space $(\mathfrak{D}^F)'$ is the space of all distributions of finite type on $\mathbb{R}^n$.

This is not the usual topology on $\mathfrak{D}$.

4.6 We want to construct the LO^k-topology from the LO-topology in the same way as above. This gives us a new topology only in the case $k = \infty$, because of the following lemma:

<u>Lemma:</u> <u>If</u> $k < \infty$ <u>then the topology on</u> $C^r(X,Y)$ $(r \geq k)$ <u>induced by the embedding</u> $j^k\colon C^r(X,Y) \to C^o(X, J^k(X,Y))$ <u>from the</u> LO^o-<u>topology</u> (<u>or the</u> LO-<u>topology</u>) <u>coincides with the</u> WO^k-<u>topology</u>.

<u>Proof:</u> The LO-topology and the LO^o-topology on $C^o(X, J^k(X,Y))$ induce the same topology on $C^r(X,Y)$ via j^k, because j^k maps into the subset of sections of the fibration $J^k(X,Y) \xrightarrow{q} X$. Compare with the proof of 4.4.

By 3.6 the LO^o-topology induces via j^k a topology on $C^r(X,Y)$ which is finer than the WO^k-topology.

Now let $f \in C^r(X,Y)$. Let $N(j^k f, L, \epsilon)$ be a basis neighbourhood of $j^k f$ in $C^o(X, J^k(X,Y))$ for the LO^o-topology as described in 3.7.1: $L = (L_\alpha)$ is a locally finite family of closed sets in X, $\epsilon = (\epsilon_\alpha)$ is a family of positive real numbers; d_k is a metric on $J^k(X,Y)$ compatible with the topology. Then $(j^k)^{-1}(N(j^k f, L, \epsilon)) = \{g \in C^r(X,Y)\colon d_k(j^k g(x), j^k f(x)) < \epsilon_\alpha \text{ for } x \in L_\alpha, \text{ for all } x\}$.

By a partition of unity argument there is a positive continuous real function $\epsilon\colon X \to \mathbb{R}$ such that $\epsilon(x) < \epsilon_\alpha$ if $x \in L_\alpha$ (since $L = (L_\alpha)$ is locally finite this is possible). Then $N(f,k,\epsilon) = \{g \in C^r(X,Y)\colon d_k(j^k g(x), j^k f(x)) < \epsilon(x) \text{ for all } x \in X\} \subseteq (j^k)^{-1}(N(j^k f, L, \epsilon))$. So WO^k is finer as $(j^k)^{-1}(LO^o)$. q.e.d.

4.7 The $\mathfrak{D}$-<u>topology</u> on $C^\infty(X,Y)$ is the topology induced by the embedding $j^\infty\colon C^\infty(X,Y) \to C^o(X, J^\infty(X,Y))$ from the LO- or the LO^o-topology, which coincide on the image of j^k (cf. 4.4). We have the following properties:

1. A basis of open sets for the $\mathfrak{D}$-topology is given by sets of the form: $M(L,U) = \{f \in C^\infty(X,Y): j^\infty f(L_n) \subseteq U_n\}$, where $L = (L_n)$ is a locally finite sequence of closed sets in X and (U_n) is a sequence of open sets in $J^\infty(X,Y)$. (Each locally finite family is essentially a sequence since X is locally compact second countable.) This follows from 3.6.

2. The following is a basis of open sets for the $\mathfrak{D}$-topology too: $M'(L,U) = \{f \in C^\infty(X,Y): j^n f(L_n) \subseteq U_n\}$, where $L = (L_n)$ is a locally finite sequence of closed sets in X and U_n is open in $J^n(X,Y)$ for all $n \in \mathbb{N}$.
Proof: $(\pi_n^\infty)^{-1}(U)$, U open in $J^n(X,Y)$, runs through a basis of open sets in $J^\infty(X,Y)$. Given any basic set of 1. one may repeat the L_n's, reorder it, put $U_n = J^n(X,Y)$ and $(\pi_l^k)^{-1}(U_l)$ into the sequence (U_n) to represent this set as a union of sets of the form $M'(L,U)$.

3. Let $K = (K_n)$ be a sequence of compact sets in X with $K_o = \emptyset$, $K_n \subseteq K_{n+1}^o$ (the open interiour), $\bigcup_n K_n = X$. Then the following system of sets is a basis for the $\mathfrak{D}$-topology: $M(U,m) = \{f \in C^\infty(X,Y): j^{m_n} f(X \setminus K_{n}^o) \subseteq U_n\}$ where $m = (m_n)$ is a sequence in $\mathbb{N}$ and $U_k \subseteq J^{m_n}(X,Y)$ is open. One may suppose m_n increasing.
Proof: Any locally finite family $L = (L_i)$ satisfies $X \setminus K_{n_1(i)}{}^o \supseteq L_i \supseteq X \setminus K_{n_2(i)}$ for suitable mappings n_1, n_2: $\mathbb{N} \to \mathbb{N}$. The "jet-order" of a set $U_i \in J^i(X,Y)$ may be lifted by $(\pi_i^k)^{-1}(U_i) \subseteq J^k(X,Y)$.

4. Let $f \in C^\infty(X,Y)$, let d be a compatible metric on $J^\infty(X,Y)$. Then all sets of the following form are a basis of neighbourhoods of f: $N(f,L,\varepsilon) = \{g \in C^\infty(X,Y): d(j^\infty g(x), j^\infty f(x)) < \varepsilon_n$ for all $x \in L_n$, for all $n\}$, where $L = (L_n)$ is a locally finite sequence of closed sets, $\varepsilon = (\varepsilon_n > 0)$ is a sequence of positive constants.

The following sets form a basis of neighbourhoods too: $N(f,\varphi) = \{g \in C^\infty(X,Y): \varphi_n(x) d(j^\infty g(x), j^\infty f(x)) < 1$ for all $x \in X$, for all $n\}$, where $\varphi = (\varphi_n)$, $\varphi_n \in C(X,[0,\infty[)$, $(\operatorname{supp} \varphi_n)$ is locally finite. This follows from 3.7.

5. Let d_k be a compatible metric on $J^k(X,Y)$ for any k, then for $f \in C^\infty(X,Y)$ the following is a basis of neighbourhoods: $V_\varphi(f) = \{g \in C^\infty(X,Y) \colon \varphi_n(x) d_n(j^n g(x), j^n f(x)) < 1$ for all $x \in X$, for all $n\}$, where again $\varphi = (\varphi_n)$, $\varphi_n \in C(X,[0,\infty[)$, with $(\operatorname{supp}\varphi_n)$ locally finite.

This can be deduced from 4. or directly from 2. (see MICHOR [2]).

6. $(C^\infty(X,Y),\mathfrak{D})$ is a Baire space. Each CO^∞-closed subset of $C^\infty(X,Y)$ is a Baire space with the $\mathfrak{D}$-topology. This follows from 3.10 and 4.2.

7. If (f_n) is a sequence in $C^\infty(X,Y)$ and $f \in C^\infty(X,Y)$, then we have: $f_n \to f$ in $\mathfrak{D}$ iff there exists a compact set $K \subseteq X$ such that f_n equals f off K for all but finitely many n's and $j^\infty f_n|K \to j^\infty f|K$ uniformly (i.e. $f_n \to f$ in CO^∞). This follows from 3.8.

8. Let $f \colon T \to (C^\infty(X,Y),\mathfrak{D})$ be a continuous mapping, where T is compact connected metric. Then there exists a compact subset $K \subseteq X$ such that $t \to f(t)(x)$ is constant on T for all $x \in X\setminus K$. See 3.9.

9. $(C^\infty(X,Y\times Z),\mathfrak{D}) = (C^\infty(X,Y),\mathfrak{D}) \times (C^\infty(X,Z),\mathfrak{D})$.

Proof: As in 4.5.1 and 2. we have $J^\infty(X,Y\times Z) \cong$
$\cong J^\infty(X,Y) \times_X J^\infty(X,Z)$ and

$$\begin{tikzcd} X \arrow[r, "(j^\infty f,\, j^\infty g)"] \arrow[dr, "j^\infty(f,g)"'] & J^\infty(X,Y)\times_X J^\infty(X,Z) \arrow[d, equal] & \subseteq J^\infty(X,Y)\times J^\infty(X,Z) \\ & J^\infty(X,Y\times Z) & \end{tikzcd}$$

commutes for all (f,g) in $C^\infty(X,Y)\times C^\infty(X,Z)$. So the LO-topology on $C^0(X,J^\infty(X,Y\times Z)) = C^0(X,J^\infty(X,Y)\times_X J^\infty(X,Z))$ induces the same topolcgy on both spaces. q.e.d.

4.8 Proposition: Let $(E,p,X,\mathbb{R}^m)$ be a C^∞-vector bundle over a manifold with corners X. Let $\Gamma_c(E)$ denote the space of all C^∞ sections with compact support of this bundle, equipped with the trace of the $\mathfrak{D}$-topology on $C^\infty(X,E)$. Then $\Gamma_c(E)$ is the maximal subspace of $\Gamma(E)$ which is a topological vector space in the $\mathfrak{D}$-topology.

$(\Gamma_c(E),\mathfrak{D})$ <u>is a complete locally convex vector space, a nuclear</u> (LF)-<u>space, dually nuclear and a Lindelöf space, hence even paracompact and normal.</u>

<u>Proof:</u> $\Gamma(E)$, the space of all C^{∞}-sections of E, is a closed subset of $C^{\infty}(X,E)$, since $p \circ s = Id_X$ is a continuous equation in s for the $\mathfrak{D}$-topology. ($p_*: C^{\infty}(X,E) \to C^{\infty}(X,X)$ is $\mathfrak{D}$-continuous; this is trivially seen in 4.5.3; it will be explicitly proved in 7. below). $\Gamma_c(E) = \{s \in \Gamma(E): \frac{1}{n} \cdot s \to 0 \text{ in } \mathfrak{D}\}$ is clearly the maximal subset of $\Gamma(E)$ which is a topological vector space.

Since X has a finite atlas (see GREUB, HALPERIN, VANSTONE I), there exists a second vector-bundle $(F,p',X,\mathbf{R}^p)$ such that the Whitney sum $E \oplus F$ is a trivial vector bundle over X, so $E \oplus F \cong X \times \mathbf{R}^{m+p}$. By 4.7.9 $\Gamma_c(E)$ is a topological subspace, even a direct summand in $\Gamma_c(E \oplus F) \cong$ $\cong \Gamma_c(X \times \mathbf{R}^{m+p}) \cong \mathfrak{D}(X)^{m+p}$, where $\mathfrak{D}(X)$ denotes the space of all smooth sections with compact support; the topology is the nuclear (LF)-topology of L. SCHWARTZ, as can be seen from comparing 4.7.3 with HORVATH, p. 170. We will give a direct explicit proof of this fact later.

So $\mathfrak{D}(X)^{m+p}$ is an (LF)-space too, i.e. a locally convex direct limit of a countable strict family of separable Fréchet spaces, which can be identified with $\mathfrak{D}_{K_l}(X)^{m+p} =$ $= \{f \in \mathfrak{D}(X)^{m+p}: \text{supp } f \subseteq K_l\}$, where $K = (K_l)$ is a sequence of compact sets $K_l \subseteq X$ with $K_l \subset \overset{\circ}{K}_{l+1}$ and $X = \bigcup_l K_l$. Each $\mathfrak{D}_{K_l}(X)^{m+p}$ is a Lindelöf space since it is separable and metric, so $\mathfrak{D}(X)^{m+p}$ and its closed subspace $\Gamma_c(E)$ are Lindelöf too. Since they are clearly completely regular, they are paracompact and normal. Since $\Gamma_c(E)$ is even a direct summand in $\mathfrak{D}(X)^{m+p}$ it is nuclear and dually nuclear and an (LF)-space too. q.e.d.

4.9 <u>Remark:</u> None of the topologies on $C^{\infty}(X,Y)$ mentioned so far is fine enough for our porposes, since $(C^{\infty}(X,Y),\mathfrak{D})$ and $(C^{\infty}(X,Y),WO^{\infty})$ are not locally contractible, not even locally arcwise connected if X is not compact: If f and $g \in C^{\infty}(X,Y)$ are connected by a continuous curve $c: [0,1] \to C^{\infty}(X,Y)$, $c(0) = f$, $c(1) = g$, then f and g differ at most on a compact set $K \subseteq X$ as can be seen from 4.7.8 and 4.4.4.

This is true for all topologies between WO^{0} and $\mathfrak{D}$. So there can be no way to make a manifold out of $C^{\infty}(X,Y)$ in any sense, if one insists that there should be open chart neighbourhoods, modelled on topological vector spaces.

4.10 Definition: Let X,Y be smooth manifolds with corners. Call $f,g \in C^{\infty}(X,Y)$ equivalent, $f \sim g$, if the set $\{x \in X: f(x) \neq g(x)\}$ is relatively compact in X. This is clearly an equivalence relation on $C^{\infty}(X,Y)$.

Now the fine-$\mathfrak{D}$-topology or $(F\mathfrak{D})$-topology on $C^{\infty}(X,Y)$ is the coarsest topology on $C^{\infty}(X,Y)$ which is finer than the $\mathfrak{D}$-topology and makes the above equivalence relation to an open one.

An equivalent description of the $(F\mathfrak{D})$-topology is the following: take all equivalence classes, induce the $\mathfrak{D}$-topology on them and take their disjoint union.
Or: Declare all equivalence classes to be open and add them to the open sets of the $\mathfrak{D}$-topology.

It is clear how the different bases and neighbourhood bases of the $\mathfrak{D}$-topology described in 4.7 give bases and neighbourhood bases of the $(F\mathfrak{D})$-topology: intersect all basic open sets with equivalence classes; intersect each basic neighbourhood of f with $\{g: g \sim f\}$.

The $(F\mathfrak{D})$-topology was called $\mathfrak{D}^{\infty}$-topology in MICHOR (1978), where it was introduced.

4.11 Remarks:

1. The $(F\mathfrak{D})$-topology has the same converging sequences and continuous curves as the $\mathfrak{D}$- and the WO^{∞}-topology, since 4.7.7 and 4.4.3, 4.7.8, 4.4.4 remain valid.

2. In the notation of 4.8 $\Gamma_c(E)$ is open in $\Gamma(E)$ for the $(F\mathfrak{D})$-topology. So $\Gamma(E)$, the space of all sections of a vector bundle, equipped with the $F\mathfrak{D}$-topology, is a

local topological affine space with model topological vector space $(\Gamma_c(E), \mathfrak{D})$: For any $s \in \Gamma(E)$ the set $s + \Gamma_c(E)$ is an open neighbourhood of s and an affine subspace which is isomorphic to $\Gamma_c(E)$. We will see later on that this is enough structure to get calculus on $\Gamma(E)$.

3. $(C^\infty(X,Y),(F\mathfrak{D}))$ is no longer a Baire space, since $\Gamma_c(E)$ is no Baire space, if X is not compact: Let (K_n) be a sequence of compacts in X with $K_0 = \emptyset$, $K_n \subseteq \overset{o}{K}_{n+1}$, $\bigcup_n K_n = X$. Then each $\Gamma_{K_n}(E) = \{s \in \Gamma(E),\ \operatorname{supp} s \subseteq K\}$ is nowhere dense in $\Gamma_c(E)$, but $\Gamma_c(E) = \bigcup_n \Gamma_{K_n}(E)$.

We will see later that $(C^\infty(X,Y),(F\mathfrak{D}))$ is locally homomorphic to spaces of type $\Gamma_c(E)$, so $C^\infty(X,Y)$ is no Baire space either, but we may conclude that $(C^\infty(X,Y),(F\mathfrak{D}))$ is paracompact and normal. This is an open problem for WO^∞ and $\mathfrak{D}$.

5 Open subsets

Let X,Y be manifolds with corners, if not explicitly stated otherwise.

5.1 Proposition:

1. The set of immersions $\mathrm{Imm}^r(X,Y)$ is WO^1-open in $C^r(X,Y)$ for each $r \geq 1$. (Therefore open in each finer topology too).

2. The set of all submersions $\mathrm{Sub}^r(X,Y)$ is WO^1-open in $C^r(X,Y)$ for each $r \geq 1$.

Proof: $f: X \to Y$ is an immersion (a submersion) iff $j^1f(x)$ has maximal rank in $J^1_{x,f(x)}(X,Y)$. The set of all 1-jets of maximal rank in $J^1(X,Y)$ is an open subset, even an open sub fibre bundle $S^1_o(X,Y)$, so $\mathrm{Imm}(X,Y)$ (resp. $\mathrm{Sub}(X,Y)) = \{f \in C^r(X,Y)\colon j^1f(X) \subseteq S^1_o(X,Y)\}$ is a basic WO^1 open set. q.e.d.

5.2 Definition: A continuous mapping $f: X \to Y$ is called proper, if $f^{-1}(K)$ is compact for each compact $K \subseteq Y$.

Proposition: The set $C^r_{prop}(X,Y)$ of all proper C^r-mappings is WO^o open and closed in $C^r(X,Y)$, $r \geq 0$. $C^r_{prop}(X,Y)$ is not empty if $\dim Y \geq 1$. For the proof we need a sublemma.

Sublemma: Let X be a manifold with corners. Then there is a complete metric on X generating the topology, such that each bounded subset is relatively compact.

Proof of the sublemma: As in the lemma in 2.7 let α be the local flow of a strictly inner vector field ξ on X, such that α_ε is everywhere defined for some $\varepsilon > 0$, then

$\alpha_\epsilon: X \to X\backslash\partial^1 X$ embeds X as a submanifold with corners of the manifold without boundary $X\backslash\partial^1 X$. Choose a complete Riemannion metric on $X\backslash\partial^1 X$ (these form a CO^∞-dense subset of the set of all Riemannion metrics, see MORROW (1970)). Geodesic distance has the stated property on $X\backslash\partial^1 X$. Since the image of α_ϵ is closed in $X\backslash\partial^1 X$, the pull bank of the geodesic distance via α_ϵ has the stated properties on X. q.e.d.

Proof of the proposition: Let $d,\bar{d}$ be complete metrics on X,Y resp. such that each bounded subset is relatively compact. Choose any constant $\epsilon > 0$. For $f \in C^\infty(X,Y)$ consider $N(f,\epsilon,0) = \{g \in C^r(X,Y): \bar{d}(f(x),g(x)) < \epsilon$ for all $x \in X\}$.

Claim: If f is proper, then any $g \in N(f,\epsilon,0)$ is proper too.

If $K \subseteq Y$ is compact, then $L = \{y \in Y: \bar{d}(y,K) \leq \epsilon\}$ is compact too, so $f^{-1}(L)$ is compact in X. If $x \notin f^{-1}(L)$ then $g(x) \notin K$ since $f(x) \notin L$ and $\bar{d}(g(x),f(x)) < \epsilon$.

So $g^{-1}(K) \subseteq f^{-1}(L)$, so $g^{-1}(K)$ is compact.

Claim: If f is not proper, then no $g \in N(f,\epsilon,0)$ is proper.

There is a compact $K \subseteq Y$ such that $f^{-1}(K)$ is not compact. Define $L = \{y \in Y: \bar{d}(y,K) \leq \epsilon\}$, then L is compact.

If $x \in f^{-1}(K)$, then $f(x) \in K$, so $g(x) \in L$, so $x \in g^{-1}(L)$. Therefore $g^{-1}(L) \supseteq f^{-1}(K)$, so $g^{-1}(L)$ contains a closed but not compact set, so $g^{-1}(L)$ is not compact either.

To prove the last claim of the proposition, let $x_0 \in X$ be fixed, let $f(x) = d(x,x_0)^2$. If d is as constructed in the sublemma, then f is C^∞ function on X. Furthermore $f^{-1}([-n,n])$ is compact in X (d-bounded and closed). Now take any embedding $c: \mathbb{R} \to Y$ and consider $c \circ f \in C^\infty(X,Y)$. This mapping is proper. q.e.d.

See MATHER (1969) and HIRSCH (1976) for alternative proofs of parts of the proposition.

5.3 Proposition: The set $E^r(X,Y)$ of all embeddings is WO^1-open in $C^r(X,Y)$, $r \geq 1$.

The set of injective immersions is not open.

Proof: Consider first U open and convex in a quadrant

Q in $\mathbb{R}^n$, let Q' be a quadrant in $\mathbb{R}^m$, let $f\colon U \to Q'$ be a C^1-embedding. By Taylor's theorem we have for $x, x+y \in U$:
$f(x+y) - f(x) = df(x).y + \int_0^1 (1-t)(df(x+ty) - df(x)).y\, dt.$
Put $U(l) = \inf\{|l(y)| : |y| = 1\}$ for $l \in L(\mathbb{R}^n, \mathbb{R}^m)$, then $U\colon L(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}^+$ is a continuous function and $U(l) > 0$ iff l is injective. Then we get (1):
$|f(x+y) - f(x)| \geq U(df(x)).|y| - \int_0^1 (1-t)\|df(x+ty) - df(x)\|\,|y|\, dt.$
Let $\epsilon(x) = U(df(x))/4$, then $\epsilon\colon U \to \mathbb{R}^+$ is continuous, $\epsilon(x) > 0$. Choose $\delta\colon U \to \mathbb{R}$ continuous, $\delta(x) > 0$, $\delta(x) < \epsilon(x)$ such that $|z| < \delta(x)$ implies $\|df(x+z) - df(x)\| < \epsilon(x)$. Then we have:
$|y| < \delta(x)$, $x, x+y \in U$ imply
$|f(x+y) - f(x)| > 4\epsilon(x).|y| - \epsilon(x)|y| = 3\epsilon(x)|y|$. Now
$\epsilon_1\colon U \to \mathbb{R}$, $\epsilon_1(x) > 0$ be a continuous function such that the following holds: if $g\colon U \to Q'$ is a C^1-function and $\|dg(x) - df(x)\| < \epsilon_1(x)$ for all $x \in U$, then $|U(dg(x)) - U(df(x))| < \epsilon(x)$. Let $W \subseteq U$ be open, convex and such that $\overline{W}$ is compact $\subseteq U$. For $x \in U$ let
$\epsilon_2(x) = \inf\{|f(x) - f(z)| : z \in \overline{W} \setminus \{x+y : |y| < \delta(x)\}\} > 0$. This is a continuous function since $f(\overline{W} \setminus \{x+y : |y| < \delta(x)\})$ is compact. Let $\epsilon_3(x) < \epsilon_1(x)$, $\epsilon_3\colon U \to \mathbb{R}$ continuous, $\epsilon_3(x) > 0$, such that $\epsilon_3(x) < \inf\{\epsilon(x+y) : |y| \leq \delta(x)\}$.
<u>Claim:</u> Let $g \in C^1(U, Q')$ be such that the following hold:
$|g(x) - f(x)| < \frac{\epsilon_2(x)}{4}$ for all $x \in U$.
$\|dg(x) - df(x)\| < \frac{\epsilon_3(x)}{2}$ for all $x \in U$. Then $g|W$ is an embedding.
Proof: $U(dg(x) \geq U(df(x)) - |U(df(x)) - U(dg(x))| >$
$> 4\epsilon(x) - \epsilon(x) > 0$, so g is an immerion on U.

Now let $x, x+y \in W$, $y \neq 0$. If $|y| < \delta(x)$, then estimate (1) for g shows:
$|g(x+y) - g(x)| \geq U(dg(x)|y| - \int_0^1 (1-t)\|dg(x+ty) - dg(x)\|\,|y|\, dt.$
But $\|dg(x+ty) - dg(x)\| \leq \|dg(x+ty) - df(x+ty)\| +$
$+ \|df(x+ty) - df(x)\| + \|df(x) - dg(x)\| \leq$
$\leq \frac{\epsilon_3(x+ty)}{2} + \epsilon(x) + \frac{\epsilon_3(x)}{2} \leq 2\epsilon(x).$
Therefore $|g(x+y) - g(x)| \geq U(dg(x))|y| - 2\epsilon(x).|y| >$

$> (3\varepsilon(x) - 2\varepsilon(x))|y| > 0.$

If on the other hand $|y| \geq \delta(x)$, then we have:

$$|g(x+y) - g(x)| \geq |f(x+y) - f(x)| - |f(x+y) - g(x+y)| - |f(x) - g(x)| \geq \frac{\varepsilon_2(x) + \varepsilon_2(x+y)}{2} - \frac{\varepsilon_2(x+y)}{4} - \frac{\varepsilon_2(x)}{4} = \frac{\varepsilon_2(x) + \varepsilon_2(x+y)}{4} > 0.$$

So g is injective. Since $\overline{W}$ is compact, $g|\overline{W}$ is am embedding, so $g|W$ is it too.

We have proven the claim.

Now we look at the general situation again: X,Y are manifolds with corners, $f: X \to Y$ is an embedding (C^r). Let d, d_1 be metrics on Y, $J^1(X,Y)$ resp. which are compatible with the topologies. Using the first part of the proof we may find the following data: a locally finite open relatively compact cover (U_α) of X, compacts K_α in U_α such that $(K_\alpha{}^o)$ is still an open cover, continuous positive functions ε_α, δ_α, defined cover $\overline{U}_\alpha$, such that the following holds: if $g \in C^1(X,Y)$ and $d(g(x),f(x)) < \varepsilon_\alpha(x)$, $d_1(j^1g(x), j^1f(x)) < \delta_\alpha(x)$ for all $x \in \overline{U}_\alpha$, then $g|U_\alpha$ is an embedding.

By a partition of unity argument we find continuous positive functions ε, δ on X such that $\varepsilon(x) < \varepsilon_\alpha(x)$ if $x \in U_\alpha$, $\delta(x) < \delta_\alpha(x)$ if $x \in U_\alpha$.

Since f is an embedding, there are disjoint open sets A_α, $B_\alpha \subseteq Y$ with $f(K_\alpha) \subseteq A_\alpha$ and $f(X \setminus U_\alpha) \subseteq B_\alpha$.

Now let $\mathfrak{B}$ be the WO^1-open neighbourhood of f, given by $\mathfrak{B} = \{g \in C^1(X,Y)\colon d(g(x),f(x)) < \varepsilon(x),\ d_1(j^1f(x), j^1g(x)) < \delta(x),\ g(K_\alpha) \subseteq A_\alpha,\ g(X\setminus U_\alpha) \subseteq B_\alpha$ for all $x \in X$, for all $\alpha\}$.

<u>Claim:</u> Each $g \in \mathfrak{B}$ is an embedding. By construction $g|U_\alpha$ is an embedding, so g is an immersion.

If x,y in X, $x \in K_\alpha$, $x \neq y$, then we have: $y \in U_\alpha$ implies $g(x) \neq g(y)$ since $g|U_\alpha$ is an embedding.

$y \in X\setminus U_\alpha$, then $g(y) \in g(X\setminus U_\alpha) \subseteq B_\alpha$, $g(x) \in g(K_\alpha) \subseteq A_\alpha$, so $g(x) \neq g(y)$. Therefore g is injective.

We have to show finally: $g: X \to g(X)$ is a homeomorphism. It suffices to show if (x_n) is a sequence in X such that $g(x_n) \to g(x)$, then $x_n \to x$.

x is contained in some K_α, so $g(x) \in A_\alpha$ which is open. So all but finitely many $g(x_n) \in A_\alpha$, only finitely many $g(x_n) \notin B_\alpha$, therefore only finitely many x_n are not in U_α. Since $g|U_\alpha$ is a homeomorphism onto its image, $x_n \to x$.

q.e.d.

5.4 Corollary: The set $E^r_{prop}(X,Y)$ of all closed embeddings is WO^1 open in $C^r(X,Y)$, $r \geq 1$.

Proof: $E^r_{prop}(X,Y) = E^r(X,Y) \cap C^r_{prop}(X,Y)$.

5.5 Let us denote the set of all surjective C^r-submersions of X onto Y by $Q^r(X,Y)$, $r \geq 1$. If $\dim X < \dim Y$, then $Q^r(X,Y) = \emptyset$, if $\dim X = \dim Y$, then $Q^r(X,Y)$ is the set of all covering mappings.

Example: Let $X = Y = [0,1]$. Let $f_t\colon [0,1] \to [0,1]$ be given by $f_t(x) = t.x$. Then $f_t \to Id_{[0,1]}$ for $t \to 1$ in $(C^\infty(X,Y),\mathfrak{D})$, but no f_t is surjective if $t \neq 1$. So $Q^r(X,Y)$ is not open.

Lemma: Let X,Y be manifolds without boundary, let $f\colon X \to Y$ be a surjective C^r-submersion, $r \geq 1$. Then there exists a WO^0-open neighbourhood of f in $C^\infty(X,Y)$, consisting entirely of surjective mappings.

Proof: Let $f\colon X \to Y$ be a surjective submersion. Using the theorem of implicit functions, we can construct the following data:

(1) (U_i,u_i), a locally finite atlas of X, such that $u_i(U_i) \supseteq D^n$, the closed unit ball in $\mathbb{R}^n$, for each i ($n = \dim X$), and $(\overline{U}_i)$ in still locally finite.

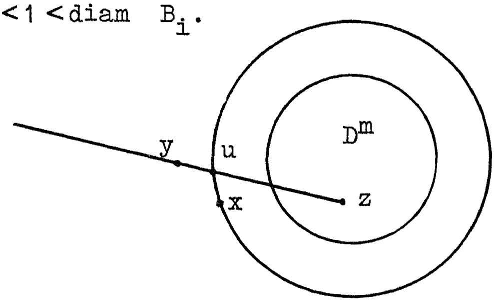
(2) (V_i,v_i) an atlas of Y such that $v_i(V_i) \supseteq D^m$ ($m = \dim Y$) and $(v_i^{-1}(D^m))$ is still a cover of Y.

(3) $f(U_i) \subseteq V_i$ for each i, and there is an m-dimensional linear subspace $L \subseteq \mathbb{R}^n$, and a linear isomorphism $l\colon \mathbb{R}^m \to L$ such $l(v_i(V_i)) \subseteq u_i(U_i)$ and $l \circ v_i \circ f \circ u_i^{-1}\colon u_i(U_i) \subseteq \mathbb{R}^n \to L$ coincides with the restriction of the orthogonal projection onto L.

Now for any i choose B_i, a closed ball of center 0 and radius > 1 in $\mathbb{R}^m$ such that $D^m \subseteq B_i^{\,o} \subseteq B_i \subseteq v_i(V_i)$.

Then choose $\varepsilon_i > 0$ in such a way, that for any $z \in D^m$, $x \in \partial B_i$ and $y \in \mathbb{R}^m$ with $|x-y| < \varepsilon_i$ the following holds:

the ray from z through y meets ∂B_i in a point u with $|x-u| < 1 < \text{diam } B_i$.



Now let $W \subseteq C^0(X,Y)$ be the following WO^0-open neighbourhood of f: $W = \{g \in C^0(X,Y): |v_i g(x) - v_i f(x)| < \epsilon_i$ for all $x \in \overline{U}_i$, for all $i\}$.

<u>Claim:</u> Each $g \in W$ is surjective. It suffices to show that $v_i \circ g(U_i) \supseteq D^m$ for then $g(U_i) \geq v_i^{-1}(D^m)$ and the $v_i^{-1}(D^m)$ still cover Y.

By condition (3) we have $v_i \circ f \circ u_i^{-1} \circ 1 = \mathrm{Id}_{v_i(V_i)}$.

Let $h_i = v_i \circ g \circ u_i^{-1} \circ 1\colon v_i(V_i) \to \mathbb{R}^m$. For any $g \in W$ we have $|h_i(x) - x| = |v_i \circ g(u_i^{-1} 1(x)) - v_i \circ f(u_i^{-1} 1(x))| < \epsilon_i$. Suppose that $h_i(B_i) \not\supseteq D^m$. Then there is some $z \in D^m \setminus h_i(B_i)$. Define $H\colon B_i \to \partial B_i$ as follows: for $x \in B_i$ let $H(x)$ be the intersection with ∂B_i of the ray from z through $h_i(x)$. H is continuous since $z \notin h_i(B_i)$. By choice of ϵ_i and $|h_i(x) - x| < \epsilon_i$ we have $H(x) \neq -x$ for all $x \in B_i$. Therefore $H|\partial B_i\colon \partial B_i \to \partial B_i$ is homotopic to the identity (a homotopy connecting $H|\partial B_i$ and $\mathrm{Id}_{\partial B_i}$ moves $H(x)$ along the shorter great circle to x). A wellknown theorem of algebraic topology (equivalent to Brouwer's fixed point theorem) says that no continuous mapping $S^{m-1} \to S^{m-1}$ which extends to a continuous mapping $D^m \to S^{m-1}$ can be homotopic to the identity. This contradiction shows that $D^m \subseteq h_i(B_i)$ for all i and proves the lemma. q.e.d.

5.6 <u>Corollary: Let</u> X,Y <u>be manifolds without boundary. Then the set</u> $Q^r(X,Y)$ <u>of all surjective submersions</u> $X \to Y$ $(r \geq 1)$ <u>is</u> WO^1 <u>open in</u> $C^r(X,Y)$.

<u>Proof:</u> Use 5.5 and 5.1.

5.7 <u>Corollary:</u> <u>Let</u> X <u>be a manifold without boundary</u>. <u>Then the set</u> $\mathrm{Diff}^r(X)$ <u>of all</u> C^r<u>-diffeomorphisms of</u> X <u>onto</u> X <u>is</u> WO^1 <u>open in</u> $C^r(X,X)$, $r \geq 1$.

5.8 <u>Lemma:</u> <u>Let</u> X,Y <u>be</u> C^0<u>-manifolds without boundary</u>, <u>let</u> $f\colon X \to Y$ <u>be a homeomorphism</u>. <u>Then there is a</u> WO^0 <u>open neighbourhood of</u> f <u>in</u> $C^0(X,Y)$ <u>consisting entirely of surjective mappings</u>.

<u>Proof:</u> If $f = \mathrm{Id}_X$, then one may carry over the proof of lemma 5.5: in this case one only needs the cover (2), so there is no need for the implicit function theorem.

Now if $\mathfrak{B} \subseteq C^0(X,X)$ is an open neighbourhood of Id_X consisting entirely of surjective mappings, and if $f\colon X \to Y$ is a homeomorphism, then by 4.5.3 $f_*\colon C^0(X,X) \to$ $\to C^0(X,Y)$ is WO^0-continuous with WO^0-continuous inverse $(f^{-1})_*$, so $f_*(\mathfrak{B}) = \{f \circ g\colon g \in \mathfrak{B}\}$ is a WO^0 open neighbourhood of f consisting entirely of surjective mappings.

q.e.d.

5.9 <u>Definition:</u> Let X,Y be manifolds with corners again. Denote by $C^r_\partial(X,Y)$ the set of all <u>"border faithful" mappings</u> in $C^r(X,Y)$, i.e. $C^r_\partial(X,Y) = \{f \in C^r(X,Y)\colon f(\partial X) \subseteq \partial Y\}$. Since ∂X and ∂Y are closed, $C^r_\partial(X,Y)$ is closed in the WO^0-topology.

5.10 The <u>double</u> of a manifold with corners: Let X be a manifold with corners. DX, the double of X, is the identification space obtained from $(X \times 0) \cup (X \times 1)$ by identifying $(x,0)$ with $(x,0)$ if $x \in \partial X$. DX is a C^0-manifold (since any quadrant is homeomorphism to a half space). If X has smooth boundary, then there is a C^∞-structure on DX inducing the given ones on the two copies of X in DX, and this structure is unique (up to diffeomorphisms): uniqueness of gluing, see e.g. HIRSCH (1976), p. 184.

Let X,Y be manifolds with corners, then we have a mapping $D\colon C^0_\partial(X,Y) \to C^0(DX,DY)$, defined by $Df = (f \times 0) \cup$ $\cup\ (f \times 1)/\sim$.

It is clear that $D\colon C^0_\partial(X,Y) \to C^0(DX,DY)$ is a topological embedding for the WO^0-topology. D has values in $C^0_\partial(DX,DY)$, i.e. the set of all f in $C^0(DX,DY)$ mapping the submanifold ∂X of DX into ∂Y.

There are two continuous projections $C^0_\partial(DX,DY) \to C^0_\partial(X,Y)$, which are left inverse to D, given by $g \to p \circ g \circ i_0$, $p \circ g \circ i_1$, where $p: DX \to X$ is induced by $p(x,0) = x$, $p(x,1) = x$, and $i_0: X \to DX$, $i_1: X \to DX$ are given by $i_0(x) = (x,0)$, $i_1(x) = (x,1)$. Since i_0, i_1 are closed embeddings, these projections are continuous (cf. § 7).

5.11 If X is a manifold with corners and $f: X \to X$ is a diffeomorphism, then automatically $f(\partial X) \subseteq \partial X$, so $\mathrm{Diff}^r(X) \subseteq C^r_\partial(X,Y)$, $r \geq 1$.

<u>Proposition:</u> <u>Let</u> X <u>be a manifold with corners</u>, <u>then</u> $\mathrm{Diff}^r(X)$ <u>is</u> WO^1 <u>open in</u> $C^r_\partial(X,X)$, $r \geq 1$.

<u>Proof:</u> If $f \in \mathrm{Diff}^r(X) \subseteq C^r_\partial(X,X)$, then $Df: DX \to DX$ is a homeomorphism. By 5.8 there is a WO^0 open neighbourhood $\mathfrak{B} \subseteq C^0(DX,DX)$ consisting entirely of surjective mappings. Then $D^{-1}(\mathfrak{B})$ is a WO^0 open neighbourhood of f in $C^0_\partial(X,X)$ consisting entirely of surjective mappings. But then $D^{-1}(\mathfrak{B}) \cap E^r(X,Y)$ (cf. 5.3) is a WO^1 open neighbourhood of f in $C^r_\partial(X,X)$, consisting entirely of diffeomorphisms.

q.e.d.

6 Transversality and dense subsets

This section lies somewhat outside the main line of development of this book. We include it since we have at hand all the necessary background on manifolds with corners and topologies on spaces of mappings. We prove the transversality theorem for manifolds with corners, in a formulation slightly more general than the usual one, thus solving two problems stated in HIRSCH (1976). See GIBSON (1979), BOLUBITZKY-GUILLEMIN (1973) and HIRSCH (1976) for proofs in the setting of manifolds without boundary.

6.1 <u>Definition:</u> Let X_1, X_2, Y be manifolds with corners, let $f_i: X_i \to Y$ be smooth mappings, $i = 1,2$.

We say that f_1 and f_2 are <u>transversal</u> at $y \in Y$ if $\operatorname{Im} T_{x_1} f_1 + \operatorname{Im} T_{x_2} f_2 = T_y Y$ whenever $f_1(x_1) = f_2(x_2) = y$. In symbols: $f_1 \pitchfork f_2$ at y. f_1, f_2 are said to be <u>transversal</u>, $f_1 \pitchfork f_2$, if they are transversal everywhere. Finally we say that f_1 and f_2 are transversal over $A \times B$ (where $A \subseteq X_1$, $B \subseteq X_2$ are subsets) if $T_{x_1} f_1 (T_{x_1} X_1) + T_{x_2} f_2 (T_{x_2} X_2) = T_{f_1(x_1)} Y$ for all $x_1 \in A$, $x_2 \in B$ such that $f_1(x_1) = f_2(x_2)$.

6.2 If $f: X \to Y$ is a submersion between manifolds without boundary, then f^{-1} (point) is a submanifold of X, whose codimension equals the dimension of Y if it is not empty. If furthermore Z is a submanifold without boundary of Y, and $f \pitchfork Z$ (i.e. $f \pitchfork i$, where $i: Z \to Y$ is the embedding), then $f^{-1}(Z)$ is a submanifold of X, whose codimension equals the codimension of Z if it is not empty. Finally let $f_i: X_i \to Y$ $(i=1,2)$ be smooth mappings between manifolds without boundary. If $f_1 \pitchfork f_2$ then the topological pullback $X_1 \times_{(Y,f_1,f_2)} X_2 = \{(x_1,x_2) \in X_1 \times X_2 : f_1(x_1) = = f_2(x_2)\}$,

$$\begin{tikzcd} X_1 \times_{(Y,f_1,f_2)} X_2 \arrow[r, "pr_2"] \arrow[d, "pr_1"] & X_2 \arrow[d, "f_2"] \\ X_1 \arrow[r, "f_1"] & Y, \end{tikzcd}$$

is a submanifold of $X_1 \times X_2$.

For $X_1 \times_{(Y,f_1,f_2)} X_2 = (f_1 \times f_2)^{-1}(\Delta)$ where $f_1 \times f_2$: $X_1 \times X_2 \to Y \times Y$ and where Δ is the diagonal of $Y \times Y$, and we have $(f_1 \times f_2) \pitchfork \Delta$ iff $f_1 \pitchfork f_2$.

All those results break down if we consider manifolds with corners.

Example: Let $f: \mathbb{R}^n \to \mathbb{R}$ be a smooth mapping such that $f^{-1}(0)$ is very bad, a Cantor-like set in $\mathbb{R}^n$, say (any closed subset of $\mathbb{R}^n$ is of the form $f^{-1}(0)$ for suitable smooth f by a partition of unity argument). Consider $g: \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$, $g(x,t) = f(x)+t$. Then g is a submersion, but $g^{-1}(0)$ has intersection $f^{-1}(0)$ with $\mathbb{R}^n \times \{0\} = \partial(\mathbb{R}^n \times \mathbb{R}_+)$.

The following lemma will serve as a substitute for results as above.

6.3 Lemma: Let X,Y,Z be manifolds with corners, Z a submanifold with corners of Y. Let $f: X \to Y$ be a smooth mapping and suppose that $f \pitchfork Z$. Let $\tilde{X}$ be a manifold without boundary of the same dimension as X containing X as a submanifold with corners (2.7).

Then $f^{-1}(Z)$, as a subset of $\tilde{X}$, is covered by at most countably many submanifolds without boundary V_j of $\tilde{X}$ of the same codimension as Z, such that for any $x \in f^{-1}(Z)$ we have $T_x f \cdot T_x V_j \subseteq T_{f(x)} Z$ whenever $x \in V_j$.

Proof: Let $x \in X$ with $f(x) \in Z \subseteq Y$. Let (U,u,Q) be a chart for Y centered at $f(x) \in Y$ and making Z to a submanifold with corners, i.e. $u(U\cap Z) = u(U)\cap Q'$, where Q' is a quadrant in $\mathbb{R}^k \subseteq \mathbb{R}^m$, $Q' \subseteq Q$. Here $m = \dim Y$, $k = \dim Z$. Consider the mapping $h = pr_2 \circ u \circ f\colon f^{-1}(U) \subseteq X \to U \to u(U) \subseteq$ $\subseteq Q \subseteq \mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^{m-k} \to \mathbb{R}^{m-k}$. This mapping is defined on an open neighbourhood of x in X. Enlarge h to an open neighbourhood of x in $\tilde{X}$, using the Whitney extension theorem (2.2). h is a submersion at x since $f \pitchfork Z$, so it is a submersion in an open neighbourhood W of x in $\tilde{X}$. Clearly $f^{-1}(Z)\cap W \subseteq h^{-1}(0)\cap W$. Put $h^{-1}(0)\cap W =: V_x$, one of the submanifolds referred to in the lemma. Clearly countably many of these V_x suffice to cover $f^{-1}(Z)$. q.e.d.

This lemma is weak but it suffices to prove the following:

6.4 <u>Lemma: Let</u> X,Y,Z,W <u>be manifolds with corners, let</u> $f\colon Z \to Y$ <u>be a smooth mapping. Let</u> $\varphi\colon W \to C^\infty(X,Y)$ <u>be a mapping. Consider</u> $\Phi\colon W\times X \to Y$, <u>given by</u> $\Phi(w,x) = \varphi(w)(x)$, <u>and assume that</u> Φ <u>is smooth.</u>

<u>If</u> $\Phi \pitchfork f$ <u>then the set</u> $\{w\in W\colon \varphi(w)\pitchfork f\}$ <u>is dense in</u> W (<u>in fact: its complement in</u> W <u>has Lebesgue measure</u> 0).

<u>Proof:</u> Consider $\Phi\times f\colon W\times X\times Z \to Y\times Y$. Since $\Phi\pitchfork f$, $\Phi\times f$ is transversal to the diagonal Δ of $Y\times Y$.

Let $\tilde{W},\tilde{X},\tilde{Z}$ be manifolds without boundary containing W,X,Z as equal dimensional submanifolds with boundary, respectively, using 2.7. By lemma 6.3 there are countably many submanifolds V_i of $\tilde{W}\times\tilde{X}\times\tilde{Y}$ without boundary whose union contains $(\Phi\times f)^{-1}(\Delta) \subseteq W\times X\times Z \subseteq \tilde{W}\times\tilde{X}\times\tilde{Z}$, having the same codimension as Δ, such that

$T_{(w,x,z)}(\Phi\times f).T_{(w,x,z)}V_i \subseteq T_{(\Phi(w,x),f(z))}\Delta$ for $(w,x,z)\in V_i\cap(\Phi\times f)^{-1}(\Delta)$.

Denote by $\pi_i\colon V_i \to \tilde{W}$ the restriction of the projection $pr_1\colon \tilde{W}\times\tilde{X}\times\tilde{Z}\to\tilde{W}$, for all i.

<u>Claim:</u> If $w\in W$ is a regular value (in $\tilde{W}$) for all π_i then $\varphi(w)\pitchfork f$. If this claim is true then we are done, since we can use Sard's theorem (for manifolds without boundary): The complement of the set of regular values of π_i in $\tilde{W}$ has

Lebesgue measure 0, so this is true in W too. The complement of the set of all $w \in W$ which are regular values of each π_i is a countable union of sets of Lebesgue measure 0 then, so is itself of Lebesgue measure 0.

Thus let $w \in W$ be regular for all π_i. If $\dim V_i =: k$ (for all i) $< \dim W$, then $w \notin \pi_i(V_i)$ for all i, so $\varphi(w)(X) \cap f(Z) = \emptyset$, so $\varphi(w) \pitchfork f$.

Suppose that $k \geq \dim W$. Let $y \in Y$. If $y \notin \varphi(w)(X) \cap f(Z)$, then $\varphi(w) \pitchfork f$ at y. So let $y \in \varphi(w)(X) \cap f(Z)$, choose $x \in (\varphi(w))^{-1}(y)$, $z \in f^{-1}(Z)$, so $x \in X$, $z \in Z$. Then $(w,x,z) \in V_i$ for some i and π_i is submersive at (w,x,z) since w is a regular value of π_i. So

$T_{(w,x,z)}\pi_i.(T_{(w,x,z)}V_i) = T_w\tilde{W} = T_wW$. Therefore we have:

$$T_{(w,x,z)}W \times X \times Z = T_{(w,x,z)}\tilde{W} \times \tilde{X} \times \tilde{Z} = T_{(w,x,z)}V_i +$$
$$+ T_{(w,x,z)}(\{w\} \times \tilde{X} \times \{z\}) + T_{(w,x,z)}(\{(w,x)\} \times \tilde{Z}) =$$
$$= T_{(w,x,z)}V_i + T_{(w,x,z)}(\{w\} \times X \times \{z\}) + T_{(w,x,z)}(\{(w,x)\} \times Z).$$

Now apply $T_{(w,x,z)}(\Phi \times f)$ to this equation:

$$T_{(w,x,z)}(\Phi \times f)(T_{(w,x,z)}(W \times X \times Z)) =$$
$$= T_{(w,x,z)}(\Phi \times f)(T_{(w,x,z)}V_i) +$$
$$+ T_{(w,x,z)}(\Phi \times f)(T_{(w,x,z)}(\{w\} \times X \times \{z\})) +$$
$$+ T_{(w,x,z)}(\Phi \times f)(T_{(w,x,z)}(\{(w,x)\} \times Z)) \subseteq T_{(y,y)}\Delta +$$
$$+ (T_x\varphi(w).T_xX) \times 0 + 0 \times (T_zf.T_zZ)$$ by the choice of V_i.

By hypothesis $(\Phi \times f) \pitchfork \Delta$, so

$$T_{(y,y)}(Y \times Y) = T_{(y,y)}\Delta + T_{(w,x,z)}(\Phi \times f)(T_{(w,x,z)}W \times X \times Z)$$

and so $T_{(y,y)}(Y \times Y) = T_{(y,y)}\Delta + (T_x\varphi(w).T_xX) \times 0 +$
$+ 0 \times (T_zf.T_zZ) = T_{(y,y)}\Delta + T_{(x,z)}(\varphi(w) \times f).T_{(x,z)}(X \times Z)$.

But this means $(\varphi(w) \times f) \pitchfork \Delta$ at (y,y) and in turn $\varphi(w) \pitchfork f$ at y. q.e.d.

6.5 <u>Lemma:</u> Let X,Y,Z be manifolds with corners, let $f: Z \to Y$ be a proper smooth mapping. Then the set $\{g \in C^\infty(X,Y): g \pitchfork f\}$ is WO^1 open in $C^\infty(X,Y)$.

<u>Proof:</u> Let $g \in C^\infty(X,Y)$, $g \pitchfork f$. Let (A_i) be a countable

locally finite compact cover of Y such that the family of open interiours $(A_i{}^o)$ is still a cover. Put $B_i := f^{-1}(A_i)$. Then each B_i is compact since f is proper (in fact: (B_i) is a locally finite cover, but we will not need this). The family $(g^{-1}(A_i{}^o))$ is an open cover of X. Let (C_α) be a locally finite compact refinement of the cover $(g^{-1}(A_i{}^o))$, let $i(\alpha)$ denote the refinement mapping, so $g(C_\alpha) \subseteq A_{i(\alpha)}{}^o$.

Now let $x \in C_\alpha$, $z \in B_{i(\alpha)}$. Then the following assertion holds:

Either $g(x) \neq f(z)$, or $g(x) = f(z)$ and the linear mapping $T_xg + T_zf: T_xX \times T_zZ \to T_{g(x)}Y$ has rank equal to dim Y.

This statement remains true for all $x' \in U_x \cap C_\alpha$, $z' \in V_z$ and $g' \in N(C_\alpha, 1, \epsilon_{xz}) = \{h \in C^\infty(X,Y): d_1(j^1h(x'), j^1g(x')) < \epsilon_{xz}$ for all $x' \in C_\alpha\}$, where U_x is an open neighbourhood of x in X, V_z is an open neighbourhood of z in Z, d_1 is a compatible metric on $J^1(X,Y)$ and $\epsilon_{x,z}$ is a constant > 0. This follows since the rank of a matrix is an upper semicontinuous function of the matrix.

Cover the compact set C_α by finitely many U_x, say $U_{x_1}, \dots, U_{x_n}$, cover the compact set $B_{i(\alpha)}$ by $V_{z_1}, \dots, V_{z_m}$ and put $\epsilon_\alpha = \min\{\epsilon_{x_i, z_j}\}$.

Let $N(C_\alpha, 1, \epsilon_\alpha) = \{h \in C^\infty(X,Y): d_1(j^1h(x), j^1g(x)) < \epsilon_\alpha$ for all $x \in C_\alpha\}$. If $g' \in N(C_\alpha, 1, \epsilon_\alpha)$ then $g' \pitchfork f$ over $C_\alpha \times B_{i(\alpha)}$ by construction.

Now let $\epsilon: X \to]0, \infty[$ be a continuous positive function such that $\epsilon(x) < \epsilon_\alpha$ for $x \in C_\alpha$ for all α. Such a function exists since (C_α) is locally finite. Put $\mathfrak{B} = \{h \in C^\infty(X,Y): d_1(j^1h(x), j^1g(x)) < \epsilon(x)$ for all $x\} \cap$ $\cap\ \{h \in C^\infty(X,Y): h(C_\alpha) \subseteq A^o_{i(\alpha)}$ for all $\alpha\}$. Then $\mathfrak{B}$ is WO^1 open and $g \in \mathfrak{B}$. We claim that any $g' \in \mathfrak{B}$ is transversal to f. Let $g' \in \mathfrak{B}$ and take $x \in X$. Then $x \in C_\alpha$ for some α.

If $z \notin B_{i(\alpha)}$ then $f(z) \notin A_{i(\alpha)}$ but $g'(x) \in A^o_{i(\alpha)}$, so $g'(x) \neq f(z)$. If $z \in B_{i(\alpha)}$ then $g' \pitchfork f$ over (x,z) since $g' \pitchfork f$ over $C_\alpha \times B_{i(\alpha)}$ (for $g' \in N(C_\alpha, 1, \epsilon_\alpha)$ for all α). So $g' \pitchfork f$ over $\{x\} \times Z$. Since x was arbitrary, $g' \pitchfork f$. q.e.d.

6.6 *Lemma: Let* X, Y, Z *be manifolds with corners, let* $f: Z \to Y$ *be a smooth mapping. The set* $\{g \in C^\infty(X,Y): g \pitchfork f$ *over* $A \times B\}$ *is* WO^1 *open in* $C^\infty(X,Y)$ *if* $A \subseteq X$ *and* $B \subseteq Z$ *are compact sets.*

Proof: This is contained in the proof of the lemma 6.5 above: Put $A = C_\alpha$, $B = B_{i(\alpha)}$ and construct $N(C_\alpha, 1, \varepsilon_\alpha)$ as in the foregoing proof, then this is a WO^1 open neighbourhood of g consisting entirely of mappings transversal to f over $A \times B$. It is even CO^1 open. q.e.d.

6.7 *Corollary: Let* X, Y, Z *be manifolds with corners, let* $f: Z \to J^k(X,Y)$ *be a proper smooth mapping. Then the set* $\{g \in C^\infty(X,Y): j^k g \pitchfork f\}$ *is* WO^{k+1} *open in* $C^\infty(X,Y)$.

Proof: $j^k: C^\infty(X,Y) \to C^\infty(X, J^k(X,Y))$ is continuous from the WO^{k+1} to the WO^1 topology by 7.1 below. But then $\{g \in C^\infty(X,Y): j^k g \pitchfork f\} = (j^k)^{-1}(\{h \in C^\infty(X, J^k(X,Y)): h \pitchfork f\})$ is WO^{k+1} open by 6.5. q.e.d.

6.8 *Theorem (Thom's transversality theorem):*
Let X, Y, Z *be manifolds with corners with* $f: Z \to J^k(X,Y)$ *a smooth mapping. Then the set* $\{g \in C^\infty(X,Y): j^k g \pitchfork f\}$ *is a residual subset of* $C^\infty(X,Y)$ *for the* WO^∞ *and the* $\mathfrak{D}$*-topology. Hence it is dense.*

Proof: We have to show that the set $\{g \in C^\infty(X,Y): j^k g \pitchfork f\}$ can be represented as a countable intersection of open dense subsets. For that end choose the following data:

1. A countable cover (A_i) of X by compact sets, each A_i contained in U_i, where (U_i, u_i, Q_i) is an atlas for X.
2. A countable cover (B_j) of Y by compact sets, each B_j contained in V_j, where (V_j, v_j, Q_j') is an atlas for Y.
3. A countable cover (C_n) of Z by compact subsets.

It suffices to show that each subset $\{g \in C^\infty(X,Y): j^k g \pitchfork f$ over $(A_i \cap g^{-1}(B_j)) \times C_n\}$ is open and dense in the two topologies considered, for their intersection is just $\{g: j^k g \pitchfork f\}$.

So fix one of these sets and forget the indices. For

$g \in C^\infty(X,Y)$ we have $j^k g \pitchfork f$ over $(A \cap g^{-1}(B)) \times C$ iff $j^k g \pitchfork f$ over $A \times (C \cap (\omega \circ f)^{-1}(B))$; here $\omega: J^k(X,Y) \to Y$ is the target projection, and the equivalence holds since: $x \in A \cap g^{-1}(B)$, $z \in C$ with $j^k g(x) = f(z)$ iff $x \in A$, $z \in C \cap$ $\cap\ (\omega \circ f)^{-1}(B)$ with $j^k g(x) = f(z)$. But then $\{g: j^k g \pitchfork f$ over $(A \cap g^{-1}(B)) \times C\} = \{g: j^k g \pitchfork f$ over $A \times (C \cap (\omega \circ f)^{-1}(B))\} =$ $= (j^k)^{-1}(\{h \in C^\infty(X, J^k(X,Y)): h \pitchfork f$ over $A \times (C \cap (\omega \circ f)^{-1}(B))\})$ is WO^{k+1} open by lemma 6.6 and continuity of j^k. So this set is WO^∞- and $\mathfrak{D}$-open.

Thus it remains to prove density. Let $h \in C^\infty(X,Y)$. We will show that we can approximate h by functions in $\{g: j^k g \pitchfork f$ over $(A \cap g^{-1}(B)) \times C\}$.

Put $D = A \cap h^{-1}(B)$, a compact set in the open chart neighbourhood U. Put $U' = h^{-1}(V) \cap U$, an open neighbourhood of D in U. Let $\lambda: u(U') \to \mathbb{R}_+$ be a non-negative smooth function with compact support in $u(U')$ such that $\lambda = 1$ on an open neighbourhood $u(U'')$ of $u(D)$.

Consider the mapping $h' = v \circ h \circ u^{-1} | u(U'): u(U') \to v(V)$. Consider the space $\mathbb{R}^m \times P^k(n,m)$ ($n = \dim X$, $m = \dim Y$) of all polynomial mappings $\mathbb{R}^n \to \mathbb{R}^m$ of degree $\leq k$, and let $E(Q,Q')$ be the subset consisting of all $\sigma \in \mathbb{R}^m \times P^k(n,m)$ with $\sigma(Q) \subseteq Q'$. We claim that $E(Q,Q')$ is a quadrant. This is seen as follows. Choose a basis for $\mathbb{R}^n$ such that Q has the form $\mathbb{R}^{n-l} \times (\mathbb{R}_+)^l$ ($l =$ index of Q) in coordinates with respect to this basis. Likewise for $\mathbb{R}^m$ and Q'. If $\sigma: \mathbb{R}^n \to \mathbb{R}^m$ is a polynomial mapping then $\sigma(Q) \subseteq Q'$ iff each monomial of each coordinate function of σ with its coefficient maps Q into Q', so $\sigma \in E(Q,Q')$ if certain of its coefficients are non-negative. For $\sigma(Q) \subseteq Q'$ iff certain of its coordinate polynomials are non-negative on Q, and a polynomial on $\mathbb{R}^n$ is non-negative on Q iff each of its monomials (with coefficient) is non-negative on Q. Now let V' be an open set in $\mathbb{R}^m$ such that $v(V) = V' \cap Q'$. Consider the set W' consisting of all σ in $\mathbb{R}^m \times P^k(n,m)$ with $(h' + \lambda\sigma)(\operatorname{supp} \lambda) \subseteq V'$. Then W' is an open neighbourhood of 0 in $\mathbb{R}^m \times P^k(n,m)$ (in the CO-topology, but this coincides with the usual topology). Put $W = W' \cap E(Q,Q')$, an open subset of a quadrant, so a manifold with corners. Note that $0 \in W$.

Now define the mapping Φ: $W \times X \to Y$ by $\Phi(\sigma,x) = v^{-1} \circ (h' + \lambda\sigma)(u(x))$ for $x \in U'$, $\Phi(\sigma,x) = h(x)$ otherwise. Then Φ is a smooth mapping. Write Φ_σ for the mapping $x \to \Phi(\sigma,x)$, $\sigma \in W$. By construction the mapping $(\sigma,x) \to j^k(\Phi_\sigma)(x)$, $W \times X \to J^k(X,Y)$, has the property that its restriction $\tilde{\Phi}$ to $W \times U''$ is a submersion. So clearly $\tilde{\Phi} \pitchfork f$ and by lemma 6.4 the set $\{\sigma \in W: \tilde{\Phi}_\sigma \pitchfork f\} = \{\sigma \in W: j^k(\Phi_\sigma) \pitchfork f$ over $U'' \times Z\}$ is dense in W, hence it contains a sequence σ_n converging to 0 in W. But then the sequence $g_n := \Phi_{\sigma_n}$ converges to h in WO^∞ and $\mathfrak{D}$ since g_n equals h off the compact set $u^{-1}(\text{supp } \lambda)$ and g_n converges to h "uniformly in all derivatives" on this compact set (4.4.3, 4.7.7).

By construction we have $j^k g_n \pitchfork f$ over $U'' \times C$. But we need $j^k g_n \pitchfork f$ over $(A \cap g_n^{-1}(B)) \times C$.

<u>Claim:</u> There is an N such that $A \cap g_n^{-1}(B) \subseteq U''$ for all $n \geq N$.

If this claim is true then we are done, since all g_n, $n \geq N$, are in the set $\{g: j^k g \pitchfork f$ over $(A \cap g^{-1}(B)) \times C\}$ and the g_n approximate h. So this set is dense.

Now we prove the claim: we have $h^{-1}(B) \cap A \subseteq U''$ by construction. This is in turn equivalent to: $A \backslash (h|A)^{-1}(B) \supseteq A \backslash U''$, and to $(h|A)^{-1}(Y \backslash B) \supseteq A \backslash U''$; thus we have $Y \backslash B \supseteq (h|A)(h|A)^{-1}(Y \backslash B) \supseteq (h|A)(A \backslash U'')$.

Now $A \backslash U''$ is compact and $Y \backslash B$ is open, $g_n \to h$ in the CO-topology too, so there is some N with $g_n(A \backslash U'') \subseteq Y \backslash B$ for all $n \geq N$. But then we have $A \backslash U'' \subseteq (g_n|A)^{-1} g_n(A \backslash U'') \subseteq (g_n|A)^{-1}(Y \backslash B)$ which is in turn equivalent to $A \backslash (g_n|A)^{-1}(B) \supseteq A \backslash U''$, and to $g_n^{-1}(B) \cap A \subseteq U''$ for $n \geq N$, what we wanted. q.e.d.

6.9 <u>Corollary:</u> (<u>Elementary transversality theorem</u>). <u>Let</u> X,Y,Z <u>be manifolds with corners with</u> f: $Z \to Y$ <u>a smooth mapping</u>. <u>Then the set</u> $\{g \in C^\infty(X,Y): g \pitchfork f\}$ <u>is a</u>

residual subset of $C^\infty(X,Y)$ *in the* WO^∞ - *and the* $\mathfrak{D}$-*topology. If* f *is proper, then this set is* WO^1-*open too.*

Proof: We have $J^0(X,Y) = X\times Y$ and $j^0 g\colon X \to J^0(X,Y)$ equals the graph-mapping $\Gamma_g\colon X \to X\times Y$, $x \to (x,g(x))$. It is easily checked that for $g\in C^\infty(X,Y)$ we have $g \pitchfork f$ iff $\Gamma_g \pitchfork (\mathrm{Id}_X\times f)$, i.e. $j^0 g \pitchfork (\mathrm{Id}_X \times f)$, where $\mathrm{Id}_X\times f\colon X\times Z \to X\times Y$. Therefore $\{g\in C^\infty(X,Y)\colon g\pitchfork f\} = \{g\in C^\infty(X,Y)\colon j^0 g\pitchfork(\mathrm{Id}_X\times f)\}$ and the corollary follows from 6.8 and 6.5. q.e.d.

6.10 The rest of this section is devoted to a generalization of transversality in jet spaces to transversality in multi jet spaces and some applications.

Let X,Y be manifolds with corners. Define $X^s = X\times X\times \ldots \times X$ (s times) and $X^{(s)} = \{(x_1,\ldots,x_s)\in X^s\colon x_i \neq x_j \text{ for } i\neq j\}$, an open submanifold of X^s. Let $\alpha\colon J^k(X,Y)\to X$ be the source projection, let $\alpha^s\colon J^k(X,Y)^s \to X^s$ denote the s-fold product mapping. Then ${}^{(s)}J^k(X,Y) := (\alpha^s)^{-1}(X^{(s)})$ is called the s-fold k-jet bündle. A multijet bundle is some s-fold k-jet bundle. ${}^{(s)}J^k(X,Y)$ is a fibre bundle over $X^{(s)}\times Y^s$.

Now let $f\colon X\to Y$ be a smooth mapping. Define ${}^{(s)}j^k f\colon X^{(s)} \to {}^{(s)}J^k(X,Y)$ in the obvious way, i.e. ${}^{(s)}j^k f(x_1,\ldots,x_s) = (j^k f(x_1),\ldots,j^k f(x_s))$.

6.11 *Lemma: Let* X,Y,Z *be manifolds with corners, let* $f\colon Z\to {}^{(s)}J^k(X,Y)$ *be a smooth mapping. Let* $A\subseteq X^{(s)}$, $B\subseteq Z$ *be compact subsets. Then the set* $\{g\in C^\infty(X,Y)\colon {}^{(s)}j^k g \pitchfork f$ *over* $A\times B\}$ *is* WO^{k+1}-*open in* $C^\infty(X,Y)$.

Proof: Let $g\in C^\infty(X,Y)$, ${}^{(s)}j^k g\pitchfork f$ over $A\times B$. Let $x = (x_1,\ldots,x_s)\in A$, $z\in B$. Then the following holds: Either ${}^{(s)}j^k g(x) \neq f(z)$, or ${}^{(s)}j^k g(x) = (j^k g(x_1),\ldots, j^k g(x_s)) = (f_1(z),\ldots,f_s(z)) = f(z)$ and each linear mapping $T_{x_i}(j^k g) + T_z f_i\colon T_{x_i}X\times T_z Z \to T_{f_i(z)}J^k(X,Y)$ has rank equal to $\dim J^k(X,Y)$.

This statement remains true for all $x' = (x_1', \dots, x_s') \in (U_{x_1} \times \dots \times U_{x_s}) \cap X^{(s)} \cap A$, $z' \in V_z$, and $g' \in \{h \in C^\infty(X,Y): d_{k+1}(j^{k+1}h(a), j^{k+1}g(a)) < \varepsilon_{x,z}$ for all $a \in \tilde{A}\}$, where d_{k+1} is a compatible metric on $J^{k+1}(X,Y)$, $\tilde{A} = \bigcup_{i=1}^{s} pr_i(A) \subseteq X$ is a compact set, $pr_i: X^{(s)} \to X$ being the i-th projection, and where $U_{x_1}, \dots, U_{x_s}$ are open neighbourhoods of $x_1, \dots, x_s$ in X respectively and V_z is an open neighbourhood of z in Z. This follows from the upper semicontinuity of the rank of a matrix. Now cover the compact set A by finitely many of these sets $U_{x_1} \times \dots \times U_{x_s}$, cover the compact set B by finitely many of the V_z's and let ε be the minimum of all the $\varepsilon_{x,z}$ corresponding to these covers. Put $\mathfrak{B} = \{h \in C^\infty(X,Y): d_{k+1}(j^{k+1}g(x), j^{k+1}h(x)) < \varepsilon$ for all $x \in \tilde{A}\}$, then for $g' \in \mathfrak{B}$ we have ${}^{(s)}j^k g' \pitchfork f$ over $A \times B$. q.e.d.

6.12 <u>Theorem</u> (<u>Multijet transversality theorem</u>). <u>Let</u> X,Y,Z <u>be smooth manifolds with corners with</u> $f: Z \to {}^{(s)}J^k(X,Y)$ <u>a smooth mapping</u>. <u>Then the set</u> $\{g \in C^\infty(X,Y): {}^{(s)}j^k g \pitchfork f\}$ <u>is a residual subset of</u> $C^\infty(X,Y)$ <u>in the</u> WO^∞ - <u>and the</u> $\mathfrak{D}$-<u>topology</u>. <u>Hence it is dense</u>.

<u>Proof:</u> Again we have to show that the set $\{g: {}^{(s)}j^k g \pitchfork f\}$ can be represented as a countable intersection of open dense subsets. The method of proof is the same as in 6.8. Choose the following data:

1. A countable cover of $X^{(s)}$ by compact sets $(A_{i1} \times \dots \times A_{is})_{i \in \mathbb{N}}$, where each A_{ij} is compact in X. Note that $A_{ij} \cap A_{ik} = \emptyset$ if $1 \leq j < k \leq s$ by the definition of $X^{(s)}$. Suppose furthermore that each A_{ij} is contained in some open set U_{ij} in X, where again $U_{ij} \cap U_{ik} = \emptyset$ for $j \neq k$, and where (U_{ij}, u_{ij}, Q_{ij}) is an atlas for X.

2. A countable cover $(B_{i1} \times \dots \times B_{is})_{i \in \mathbb{N}}$ of Y^s by compact sets, each B_{ij} contained in some V_{ij} where $(V_{ij}, v_{ij}, Q'_{ij})$ is an atlas for Y.

3. A countable cover $(C_i)_{i\in\mathbb{N}}$ of Z by compact subsets.

It suffices to show that each set $\{g \in C^\infty(X,Y): {}^{(s)}j^k g \pitchfork f$ over $((A_{i1} \cap g^{-1}(B_{j1})) \times \dots \times (A_{is} \cap g^{-1}(B_{js})) \times C_n\}$ (for any $i,j,n \in \mathbb{N}$) is open and dense in the two topologies, for their intersection is the set of all g such that ${}^{(s)}j^k g \pitchfork f$ over $\bigcup\limits_{i,j,n} ((A_{i1} \cap g^{-1}(B_{j1})) \times \dots \times ((A_{is} \cap g^{-1}(B_{js})) \times C_n = X^{(s)} \times Z$.

So fix one of these sets and forget the first indices, for convenience's sake, so $A := A_1 \times \dots \times A_s$ is compact in $X^{(s)}$, $B := B_1 \times \dots \times B_s$ is compact in Y^s, C is compact in Z.

For $g \in C^\infty(X,Y)$ we have ${}^{(s)}j^k g \pitchfork f$ over $(A \cap (g^s)^{-1}(B)) \times C$ iff ${}^{(s)}j^k g \pitchfork f$ over $A \times (C \cap (\omega^s \circ f)^{-1}(B))$; here $g^s: X^s \to Y^s$ is the s-fold product mapping, $\omega^s: {}^{(s)}J^k(X,Y) \to J^k(X,Y)^s \to Y^s$ is the s-fold target projection. This follows from the argument used in the proof of 6.8. But then the set $\{g: {}^{(s)}j^k g \pitchfork f$ over $(A \cap (g^s)^{-1}(B)) \times C\} = \{g: {}^{(s)}j^k g \pitchfork f$ over $A \times (C \cap (\omega^s \circ f)^{-1}(B))\}$ is WO^{k+1}-open in $C^\infty(X,Y)$ by lemma 6.11.

So it remains to prove density. Let $h \in C^\infty(X,Y)$. We will show that we can approximate h by functions in $\{g: {}^{(s)}j^k g \pitchfork f$ over $(A \cap (g^s)^{-1}(B)) \times C\}$. Now, for $1 \le j \le s$, put $D_j = A_j \cap h^{-1}(B_j)$, a compact set in U_j. Put $U_j' = h^{-1}(V_j) \cap U_j$, an open neighbourhood of D_j in U_j. Let $\lambda_j: u_j(U_j) \to \mathbb{R}$ be a non-negative smooth function with compact support in the set $u_j(U_j')$ which is open in $Q_j \subseteq \mathbb{R}^n$ $(n = \dim X)$, such that $\lambda_j = 1$ on an open neighbourhood $u_j(U_j'')$ of $u_j(D_j)$ in $u_j(U_j')$.

Consider the mapping $h_j = v_j \circ h \circ u_j^{-1} | u_j(U_j'): u_j(U_j') \to v_j(V_j)$.

Consider $\mathbb{R}^m \times P^k(n,m)$ $(n = \dim X,\ m = \dim Y)$ of all polynomial mappings $\mathbb{R}^n \to \mathbb{R}^m$ of degree $\le k$ and let $E(Q_j, Q_j')$ be the subset of all $\sigma \in \mathbb{R}^m \times P^k(n,m)$ such that $\sigma(Q_j) \subseteq Q_j'$. Then $E(Q_j, Q_j')$ is a quadrant, by the argument in 6.8. Let W_j' be the set of all $\sigma \in \mathbb{R}^m \times P^k(n,m)$ such that $(h_j + \lambda_j \sigma)(\operatorname{supp} \lambda_j) \subseteq V_j'$, where V_j' is an open subset of $\mathbb{R}^m$ with $v_j(V_j) = V_j' \cap Q_j'$. Then W_j' is an open neighbourhood of 0 in $\mathbb{R}^m \times P^k(n,m)$. Let $W_j = W_j' \cap E(Q_j,Q_j')$, a submanifold with corners of $\mathbb{R}^m \times P^k(n,m)$ containing 0. Put $W = W_1 \times W_2 \times \ldots \times W_s$ and define Φ: $W \times X \to Y$ by $\Phi(\sigma_1,\ldots,\sigma_s,x) = v_j^{-1} \circ (h_j + \lambda_j\sigma_j)(u_j(x))$ if $x \in U_j$, $\Phi(\sigma,x) = \Phi(\sigma_1,\ldots,\sigma_s,x) = h(x)$ otherwise. Φ is a smooth mapping since the open sets $U_1,\ldots,U_j$ of X are pairwise disjoint. Write Φ_σ for the mapping $X \to Y$, $x \to \Phi(\sigma,x) = = \Phi(\sigma_1,\ldots,\sigma_s,x)$ for $\sigma \in W$. By construction the mapping $(\sigma,x) \to {}^{(s)}j^k(\Phi_\sigma)(x) = (j^k(\Phi_\sigma)(x_1),\ldots,j^k(\Phi_\sigma)(x_s))$, $W \times X^{(s)} \to {}^{(s)}J^k(X,Y)$, is smooth and has the property that its restriction $\tilde{\Phi}$ to $\dot{W} \times U'' = W \times (U_1'' \times \ldots \times U_s'')$ is a submersion (on these sets all the λ_j's equal 1). So clearly $\tilde{\Phi} \pitchfork f$ and by lemma 6.4 again the set $\{\sigma = (\sigma_1,\ldots,\sigma_s)\colon \tilde{\Phi}_\sigma \pitchfork f\} = \{\sigma \in W\colon {}^{(s)}j^k(\Phi_\sigma) \pitchfork f$ over $U'' \times Z\}$ is dense in W, hence it contains a sequence $\sigma^{(n)} = = (\sigma_1^{(n)},\ldots,\sigma_s^{(n)})$ converging to $0 \in W$. But then the sequence $g_n := \Phi_{\sigma^{(n)}}$ in $C^\infty(X,Y)$ converges to h in the WO^∞-topology and in the $\mathfrak{D}$-topology since g_n equals h off the compact set $\bigcup_{j=1}^{s} u_j^{-1}(\operatorname{supp}\lambda_j)$ and g_n converges to h "uniformly in all derivatives" on this compact set. By construction we have ${}^{(s)}j^k g_n \pitchfork f$ over $U'' \times Z$, so over $U'' \times C$. But again we need ${}^{(s)}j^k g_n \pitchfork f$ over $(A \cap (g_n{}^s)^{-1}(B)) \times C$. That this is true for n sufficiently large follows from the argument used at the end of the proof of 6.8. q.e.d.

6.13 <u>Corollary:</u> <u>Let</u> X,Y <u>be manifolds with corners</u>. <u>Then the set</u> Imm(X,Y) <u>of immersions is open and dense in</u> $C^\infty(X,Y)$ <u>for the</u> WO^∞-<u>topology and for the</u> $\mathfrak{D}$-<u>topology if</u> $\dim Y \geq 2 \dim X$.

<u>Proof:</u> Imm(X,Y) is always open by 5.1. It remains to show that it is dense. Let $n = \dim X$, $m = \dim Y$. Let $R_k = R_k(X,Y) \subseteq J^1(X,Y)$ be the subsets of 1-jets of rank k. Locally, in $J^1(u(U),v(V)) = u(U) \times v(V) \times L(\mathbb{R}^n,\mathbb{R}^m)$ ($(U,u,Q),(V,v,Q')$ being charts of X,Y resp.) we have $R_k(u(U),v(V)) = J^1(u^{-1},v)(R_k \cap J^k(U,V)) = u(U) \times v(V) \times L_k(n,m)$,

where $L_k(n,m)$ is the space of linear mappings of rank k from $\mathbb{R}^n$ to $\mathbb{R}^m$, a submanifold of $L(\mathbb{R}^n,\mathbb{R}^m)$ of dimension $k(n+m)-k^2$ (see below); so $R^k(X,Y)$ is a sub fibre bundle of $J^1(X,Y)$, a submanifold of dimension $k(n+m)-k^2+n+m$.

$f \in C^\infty(X,Y)$ is an immersion iff $j^1 f$ misses $R_0, R_1, \dots, R_{n-1}$. Let us suppose that f is transversal to each R_k, $0 \le k \le n-1$. Then f misses these sets if $\dim R_k + \dim X < \dim J^1(X,Y)$ for $0 \le k \le n-1$, i.e. if $\dim R_k + \dim X \le \dim R_{n-1} + \dim X < \dim J^1(X,Y)$, i.e. $[(n-1)(m+n) - (n-1)^2] + n < n+m+nm$ or $m \ge 2n$. q.e.d.

It remains to show the

<u>Sublemma:</u> $L_k(n,m)$ is a submanifold of $L(n,m)$ of dimension $k(n+m)-k^2$.

<u>Proof:</u> Let $E \in L_k(n,m)$ be given. Choose bases for $\mathbb{R}^n, \mathbb{R}^m$ such that the matrix of E has the form $\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$ where I_k is the $k \times k$ unit matrix. Choose an element near E in $L(\mathbb{R}^n,\mathbb{R}^m)$ with matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, near enough such that $\det A \neq 0$. Then

$$\operatorname{rank}\begin{pmatrix} A & B \\ C & B \end{pmatrix} = \operatorname{rank}\left[\begin{pmatrix} A^{-1} & 0 \\ -CA^{-1} & I_{m-k} \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right] =$$

$$= \operatorname{rank}\begin{pmatrix} I_k & A^{-1}B \\ 0 & -CA^{-1}B+D \end{pmatrix} .$$

So $\operatorname{rank}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = k$ iff $D = CA^{-1}B$. So the dimension of $L_k(n,m)$ is just the number of free entries in A, B, C, $\dim L_k(n,m) = k^2 + k(n-k) + k(m-k) = k(n+m) - k^2$. q.e.d.

6.14 <u>Corollary:</u> <u>Let</u> X, Y <u>be manifolds with corners such that</u> $\dim Y \ge 2 \dim X + 1$. <u>Then the set of injective immersions</u> $X \to Y$ <u>is dense in</u> $C^\infty(X,Y)$ <u>in the</u> WO^∞ - <u>and the</u> $\mathfrak{D}$<u>-topology</u>.

<u>Proof:</u> Since the set of immersions is open and dense it remains to show that the set of injective mappings is residual. ${}^{(2)}J^0(X,Y) = X^{(2)} \times Y^2$ and $f \in C^\infty(X,Y)$ is injective iff ${}^{(2)}j^0 f \colon X^{(2)} \to {}^{(2)}J^0(X,Y)$ does not meet $X^{(2)} \times \Delta_Y$, Δ_Y being the diagonal in $Y \times Y$. Now codim $X^{(2)} \times \Delta_Y = \operatorname{codim} \Delta_Y = \dim Y$. So ${}^{(2)}j^0 f \pitchfork X^{(2)} \times \Delta_Y$ implies f

injective if dim $Y > \dim X^{(2)} = 2 \dim X$. q.e.d.

7 Continuity of certain canonical mappings

We will use only C^∞-mappings and only the $\mathfrak{D}$- and $(\mathfrak{FD})$-topology in the next chapters. Therefore we will prove continuity of composition etc. only for these topologies. The proofs for the WO^k-topologies are often earier and can be found in the literature: MATHER (1969), GOLUBITSKY-GUILLEMIN (1973) X,Y,Z designate manifolds with corners, if not stated explicitly otherwise.

7.1 Proposition: For any $k \geq 0$ the mapping $j^k: C^\infty(X,Y) \to \to C^\infty(X, J^k(X,Y))$ is continuous for the $\mathfrak{D}$- and the $(\mathfrak{FD})$-topology (on both spaces).

Remark: $T: C^\infty(X,Y) \to C^\infty(TX,TY)$ is not continuous: We use 4.2.3. Let $t \to f_t$ be a continuous curve in $C^\infty(X,Y)$, $t \in [0,1]$, then $f_t, f_{t'}$ differ only on a compact set $K \subseteq X$, but $Tf_t, Tf_{t'}$ differ on whole fibres of TX, so $t \to Tf_t$ is no continuous curve anymore.

Proof: For any $l \in \mathbb{N}$ we consider the mapping $\alpha_{k,l}: J^{k+l}(X,Y) \to J^l(X, J^k(X,Y))$ defined as follows: for $\sigma \in J^{k+l}(X,Y)$ with $\alpha(\sigma) = x$ choose a representant $f \in C^\infty(\tilde{U},\tilde{V})$, $\tilde{U}$ an open neighbourhood of x in a manifold without boundary containing X as a submanifold with corners of the same dimension, $\tilde{V}$ an open neighbourhood of $\omega(\sigma)$ in a like manifold containing Y. Then put $\alpha_{k,l}(\sigma) = = j^l(j^k f)(x)$. In local charts $J^{k+l}(u(U),v(V)) = = u(U) \times v(V) \times P^{k+l}(n,m)$ $(n = \dim X,\ m = \dim Y)$, $J^l(u(U), J^k(u(U),v(V))) = u(U) \times (u(U) \times v(V) \times P^k(n,m)) \times \times P^l(n, m(\binom{n+k}{k} - 1))$ $(\dim P^k(n,m) = m(\binom{n+k}{k} - 1))$; this

mapping looks as follows:
$(x,y,\overline{\sigma}) \to (x,(x,y,j^k\overline{\sigma}(0) = \pi_k^{k+1}\overline{\sigma}),\ j^1(j^k\overline{\sigma})(0))$ (cf. 2.11, 2.12).

It is clear that this gives a smooth well defined mapping, in fact, an embedding.

Let now $M'(L,U)$ be one of the open basic sets for $\mathfrak{D}$ from 4.7.2. in $C^\infty(X,J^k(X,Y))$, i.e. $U=(U_n)$, U_n open in $J^n(X,J^k(X,Y))$, $L=(L_n)$ a locally finite closed family in X and $M'(L,U) = \{f \in C^\infty(X,J^k(X,Y)) : j^n f(L_n) \subseteq U_n\}$. Put $L_n{}' = \emptyset$ for $n=0,\dots,k-1$, $L_{k+l}{}' = L_l$, $l=0,1,\dots, L'=(L_n{}')$, put $U_n{}' = \emptyset$, $n=0,\dots,k-1$, $U_{k+l}{}' = (\alpha_{k,l})^{-1}(U_l)$, $l=0,1,2,\dots,(U_n{}')=U'$. Then $M'(L',U')$ is a basic open set in $(C^\infty(X,Y),\mathfrak{D})$ and clearly $(j^k)^{-1}(M'(L,U)) = M'(L',U')$. So $j^k\colon (C^\infty(X,Y),\mathfrak{D}) \to (C^\infty(X,J^k(X,Y)),\mathfrak{D})$ is continuous. For $(F\mathfrak{D})$ the result follows: since $f \sim g$ iff $j^k f \sim j^k g$. q.e.d.

7.2 Continuity of a certain restriction of the composition is our next aim. But first some preparations.

Let A,B,P be topological Hausdorff spaces, let $\pi_A\colon A \to P$, $\pi_B\colon B \to P$ be continuous mappings. Let $A \times_P B = \{(a,b) \in A \times B,\ \pi_A(a) = \pi_B(b)\}$ be the topological pullback, with the topology induced from $A \times B$:

$$\begin{array}{ccc} A \times_P B & \xrightarrow{pr_2} & B \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle \pi_B} \\ A & \xrightarrow{\pi_A} & P \end{array}$$

A Kelley topological space S is a Hausdorff topological space bearing the inductive limit topology with respect to all the embeddings of its compacta. So a subset W in S is closed (open) iff $W \cap K$ is closed (open) in K for each compact set K in S.

Metric spaces are Kelley spaces, as are locally compact spaces. Closed subsets of Kelley spaces are again Kelley spaces, as are topological inductive limits of Kelley spaces. $\mathfrak{D}(\Omega)$ (Ω an open subset of some $\mathbb{R}^n$), the space of smooth functions with compact support on Ω, is <u>not</u> a

Kelley space (if $n \geq 1$), see VALDIVIA (1974).

Lemma: Let A, B, P be Hausdorff topological spaces, P a paracompact Kelley space. Let $\pi_A \colon A \to P$ and $\pi_B \colon B \to P$ be continuous mappings. Let $K \subseteq A$, $L \subseteq B$ be subsets such that $\pi_A | K$ and $\pi_B | L$ are proper (5.2). Let U be an open neighbourhood of $K \times_P L$ in $A \times_P B$. Then there are open neighbourhoods V of K in A and W of L in B such that $K \times_P L \subseteq V \times_P W \subseteq U \subseteq A \times_P B$.

This lemma (with P locally compact and paracompact) is due to MATHER (1969). For the next theorem we need this lemma only in the restricted form. For the proof we need a

Sublemma: Let T be a Hausdorff space, let R be a Kelley space. If $f \colon T \to R$ is continuous and proper, then f is a closed mapping (i.e. closed sets have closed image). If furthermore R is locally compact than T is, too.

Proof of the sublemma: Let $S \subseteq T$ be a closed subset. Let $K \subseteq R$ be a compact subset. Then $f^{-1}(K)$ is compact in T since f is proper. So $S \cap f^{-1}(K)$ is compact, so $f(S) \cap K = f(S \cap f^{-1}(K))$ is compact in R. Since this holds for any compact $K \subseteq R$, $f(S)$ is closed in R.

Now if R is locally compact, then the inverse image of a compact neighbourhood of $f(x)$ is a compact neighbourhood of x in T, so T is locally compact. q.e.d.

Proof of the lemma: $A \times_P B = (\pi_A \times \pi_B)^{-1}$ (diagonal in $P \times P$), so $A \times_P B$ is closed in $A \times B$. So $E := A \times B \setminus A \times_P B$ is open in $A \times B$. For $p \in P$ let $K_p := K \cap (\pi_A)^{-1}(p)$, $L_p := L \cap (\pi_B)^{-1}(p)$. Then $K_p \times L_p = K_p \times_P L_p \subseteq K \times_P L \subseteq U$. By hypothesis K_p and L_p are compact. $U \cup E$ is open in $A \times B$ (U alone is not open). We claim that there are open neighbourhoods V_p of K_p in A and W_p of L_p in B such that $V_p \times W_p \subseteq U \cup E$.

This is seen as follows: For any $(k,l) \in K_p \times L_p$ choose open neighbourhoods $V^{k,l}$ of k in A and $W^{k,l}$ of l in B such that $V^{k,l} \times W^{k,l} \subseteq U \cup E$ (by definition of the product topology). For any fixed k the family $(W^{k,l})_{l \in L_p}$ is an open cover of L_p, so there is a finite subcover

$(W^{k,l_1},\ldots,W^{k,l_m})$. Put $V^k = V^{k,l_1}\cap\ldots\cap V^{k,l_m}$ and $W^k = W^{k,l_1}\cup\ldots\cup W^{k,l_m}$.

Then $V^k\times W^k \subseteq \bigcup_{j=1}^{m} V^{k,l_j}\times W^{k,l_j} \subseteq U\cup E$, and $W^k \supseteq L_p$. $(V^k)_{k\in K_p}$ is an open cover of K_p, so there is a finite subcover $(V^{k_1},\ldots,V^{k_n})$. Put $W_p = W^{k_1}\cap\ldots\cap W^{k_n}$ and $V_p = V^{k_1}\cup\ldots\cup V^{k_n}$, then $W_p \supseteq L_p$, $V_p \subseteq K_p$ and $V_p\times W_p \subseteq \subseteq \bigcup_i V^{k_i}\times W^{k_i} \subseteq U\cup E$. Now we proceed with the proof of the lemma. $\pi_A|K$, $\pi_B|L$ are closed mappings by the sublemma, therefore $\pi_A(K\setminus V_p)$ and $\pi_B(L\setminus W_p)$ are closed in P, so $P_p := P\setminus(\pi_B(L\setminus W_p)\cup\pi_A(K\setminus U_p))$ is open, $p\in P_p$. So $(P_p)_{p\in P}$ is an open cover of P. Since P is paracompact, there exists a locally finite refinement of this cover, which we call (P_α). For α let $p(\alpha)\in P$ such that $P_\alpha \subseteq P_{p(\alpha)}$ (refining mapping). Put
$$V_\alpha = V_{p(\alpha)}\cup(\pi_A)^{-1}(P\setminus P_\alpha)$$
$$W_\alpha = W_{p(\alpha)}\cup(\pi_B)^{-1}(P\setminus P_\alpha),$$
and $V=\bigcap_\alpha V_\alpha$, $W=\bigcap_\alpha W_\alpha$.

We claim, that: $K\subset V$, $L\subset W$, V, W open in A, B resp., $V\times_P W\subset U$.

(1) $K\subset V$: It suffices to show that $K\subset V_\alpha$ for all α, i.e.: if $k\in K$ then $k\in V_{p(\alpha)}$ or $k\in(\pi_A)^{-1}(P\setminus P_\alpha)$. If $k\notin V_{p(\alpha)}$, then $\pi_A(k)\in\pi_A(K\setminus V_{p(\alpha)})$, so $\pi_A(k)\notin P_{p(\alpha)}$ by construction.

(2) $L\subset W$: the same argument.

(3) V is open: Let $v\in V$. (P_α) is locally finite, so there is an open neighbourhood N of $\pi_A(v)$ in P such that $N\cap P_\alpha=\emptyset$ for all α but $\alpha_1,\ldots,\alpha_r$, say. Put $N' = (\pi_A)^{-1}(N)\cap V_{\alpha_1}\cap\ldots\cap V_{\alpha_r}$; this is an open neighbourhood of v. If $\alpha\notin\{\alpha_1,\ldots,\alpha_r\}$, then $N\cap P_\alpha=\emptyset$, so $N'\cap(\pi_A)^{-1}(P_\alpha)=\emptyset$, so $N'\subseteq\pi_A{}^{-1}(P\setminus P_\alpha)\subseteq V_\alpha$.

So $N'\subseteq V_\alpha$ for all α, so $N'\subseteq V$. Thus V is open.

(4) W is open: the same argument.

(5) $V\times_P W\subseteq U$: Let $(v,w)\in V\times_P W$, $p=\pi_A(v)=\pi_B(w)$. There is an α' with $p\in P_{\alpha'}$. Then $v\in V=\bigcap_\alpha V_\alpha\subset V_{\alpha'}$, but

$v \notin (\pi_A)^{-1}(P\setminus P_{\alpha'})$, so $v \in V_{\alpha'} \setminus (\pi_A)^{-1}(P\setminus P_{\alpha'}) \subseteq V_{p(\alpha')}$. In the same way it follows that $w \in W_{p(\alpha')}$. Therefore $(v,w) \in V_{p(\alpha')} \times W_{p(\alpha')} \subset U \cup E$, but $(v,w) \notin E$, so $(v,w) \in U$. Thus $V \times_p W \subseteq U$. q.e.d.

7.3 <u>Theorem: Let</u> X,Y,Z <u>be smooth manifolds with corners. Then composition</u> $\mathrm{Comp}: C^\infty(Y,Z) \times C^\infty_{\mathrm{prop}}(X,Y) \to C^\infty(X,Z)$ <u>is continuous in the</u> $\mathfrak{D}$- <u>and the</u> $(\mathcal{F}\mathfrak{D})$-<u>topology</u>.

For the WO^k-topologies this theorem is due to MATHER (1969).

<u>Proof:</u> Let $(g,f) \in C^\infty(Y,Z) \times C^\infty_{\mathrm{prop}}(X,Y)$, let $M'(L,U)$ be a basic open neighbourhood of $g \circ f$ in $(C^\infty(X,Z),\mathfrak{D})$, as described in 4.7.2., i.e. $L = (L_n)$ is a locally finite closed family in X, $U = (U_n)$ with U_n open in $J^n(X,Y)$, $g \circ f \in M'(L,U) = \{h \in C^\infty(X,Z): j^n h(L_n) \subseteq U_n \text{ for all } n\}$. So we have $j^n(g \circ f)(L_n) \subseteq U_n$. For any $n \geq 0$ consider the topological pullback as described in 7.2:

$$\begin{array}{ccc} J^n(Y,Z) \times_Y J^n(X,Y) & \xrightarrow{pr_2} & J^n(X,Y) \\ \big\downarrow{\scriptstyle pr_1} & & \big\downarrow{\scriptstyle \omega} \\ J^n(Y,Z) & \xrightarrow{\alpha} & Y \end{array}$$

The mappings $\gamma_n: J^n(Y,Z) \times_Y J^n(X,Y) \to J^n(X,Z)$, $\gamma_n(\sigma,\tau) = \sigma \circ \tau$ (cf. 1.4) are well defined (since $\alpha(\sigma) = = \omega(\tau)$) and smooth. We have:

$\gamma_n(j^n g(Y) \times_Y j^n f(L_n)) = \gamma_n(\{(j^n g(f(x)), j^n f(x)): x \in L_n\}) = = \{j^n(g \circ f)(x): x \in L_n\} = j^n(g \circ f)(L_n) \subseteq U_n$.

This means

$j^n g(Y) \times_Y j^n f(L_n) \subseteq \gamma_n^{-1}(U_n)$ for all n. $\alpha | j^n g(Y)$ is proper, since it is a diffeomorphism, inverse to $j^n g: Y \to j^n g(Y)$.

$\omega | j^n f(L_n)$ is proper: if $C \subseteq Y$ is compact, then $(\omega | j^n f(L_n))^{-1}(C) = j^n f(L_n \cap f^{-1}(C)) = j^n f$ (compact) $=$ $=$ compact, since f is proper. So all hypotheses of lemma 7.2 are fulfilled, therefore we may find open neighbourhoods V_n of $j^n g(Y)$ in $J^n(Y,Z)$ and W_n' of $j^n f(L_n)$ in $J^n(X,Y)$ such that $j^n g(Y) \times_Y j^n f(L_n) \subseteq V_n \times_Y W_n' \subseteq \gamma_n^{-1}(U_n)$. Since f is proper and Y is locally compact and (L_n) is

locally finite, the family $(f(L_n))$ is again closed (sublemma 7.2) and locally finite. There is a closed locally finite family (K_n) in Y with $f(L_n) \subset K_n^{\,o}$ (open interiour). So $\omega\,(j^n f(L_n)) = f(L_n) \subset K_n^{\,o}$ and $K_n^{\,o}$ is open, so $\omega^{-1}(K_n^{\,o})$ is open in $J^n(X,Y)$. Let $W_n = W_n' \cap \omega^{-1}(K_n^{\,o})$, then W_n is open and $j^n f(L_n) \subseteq W_n$; furthermore
$j^n g(Y) \times_Y j^n f(L_n) \subseteq V_n \times_Y W_n \subseteq V_n \times_Y W_n' \subseteq \gamma_n^{-1}(U_n)$.
Put $K = (K_n)$, $V = (V_n)$, $W = (W_n)$. Then $g \in M'(K,V)$ since $j^n g(K_n) \subset j^n g(Y) \subset V_n$, and $f \in M'(L,W)$ since $j^n f(L_n) \subset W_n$.
<u>Claim:</u> $\mathrm{Comp}(M'(K,V), M'(L,W)) \subseteq M'(L,U)$. Let $g' \in M'(K,V)$, $f' \in M'(L,W)$, then for any n and $x \in L_n$ we have:
$f'(x) = \omega\; j^n f'(x) \in \omega\; j^n f'(L_n) \subset \omega(W_n) \subset K_n^{\,o} \subset K_n$, so
$(j^n g'(f'(x)), j^n f'(x)) \in V_n \times_Y W_n \subseteq \gamma_n^{-1}(U_n)$, so
$j^n(g' \circ f')(x) = \gamma_n(j^n g'(f'(x)), j^n f'(x)) \in U_n$ so
$g' \circ f' \in M'(L,U)$.

So Comp is continuous for the $\mathfrak{D}$-topologies. Now if $f \sim f'$, $g \sim g'$, then f' is proper too and thus $g \circ f \sim g' \circ f'$.

q.e.d.

7.4 <u>Proposition:</u>

1. <u>Let</u> $f\colon X' \to X$ <u>be a proper smooth mapping, then</u> $f^* = C^\infty(f,Y)\colon C^\infty(X,Y) \to C^\infty(X',Y)$, $f^*(g) = g \circ f$, <u>is continuous in the</u> $\mathfrak{D}$- <u>and</u> $(\mathfrak{F}\mathfrak{D})$-<u>topology</u>.

2. <u>Let</u> $h\colon Y \to Y$ <u>be a smooth mapping</u>. <u>Then</u> $h_* = C^\infty(X,h)\colon C^\infty(X,Y) \to C^\infty(X,Y')$ <u>is continuous for the</u> $\mathfrak{D}$- <u>and</u> $(\mathfrak{F}\mathfrak{D})$-<u>topology</u>.

<u>Proof:</u> 1. follows from 7.3.
2. 7.3 shows, that $h_*\colon C^\infty_{\mathrm{prop}}(X,Y) \to C^\infty(X,Y')$ is continuous, but we want more. Let $M'(L,U)$ be a basic open set in $(C^\infty(X,Y'),\mathfrak{D})$, as in 4.7.2: $L = (L_n)$ is a locally finite closed family, $U = (U_n)$, U_n open in $J^n(X,Y')$. $J^n(X,h)\colon J^n(X,Y) \to J^n(X,Y')$ is smooth (2.14 resp. 1.10). Put $V_n = (J^n(X,h))^{-1}(U_n)$, $V = (V_n)$. Then $(h_*)^{-1} M'(L,U) = M'(L,V) \subseteq C^\infty(X,Y)$. For the $(\mathfrak{F}\mathfrak{D})$-topology one remarks that $f \sim f'$ implies $h \circ f \sim h \circ f'$. q.e.d.

7.5 Let X be a manifold with corners. For any $k \geq 1$ we consider the open sub fibre bundle $J^k_{\mathrm{inv}}(X,X)$ of the fibre bundle $J^k(X,X)$ over $X\times X$, consisting of all "invertible k-jets". In local coordinates $(U,u),(V,v)$ on X we have $J^k(u(U),v(V)) = u(U)\times v(V)\times P^k(n,n) =$ $= u(U)\times v(V)\times L^1_{sym}(\mathbb{R}^n,\mathbb{R}^n)\times\dots\times L^k_{sym}(\mathbb{R}^n,\mathbb{R}^n)$, and $J^k_{inv}(u(U),v(V))$ consist of all $(x,y,\sigma) = (x,y,\sigma_1,\sigma_2,\dots,\sigma_k)$ such that σ is invertible with respect to the truncated composition, i.e. such that $\sigma_1 \in GL(n,\mathbb{R})$.

<u>Lemma: The mapping</u> inv: $J^k_{inv}(X,X)\to J^k_{inv}(X,X)$, <u>given by</u> $inv(\sigma) = \sigma^{-1}$, <u>is a smooth fibre respecting mapping over</u> $(x,y)\to(y,x)$, $X\times X\to X\times X$.

<u>Proof:</u> In local coordinates we have:
inv: $J^k_{inv}(u(U),v(V))\to J^k_{inv}(v(V),u(U))$ is given by $inv(x,y,\sigma) = (y,x,inv^k\sigma)$, where $inv^k\sigma$ is the inverse power series for the polynomial σ, truncated at order k. Since the coefficients of $inv^k\sigma$ are rational functions of the coefficients of σ, inv is a smooth mapping. q.e.d.

7.6 <u>Theorem: The mapping</u> Inv: $Diff(X)\to Diff(X)$, <u>given by</u> $Inv(f) = f^{-1}$, <u>is continuous for the</u> $\mathfrak{D}$- <u>and the</u> $(\mathfrak{F}\mathfrak{D})$-<u>topology</u>.

<u>Proof:</u> Let $M'(L,U)$ be a basic open neighbourhood of f^{-1} in $C^\infty(X,X)$, where $f\in Diff(X)$, where $L=(L_n)$ is a locally finite closed family in X, $U=(U_n)$, U_n open in $J^n(X,X)$. We may assume that U_n is open in $J^n_{inv}(X,X)$, and that each $X\setminus L_n{}^o$ is compact (this is possible, see 4.7.3 and 4.7.3). We want to construct a $\mathfrak{D}$-open neighbourhood $\mathfrak{B}$ of f in $C^\infty(X,X)$ such that $Inv(\mathfrak{B}\cap Diff(X))\subseteq M'(L,U)$. Since $f^{-1}\in M'(L,U)$ we have $(j^n(f^{-1}))^{-1}(U_n)\supseteq L_n$ for each n. Let $(L_n{}')$ be a locally finite sequence of closed set in X, such that $(j^n(f^{-1}))^{-1}(U_n)\supseteq L_n{}'\supseteq L_n'^{o}\supseteq L_n$ for all n. Since $X\setminus L_n'^{o}\subseteq X\setminus L_n{}^o$ we have $X\setminus L_n'^{o}$ compact too for each n. Put $K_n{}' = f^{-1}(L_n{}')$, $K_n = f^{-1}(L_n)$, $K' = (K_n{}')$, $K = (K_n)$. K',K are again locally finite closed families in X and $K_n\subseteq K_n'^{o}$.

Let d be a matric on X, compatible with the topology, let $\varepsilon: X\to\mathbb{R}_+$ be a strictly positive continuous function on X such that $0<\max\{\varepsilon(x): x\in X\setminus K_n{}^o\}<$distance between the compact $X\setminus L_n'^{\circ}$ and the disjoint closed set L_n, for each n. This exists since (K_n) is locally finite, $X\setminus K_n^{\circ}$ is compact and (L_n) is locally finite. Put $V_n = \mathrm{inv}(U_n)$ (7.4), an open set in $J^n_{\mathrm{inv}}(X,X)$, $V = (V_n)$. Consider the basic $\mathfrak{D}$-open set $M'(K',V)$. We claim that it contains f:

For $n \in \mathbb{N}$ and $x \in K_n'$ we have $j^n f(x) = \mathrm{inv}(j^n(f^{-1})(f(x)))$, $f(x) \in f(K_n') = L_n'$, so $j^n(f^{-1})(f(x)) \in U_n$ by the choice of L_n'. Thus $j^n f(x) \in \mathrm{inv}(U_n) = V_n$, so $f \in M'(K',V)$. Now let $N(f,0,\varepsilon) = \{g \in C^{\infty}(X,X): d(f(x),g(x)) < \varepsilon(x) \text{ for all } x \in X\}$ and put $\mathfrak{B} = M'(K',V) \cap N(f,0,\varepsilon)$. Then $\mathfrak{B}$ is open, $f \in \mathfrak{B}$.

We finally claim that $\mathrm{Inv}(\mathfrak{B} \cap \mathrm{Diff}(X)) \subseteq M'(L,U)$: Let $g \in \mathfrak{B} \cap \mathrm{Diff}(X)$.

Then $g(K_n') \subseteq L_n$ for each n, since $d(f(x),g(x)) < \varepsilon(x)$ and for $x \in X\setminus K_n'$ we have $f(x) \in X\setminus L_n' \subseteq X\setminus L_n'^{\circ}$ and $\varepsilon(x) <$ < distance between $X\setminus L_n'^{\circ}$ and L_n; so $g(x) \notin L_n$, $g(x) \in X\setminus L_n$.

Therefore $g(X\setminus K_n') \subseteq X\setminus L_n$, i.e. $g(K_n') \subseteq L_n$ as asserted, by $g \in \mathrm{Diff}(X)$.

Now for $x \in L_n$ we have $j^n(g^{-1})(x) = \mathrm{inv}(j^n g(g^{-1}(x))) \subseteq$ $\subseteq \mathrm{inv}(V_n) = U_n$ since $g \in M'(K',V)$ and $g^{-1}(L_n) \subseteq K_n'$, so $j^n g(g^{-1}(x)) \subseteq j^n g(K_n') \subseteq V_n'$.

This says that $g^{-1} \in M'(L,U)$. We have proved that Inv is continuous for the $\mathfrak{D}$-topology. To obtain the same result for the $(F\mathfrak{D})$-topology just note that $f \sim f'$ iff $f^{-1} \sim f'^{-1}$. q.e.d.

7.7 <u>Corollary:</u> Diff(X) <u>is a topological group in the</u> WO^k<u>-topology</u>, $k \geq 1$, <u>in the</u> $\mathfrak{D}$<u>-topology and in the</u> $(F\mathfrak{D})$-<u>topology</u>. <u>Denote by</u> $\mathrm{Diff}_c(X) = \{f \in \mathrm{Diff}(X): f \sim \mathrm{Id}_X\}$ <u>the set of all diffeomorphisms with compact support</u>. <u>Then</u> $\mathrm{Diff}_c(X)$ <u>is a closed normal subgroup in</u> $(\mathrm{Diff}(X), WO^k, k \geq 1, \mathfrak{D})$ <u>and is an open subgroup in</u> $(\mathrm{Diff}(X), (F\mathfrak{D}))$.

<u>Remark</u>: $(\mathrm{Diff}_c(X), \mathfrak{D})$ has been studied by several authors: EPSTEIN (1970), MATHER (1974,1975), BANYAGA (1978), CALABI (1970).

Mather has shown that the subgroup of $\mathrm{Diff}_c(X)$ consisting of all diffeomorphisms diffeotopic to Id_X (i.e. homotopic in $\mathrm{Diff}_c(X)$) is perfect, i.e. coincides with its

commutator group. We will see later that this is exactly the connected component of Id_X in $(Diff_c(X),\mathfrak{D})$.
BANYAGA proved the same for symplectic diffeomorphisms, CALABI gave an erroneous proof of this.

8 Differential calculus on locally convex spaces

Ordinary differential calculus as beginning students of mathematics learn in the introductory courses on Analysis generalizes rather wonderfully up to Banach spaces (only partitions of unity are lost in the process). By this I mean that there is essentially one "good" definition of C^k-mappings and that important theorems of calculus continue to hold on Banach spaces in essentially the same form as they do on $\mathbb{R}^n$, including the implicit function theorem. Therefore there is nearly no difficulty in generaling manifold theory to Banach spaces (see S. LANG, 1972, for a wonderful account) and one even gets the "best" formulations for finite dimensional differential geometry by writing it down for Banach spaces in some cases.

There is a definite end to this beautiful theory at Banach spaces. This is mainly due to the fact that the usual norm topology on $L(E,F)$ for normed spaces E,F does not have a canonical extension to the wider category of locally convex spaces; what is worse: if E is not normable, then there does not exist a compatible topology on $L(E,F)$ such that for example the evaluation map $ev: E \times L(E,F) \to F$ would be continuous. So if one wants to have a theory of differentiation such that the chain rule holds, one has to leave the realms of topology and use convergence structures instead. A whole hort of mutually inequivalent definitions of differentiability therefore appears in the literature. But however, as H.H. KELLER

(1974) has shown, many of these notions coincide, if one looks at C^∞ mappings, and more so, if one restricts the spaces. We will use the simplest of the good notions of KELLER, the notion $C_c^\infty = C_\pi^\infty$.

Recently U. SEIP (1979) has shown that the compactly generated analogue of C_c^∞, restricted to a carefully chosen category of compactly generated linear spaces (those which come from sequentially complete locally convex vector spaces) gives a cartesion closed category of smooth mappings (i.e. $C^\infty(E, C^\infty(F,G)) = C^\infty(E \pi F, G)$ holds generally), a so called "convenient setting" for differential calculus. By generalizing the notion of manifold considerably (they need not have charts homeomorphic to open subsets of vector spaces; they are just required to have something like a fibre linear tangent bundle) he is even able to get a cartesian closed category of smooth mappings and "manifolds" (U. SEIP. preprint).

We will stick to the traditional notion of manifold as having an atlas consisting of charts in this book, since we are essentially interested in getting as much "differential geometry" on manifolds of mappings as possible.

In the following we denote by E,F,G,... complete locally convex vector spaces.

8.1 <u>Definition:</u> Let $U \subseteq E$ be an open subset. A mapping $f: U \to F$ is said to be C_c^1 on U iff the following two conditions hold:

1. $\lim_{\lambda \to 0} \frac{1}{\lambda}(f(x+\lambda y) - f(x)) = Df(x).y$ in F where $Df(x): E \to F$ is a linear map, for x in U, $y \in E$, $\lambda \in \mathbb{R}$.

2. The map $(x,y) \to Df(x)y$ is jointly continuous, $U \times E \to F$.

<u>Remark:</u> Condition 1 says that all directional derivatives of f should exist, condition 2 says that these fit together continuously. Condition 2 cannot be expressed in the following manner: $Df: U \to L(E,F)$ is continuous, where L(E,F) has some carefully chosen topology (cf. KELLER

1974). This is possible however in the compactly generated setting of SEIP 1979.

It can easily be shown that any C^1_c mapping is continuous.

Let us denote the set of all C^1_c mappings from U to F by $C^1_c(U,F)$ Clearly $C^1_c(U,F)$ is a linear space and $C^1_c(U,\mathbb{R})$ is an algebra and $C^1_c(U,F)$ is a module over this algebra.

8.2 <u>Lemma:</u> <u>Let</u> $U \subseteq E$, $V \subseteq F$ <u>be open</u>, <u>let</u> $f \in C^1_c(U,F)$, $g \in C^1_c(V,G)$, $f(U) \subseteq V$. <u>Then</u> $g \circ f \in C^1_c(U,G)$ <u>and we have</u> $D(g \circ f)(x)y = Dg(f(x)).Df(x).y$ <u>for any</u> $x \in U$ <u>and</u> $y \in E$.

<u>Proof:</u> The limit condition 8.1.1 can be computed as in any analysis course. $D(g \circ f)(x)y$ is jointly continuous in x and y as can be seen from the right hand side of the above equation.

8.3 <u>Lemma</u> (<u>partial derivatives</u>): <u>Let</u> $f\colon E \times F \to G$ <u>be a mapping</u>. <u>Then</u> $f \in C^1_c(E \times F,G)$ <u>iff the following conditions are satisfied</u>: $x_1 \to f(x_1,x_2)$, $x_2 \to f(x_1,x_2)$ <u>are of class</u> C^1_c <u>for fixed</u> x_2,x_1 <u>respectively with derivatives</u> $D_1f(x_1,x_2)y_1$ <u>and</u> $D_2f(x_1,x_2)y_2$ <u>which are jointly continuous in all appearing variables</u>.

<u>The derivative of</u> f <u>is then given by</u> $Df(x_1,x_2)(y_1,y_2) = D_1f(x_1,x_2)y_1 + D_2f(x_1,x_2)y_2$.

<u>Clearly the same result holds if</u> f <u>is only defined in an open subset</u> of $E \times Y$.

<u>Proof:</u> Necessity:

$$D_1f(x_1,x_2)y_1 = \lim_{\lambda\to 0} \frac{1}{\lambda} (f(x_1+\lambda y_1,x_2) - f(x_1,x_2) =$$
$$= \lim_{\lambda\to 0} \frac{1}{\lambda} (f((x_1,x_2) + \lambda(y_1,0)) - f(x_1,x_2)) = Df(x_1,x_2)(y_1,0),$$

so D_1f is jointly continuous in all appearing variables. Similarly for D_2f.

Sufficiency: $D_1f(x_1,x_2)y_1 + D_2f(x_1,x_2)y_2 =$

$$= \lim_{\lambda\to 0} D_1f(x_1,x_2+\lambda y_2)y_1 + D_2f(x_1,x_2)y_2 =$$
$$= \lim_{\lambda\to 0}\lim_{\mu\to 0} \frac{1}{\mu} [f(x_1+\mu y_1,x_2+\lambda y_2) - f(x_1,x_2+\lambda y_2)] +$$

$$+ \lim_{\mu\to 0} \frac{1}{\mu} [f(x_1, x_2 + \mu y_2) - f(x_1, x_2)] =$$
$$= \lim_{\mu\to 0} \frac{1}{\mu} [f(x_1 + \mu y_1, x_2 + \mu y_2) - f(x_1, x_2 + \mu y_2)] +$$
$$+ \lim_{\mu\to 0} \frac{1}{\mu} [f(x_1, x_2 + \mu y_2) - f(x_1, x_2)] =$$
$$= \lim_{\mu\to 0} \frac{1}{\mu} [f(x_1 + \mu y_1, x_2 + \mu y_2) - f(x_1, x_2)] =$$
$$= Df(x_1, x_2)(y_1, y_2).$$

So the joint continuity of Df in all variables is equivalent to the joint continuity of $D_1 f$, $D_2 f$ in all variables. q.e.d.

8.4 We do not prove a mean-value-lemma, since we will always take recourse to the following simple fact in situations traditionally mastered with the mean value lemma.

Lemma: *Let $f: U \to F$ be C^1_c, $x, y \in U$ and $[x,y]$ (i.e. the segment from x to y: $\{tx + (1-t)y:\ 0 \le t \le 1\}) \subseteq U$, where $U \subseteq E$ is open. Then $f(y) - f(x) = \int_0^1 Df(x + t(y-x))(y-x)dt$, where the Integral is the ordinary Bochner Integral (even Riemann-sums converge in F).*

Proof: The integral converges in F since the function is continuous in t. The formula follows by considering the function $g(t) = f(x + t(y-x))$, $\mathbb{R} \to F$; we may even assume that g has values in $\mathbb{R}$ by using the Hahn-Banach theorem and the chain rule. q.e.d.

Remark: Clearly $f(y) - f(x)$ is contained in closed convex set containing all $Df(x + t(y-x))(y-x)$, $t \in [0,1]$; this is normally alluded to be the mean value theorem.

8.5 Definition: $f: U \subseteq E \to F$ is called C^2_c if f is C^1_c and $Df: U \times E \to F$ is again C^1_c.

To define $D^2 f$ we compute as follows, using 8.3:

$$D(Df)(x_1, x_2)(y_1, y_2) =$$
$$= D_1(Df)(x_1, x_2)y_1 + D_2(Df)(x_1, x_2)y_2 =$$
$$= \lim_{\lambda\to 0} \frac{1}{\lambda} [(Df(x_1 + \lambda y_1)x_2 - Df(x_1)x_2] +$$
$$+ \lim_{\mu\to 0} \frac{1}{\mu} [Df(x_1)(x_2 + \mu y_2) - Df(x_1)x_2] =$$

$$= \lim_{\lambda\to 0} \frac{1}{\lambda} [Df(x_1 + \lambda y_1)x_2 - Df(x_1)x_2] +$$
$$+ \lim_{\mu\to 0} \frac{1}{\mu} Df(x_1)(\mu y_2)$$
$$=: D^2f(x_1)(y_1,x_2) + Df(x_1)y_2$$

where we defined

$$D^2f(x_1)(y_1,y_2): = \lim_{\lambda\to 0} \frac{1}{\lambda} [Df(x_1 + \lambda y_1)y_2 - Df(x_1)y_2] =$$
$$= D_1(Df)(x_1,y_2)y_1 =$$
$$= D(Df)(x_1,y_2)(y_1,0) =$$
$$= D(Df)(x_1,y_2)(y_1,z) - Df(x_1).z.$$

Clearly $D^2f\colon U\times E\times E\to F$ is jointly continuous.

We use this recursively to define:

Definition: $f\colon U\subseteq E\to F$ is called C^p_c iff $D^{p-1}f\colon U\times(\underbrace{E\times\ldots\times E}_{p-1})\to F$ is C^1_c;

We define recursively

$$D^pf(x)(y_1,\ldots,y_p): = D_1(D^{p-1}f)(x,y_2,\ldots,y_p).y_1 =$$
$$= D(D^{p-1}f)(x,y_2,\ldots,y_p)(y_1,0,\ldots,0) =$$
$$= D(D^{p-1}f)(x,y_2,\ldots,y_p)(y_1,z_2,\ldots,z_p) -$$
$$- D^{p-1}f(x)(z_2,\ldots,z_p).$$

$D^pf\colon U\times(E\times\ldots\times E)\to F$ is jointly continuous by recursion, and p-linear. It is even symmetric in the p-factors. This can easily be seen by restricting $D^pf(x)$ to the p-dimensional linear subspace of E containing $y_1,\ldots,y_p$; this is then the ordinary p-th derivative at x of f restricted to the p-dimensional affine subspace through x parallel to the one just chosen and symmetry follows.

Definition: Let $C^p_c(U,F)$ denote the space of all C^p_c-mappings $U\subseteq E\to F$. Let $C^\infty_c(U,F) = \bigcap_{p\geq 1} C^p_c(U,F)$.

Remark: We refrain from putting a topology on the space $C^\infty_c(U,F)$. If $U\subseteq E$ and E is infinite-dimensional then all topologies considered in § 4 with the exception of the compact-C^∞-topology become zero-dimensional. 4.4.4 shows that there are no nonconstant continuous curves in

$C_c^\infty(U,F)$, since there are of course no smooth mappings with compact support. The problem of putting a topology on $C_c^\infty(U,F)$ has been successfully solved by U. SEIP.

8.6 <u>Theorem:</u> <u>Let</u> $E\to X$ <u>be a finite dimensional vector bundle over a second countable smooth manifold with corners</u> X. <u>Then the space</u> $\Gamma_c(E)$ <u>of all smooth sections of</u> E, <u>bearing the</u> $\mathfrak{D}$<u>-topology</u> (<u>cf. 4.8</u>) <u>admits</u> C_c^∞ <u>partitions of unity</u>. <u>In particular it is paracompact</u>.

The last assertion has already been proved in 4.8. We repeat the proof of this. We begin with a sublemma.

<u>Sublemma:</u> $\Gamma_c(E)$ is a Lindelöf space, i.e. each open cover of $\Gamma_c(E)$ has a countable subcover.

<u>Proof</u> of the sublemma: Let (K_n) be a sequence of compact subsets in X such that $K_n\subset K_{n+1}{}^{o}$ and $X=\bigcup_n K_n$. Denote by $\Gamma_{K_n}(E)$ the subspace $\{s\in\Gamma_c(E)\colon \operatorname{supp} s\subseteq K_n\}$. Then $\Gamma_{K_n}(E)$ with the $\mathfrak{D}$-topology is a separable nuclear Fréchet space; we will use separable Fréchet here. So each $\Gamma_{K_n}(E)$ is a Lindelöf space. Let $\mathfrak{U}=(\mathfrak{U}_i)_{i\in I}$ be an open cover of $\Gamma_c(E)$. Then $\mathfrak{U}$ covers the closed linear subspace $\Gamma_{K_n}(E)$ too, so there is a countable subfamily $\mathfrak{U}_n$ of $\mathfrak{U}$ covering $\Gamma_{K_n}(E)$. Then $\bigcup_n \mathfrak{U}_n$ covers $\bigcup_n \Gamma_{K_n}(E)=\Gamma_c(E)$ and $\bigcup_n \mathfrak{U}_n$ is countable.

q.e.d.

<u>Proof</u> of the theorem: The constructions to come follow closely the method of BOURBAKI, General topology, IX, § 5, but we will carry along more information in the proof.

It is well known that $\Gamma_c(E)$ is a nuclear space (GROTHENDIECK, 1955). We will give an explicit proof of this fact in the next volume.

So we may assume that there is a system of seminorms on $\Gamma_c(E)$, $P=\{p_i\}_{i\in I}$, generating the topology of $\Gamma_c(E)$, such that for any two seminorms $p_i,p_j\in P$ there is a third one p_k with $p_k\geq p_i$, $p_k\geq p_j$, and such that the completion $\overline{\Gamma_c(E)/p_i{}^{-1}(0)}$ of each factor space $\Gamma_c(E)/p_i{}^{-1}(0)$ in the norm topology $\tilde{p}_i$ derived from the seminorm p_i is a

Hilbert space. Let us denote by $q_i: \Gamma_c(E) \to \overline{\Gamma_c(E)/p_i^{-1}(0)}$ the projection.

It is well known that the square of the norm in a Hilbert space is a C^∞-function (by bilinearity), therefore for each $i \in I$ the mapping $p_i^2 = (\tilde{p}_i)^2 \circ q_i$ is a C_c^∞-function on $\Gamma_c(E)$ (by the chain rule, since q_i is linear and continuous).

So we have the following data: $P = \{p_i\}_{i \in I}$ is a system of seminorms on $\Gamma_c(E)$, generating the $\mathfrak{D}$-topology, complete (so $p_i, p_j \leq p_k$ for all i, j for some $k = k(i,j)$) and such that p_i^2 is C_c^∞ on $\Gamma_c(E)$.

Now let $\mathfrak{U} = (U_\alpha)_{\alpha \in A}$ be an arbitrary open cover of $\Gamma_c(E)$. For each $x \in U_\alpha$ choose $p_i \in P$ and $\epsilon > 0$ such that $V_{x,\epsilon} := \{y \in \Gamma_c(E): p_i(x-y) < \epsilon\} \subseteq \overline{V_{x,\epsilon}} \subseteq U_\alpha$. Then $(V_{x,\alpha})$, $x \in U_\alpha$, $\alpha \in A$ is an open cover of $\Gamma_c(E)$ refining $\mathfrak{U}$. By the sublemma there is a countable subcover $(V_n)_{n \in \mathbb{N}}$ of $(V_{x,\alpha})$; then $(V_n)_{n \in \mathbb{N}}$ is a countable open cover of $\Gamma_c(E)$ refining $\mathfrak{U}$, and each V_n is of the form

$V_n = \{y \in \Gamma_c(E): p_{i_n}(y - x_n) < \epsilon_n\}$ for suitable $x_n \in \Gamma_c(E)$, $\epsilon_n > 0$, $p_{i_n} \in P$.

From now on we adapt the proof for Hilbert spaces of S. LANG (1972), p. 35 f.

Define a cover $(W_n)_{n \in \mathbb{N}}$ of $\Gamma_c(E)$ refining $(V_n)_{n \in \mathbb{N}}$ recursively as follows: Let $W_0 = V_0$.

Having defined W_{n-1}, let

$r_{0,n} = \epsilon_0 - \frac{1}{n}, \ldots, r_{n-1,n} = \epsilon_{n-1} - \frac{1}{n}$, let $A_{j,n} = \{y \in \Gamma_c(E): p_{i_j}(y - x_j) > r_{jn}\}$ for $0 \leq j < n$, and let

$W_n = V_n \cap A_{0,n} \cap \ldots \cap A_{n-1,n}$.

<u>Claim:</u> (W_n) is an open cover of $\Gamma_c(E)$ and $W_n \subseteq V_n$.

Open, $W_n \subseteq V_n$ is clear. Let $y \in \Gamma_c(E)$. Let n be the smallest index such that $y \in V_n$. If y were not in W_n, then

$y \in \Gamma_c(E) \setminus W_n = (\Gamma_c(E) \setminus V_n) \cup (\Gamma_c(E) \setminus A_{0n}) \cup \ldots \cup (\Gamma_c(E) \setminus A_{n-1,n})$;

so there is some $j < n$ with

$y \in \Gamma_c(E) \setminus A_{jn} = \{z \in \Gamma_c(E): p_{i_j}(y - x_j) \leq r_{jn}\} \subseteq V_j$ since $r_{jn} < \epsilon_j$.

This contradicts the minimality of n for $y \in V_j$. Therefore $y \in W_n$.

<u>Claim:</u> $(\overline{W_n})$ is locally finite. Let $x \in \Gamma_c(E)$. Then $x \in V_n$ for some n, i.e. $p_{i_n}(x - x_n) < \varepsilon_n$. Let $0 < \eta < \frac{1}{2}(\varepsilon_n - p_{i_n}(x - x_n))$. Let $B = \{y \in \Gamma_c(E): p_{i_n}(y-x) < \eta\}$. Now $B \subseteq \Gamma_c(E) \backslash \overline{A_{n,k}}$ for k sufficiently large since $\Gamma_c(E) \backslash \overline{A_{nk}} = \{y: p_{i_n}(y - x_n) < r_{n,k} = \varepsilon_n - \frac{1}{k}\}$ and $p_{i_n}(x - x_n) + \eta < \varepsilon_n - \frac{1}{k}$ for k large. But this means that $B \cap \overline{W_k} = \emptyset$ for k large enough, so we have found a neighbourhood B of x meeting only finitely many $\overline{W_n}$'s.

<u>Claim:</u> For each W_n there a function $\varphi_n' \in C_c^\infty(\Gamma_c(E),\mathbb{R})$ such that $\varphi_n'(x) > 0$ if $x \in W_n$ and $\varphi_n'(x) = 0$ if $x \notin W_n$.

Let $\alpha(t)$, $\beta(t)$ be C^∞-functions on $\mathbb{R}$ such that $\alpha(t) > 0$ if $|t| < 1$ and $\alpha(t) = 0$ if $|t| \geq 1$, $\beta(t) > 0$ if $|t| > 1$ and $\beta(t) = 0$ if $|t| \leq 1$.

Then $x \mapsto \alpha(\varepsilon_n^{-2} \cdot p_{i_n}{}^2(x - x_n))$ is a C_c^∞-function on $\Gamma_c(E)$, > 0 if $x \in V_n$, $= 0$ if $x \notin V_n$, and $x \mapsto \beta(r_{j,n}{}^{-2} \cdot p_{ij}{}^2(x - x_j))$ is a C_c^∞-function on $\Gamma_c(E)$, > 0 if $x \in A_{j,n}$, $= 0$ if $x \notin A_{jn}$ (if $r_{j,n} \leq 0$ adjust the definition suitably).

Then $\varphi_n'(x) = \alpha(\varepsilon_n{}^{-2} p_{i_n}{}^2(x - x_n)) \cdot \prod_{1 \leq j \leq n} \beta(r_{jn}{}^{-2} \cdot p_{ij}{}^2(x - x_j))$ has the required properties.

So $\varphi_n' \geq 0$, $(\operatorname{supp} \varphi_n') = (\overline{W_n})$ is locally finite and for each x there is an n with $\varphi_n'(x) > 0$. So $x \to \sum_n \varphi_n'(x)$ is well defined and C_c^∞ on $\Gamma_c(E)$ and > 0 everywhere, therefore

$$\varphi_n(x) = \frac{\varphi_n'(x)}{\sum_j \varphi_j'(x)}$$

is the required partition of unity subordinated to $(\overline{W_n})$, so to $(\overline{V_n})$ and (U_α). q.e.d.

8.7 <u>Theorem</u> (**Ω-lemma**): <u>Let</u> (E_i, p_i, X, F_i), $i = 1,2$, <u>be finite dimensional smooth vector bundles over a manifold with corners</u> X. <u>Let</u> $U \subseteq E_1$ <u>be an open neighbourhood of the</u>

image of a section $s_o \in \Gamma_c(E_1)$, *let* $\alpha\colon U \to E_2$ *be a smooth fibre respecting mapping such that* $\alpha \circ s_o$ *has compact support.*

Then the mapping $\alpha_*\colon V \subseteq \Gamma_c(E_1) \to \Gamma_c(E_2)$, $\alpha_*(s) = \alpha \circ s$, *is a* C_c^∞*-mapping, where* $V = \{s \in \Gamma_c(E_1)\colon s(X) \subseteq U\}$ *is open in* $\Gamma_c(E_1)$.

We have $D(\alpha_*) = (d_F\alpha)_*$, *where* $d_F\alpha\colon U \times E_1 \to E_2$ *is the fibre derivative of* α (cf. 1.16), i.e. $d_F\alpha(\eta_x) = d(\alpha|(E_1)_x \cap U)(\eta_x)$ *for* $\eta_x \in E_x \cap U$, $x \in X$.

Remark: We will need this theorem in a slightly more general form; E_i will be TX or a pull back of this bundle, and U will be open in ^{i}TX only. Then V is no more open in $\Gamma_c(E_1)$, only open in a "quadrant of infinite index".

Proof: It suffices to show that $D(\alpha_*)(s)(s') = (d_F\alpha)_*(s,s') = d_F\alpha \circ (s,s')$, since then α_* is of class C_c^∞ already: $(d_F\alpha)_*\colon V \times \Gamma_c(E_1) \to \Gamma_c(E_2)$ is continuous by 7.4.2, so α_* is of class C_c^1. But $D(\alpha_*) = (d_F\alpha)_*\colon \Gamma_c(E_1 \oplus E_1) \to \Gamma_c(E_2)$ is of the same form as α_*, so α_* is C_c^2. By recursion α_* is C_c^∞. So we have to show that for $s, s' \in \Gamma_c(E_1)$ (we ignore V from now on) the following holds:

(1) $\lim\limits_{\lambda\to 0} \dfrac{\alpha_*(s+\lambda s') - \alpha_*(s)}{\lambda} = (d_F\alpha)_*(s,s')$ in $(\Gamma_c(E_2),\mathfrak{D})$, for $\lambda \in \mathbb{R}$. We will use 4.7.6, 4.7.7. For $x \in X$ we have

(2) $\frac{1}{\lambda}\,[\alpha_*(s+\lambda s') - \alpha_*(s)](x) = \frac{1}{\lambda}\,[\alpha(s(x) + \lambda s'(x)) - \alpha(s(x))]$

(3) $[(d_F\alpha)_*(s,s')](x) = d_F\alpha(s(x)).s'(x)$.

If x is not in the compact support of s', then (2) and (3) are both zero, so (1) holds there.

It suffices to show, that on the compact support of s' "all partial derivatives" of (2) with respect to x converge uniformly to those of (3) for $\lambda \to 0$. For that it suffices to show that for any x_o this is true on a neighbourhood of x_o. So we may restrict to a chart W centered at x_o and trivializing for both bundles, to get the following situation:

$$\begin{tikzcd} W\times F_1 \arrow[rr, "\alpha"] & & W\times F \\ & x_0\in W\subseteq Q\subseteq \mathbb{R}^n. \arrow[ul, "{s,s'}"] \arrow[ur, dashed] & \end{tikzcd}$$

Then we have $s(x)=(x,t(x))$, $s'(x)=(x,t'(x))$, $\alpha(x,y)=(x,\beta_x(y))=(x,\beta(x,y))$ for $x\in W$, $t,t'\in C^\infty(W,F_1)$ and $\beta\in C^\infty(W\times F_1,F_2)$, and we have to show, that each derivative with respect to x of

$$(4)\quad \frac{\beta(x,t(x)+\lambda t'(x))-\beta(x,t(x))}{\lambda}$$

converges to the corresponding derivative with respect to x of

$$(5)\quad d(\beta_x)(t(x)).t'(x)=d_2\beta(x,t(x)).t'(x),$$

uniformly on a neighbourhood of x_o. By Taylor's theorem we have

$$\frac{\beta_x(t(x)+\lambda t'(x))-\beta_x(t(x))}{\lambda}=\frac{1}{\lambda}\,d(\beta_x)(t(x))(\lambda t'(x))+$$
$$+\frac{1}{\lambda}\int_0^1 (1-\mu)d^2(\beta_x)(t(x)+\mu\lambda t'(x))(\lambda t'(x),\lambda t'(x))d\mu.$$

So it remains to show that each derivative with respect to x of

$$(6)\quad \lambda\int_0^1 (1-\mu)d^2(\beta_x)(t(x)+\mu\lambda t'(x))(t'(x),t'(x))d\mu$$

converges to 0 uniformly on a neighbourhood of x_o for $\lambda\to 0$. For $|\lambda|\le 1$ e.g. the integrand is bounded on a compact neighbourhood of x_o, so converges to 0 uniformly with $\lambda\to 0$. Any derivative d^k with respect to x commutes with the integral in (6); after that the argument may be repeated. q.e.d.

9 Manifolds modelled on locally convex spaces

Here we present the main concepts in a form and notation suitable for our purposes. The second part is devoted to a simple example.

9.1 Definition: By a C_c^∞-manifold we mean the following data:

1. A Hausdorff topological vector space M, together with a family $(U_i, u_i, E_i)_{i\in I}$, where (U_i) as an open cover of M, $u_i : U_i \to u_i(U_i) \subseteq E_i$ is a homeomorphism onto an open subset $u_i(U_i)$ of a complete locally convex vector space E_i for each i.

2. If $U_{ij} := U_i \cap U_j \neq \emptyset$ then the mapping

$$u_{ij} = u_i \circ u_j^{-1} : \underset{\displaystyle E_j}{\underset{\cap}{u_j(U_{ij})}} \to U_{ij} \to \underset{\displaystyle E_i}{\underset{\cap}{u_i(U_{ij})}}$$

is required to be a C_c^∞-mapping. It follows that it is a C_c^∞ diffeomorphism and that E_i is linearly isomorphic to E_j.

Each (U_i, u_i, E_i) is called a chart for the C^∞-manifold M, the collection $(U_i, u_i, E_i)_{i\in M}$ is called the defining atlas. Any family as in 1. satisfying 2. is called an atlas, two atlasses are called equivalent, if their union is an atlas too (i.e. satisfies 2.).

By a C_c^∞-manifold we will always mean a manifold defined as above, a C^∞-manifold or smooth manifold will always be finite-dimensional (it is a C_c^∞-manifold too then).

C_c^∞-mappings between C_c^∞-manifolds will be mappings that are C_c^∞ when composed with chart mappings.

9.2 Let M be a C_c^∞-manifold, let $N \subseteq M$ be a subset. N is called a <u>splitting C_c^∞-submanifold</u> of M, if for each $x \in N$ there is a chart (U,u,E) of M with $x \in U$, $u(x) = 0 \in E$ and a closed direct summand $F \subseteq E$ (i.e. F is a closed linear subspace having a closed topological complemtary subspace in E) such that $u(U \cap N) = u(U) \cap F$. The collection of all $(U \cap N, u|U \cap N, F)$, (U,u,E) as above, is an atlas for N, making it to a C_c^∞-manifold itself.

If one drops the requirement that F has to be a direct summand (so F is only required to be a closed linear subspace), then the subset N is called a <u>non splitting C^∞-submanifold</u> (short for: not necessarily splitting C_c^∞-submanifold). We will have the chance to meet specimens of both kinds later on.

9.3 <u>Tangent bundle.</u> Let M be a C_c^∞-manifold with an atlas $(U_i,u_i,E_i)_{i\in I}$ defining it. A tangent vector at $x \in M$ should be a natural way to define "directional derivatives of functions at x". So if $f: M \to \mathbb{R}$ is C_c^∞, one may try $D(f \circ u_i^{-1})(u_i(x)).v$, where $v \in E_i$ is arbitrary. So we choose the following definition:

A <u>tangent vector</u> on the C_c^∞-manifold M is an equivalence class of 5-tuples (v,U_i,u_i,E_i,x), where (U_i,u_i,E_i) is a (compatible) chart on M, $v \in E_i$, and $x \in U_i$; two such tuples (v,U_i,u_i,E_i,x) and (w,U_j,u_j,E_j,y) are equivalent iff $x = y$ and $D(u_i \circ u_j^{-1})(u_j(y)).w = v$. The unique point x in each 5-tuple of a class is called is <u>foot point</u> of the tangent vector; T_xM is the space of all tangent vectors with foot point x, and TM denotes the space of all tangent vectors on M. Let $\pi = \pi_M$: $TM \to M$ the mapping associating its foot point to each tangent vector. Choose a chart (U_i,u_i,E_i) of M. Then one gets a chart $(\bar{U}_i,\bar{u}_i,E_i \times E_i)$ of TM as follows:

$\overline{U}_i = \pi_M^{-1}(U_i)$,
$\overline{u}_i(\xi) = (u_i(x),v)$ if $(v,U_i,u_i,E_i,x) \in \xi$.

It is easily checked that the chart change $\overline{u_j} \circ \overline{u_i}^{-1}$ is given by $(y,v) \to (u_j u_i^{-1}(y), D(u_j \circ u_i)(y).v)$, which is a C_c^∞ mapping nearly by definition. Now induce the (unique) topology on TM which makes each $\overline{u}_i$ to a homeomorphism. It is clear that π_M is continuous for this topology. It remains to show that this topology is Hausdorff:
Let $\xi,\eta \in TM$, $\xi \neq \eta$. If $\pi_M(\xi) = \pi_M(\eta)$, then for any canonical chart $(\overline{U}_i, \overline{u}_i, E_i \times E_i)$ we have $\overline{u}_i(\xi) = (y,v)$, $\overline{u}_i(\eta) = (y,w)$; separate v,w in E_i by disjoint open sets V,W, $v \in V$, $w \in W$, then $\overline{u}_i^{-1}(E_i \times V)$, $\overline{u}_i^{-1}(E_i \times W)$ separate ξ,η. If $\pi_M(\xi) \neq \pi_M(\eta)$, then separate these images by disjoint open set V,W in M, so $\pi_M^{-1}(V)$, $\pi_M^{-1}(W)$ separate ξ,η.

Thus we have proved that TM is again a C_c^∞-manifold with the atlas $(\overline{U}_i, \overline{u}_i, E_i \times E_i)$, which we call the canonical atlas.

9.6 <u>Definition:</u> By a C_c^∞-vector bundle we mean the following data:

1. A triple $(\mathfrak{E},p,M)$, where $\mathfrak{E}$ and M are C_c^∞-manifolds and $\pi\colon \mathfrak{E} \to M$ is a C_c^∞-mapping.

2. A family $(U_i,\varphi_i,F_i)_{i \in I}$ where (U_i) is an open cover of M, $\varphi_i\colon U_i \times F_i \to p^{-1}(U_i)$ is a C_c^∞-diffeomorphism and F_i is a complete locally convex vector space. φ_i is required to be "fibre-respecting", i.e. $p\,\varphi_i(x,y) = x$ for $x \in U_i$, $y \in F_i$, or

$$\begin{tikzcd} U_i \times F_i \arrow[rr, "\varphi_i"] \arrow[dr, "pr_1"'] & & p^{-1}(U_i) =: \mathfrak{E}/U_i \arrow[dl, "p"] \\ & U_i & \end{tikzcd}$$

commutes.

Furthermore for each $x \in U_{ij} = U_i \cap U_j$ the mapping $\varphi_{ij}(x) = \varphi_i^{-1} \circ \varphi_j \,|\, \{x\} \times F_j\colon F_j \to F_i$ is required to be a linear isomorphism. More exactly: $\varphi_i^{-1}\varphi_j(x,y) =$
$= (x, \varphi_{ij}(x).y)$, this defines $\varphi_{ij}\colon F_j \to F_i$ which is required to be linear (isomorphism follows).

A family $(U_i,\varphi_i,F_i)_{i\in I}$ as in 2. is called a vector bundle atlas, each (U_i,φ_i,F_i) is called a vector bundle chart. Two vector bundle atlasses are called equivalent if their union is again a vector bundle atlas. So more exactly a C^∞_c vector bundle is given by an equivalence class of vector bundle atlasses on $(\mathfrak{E},p,M)$.

Given an atlas $(U_i,u_i,E_i)_{i\in I}$ of a C^∞_c-manifold M, consider the canonical atlas $(\overline{U}_i,\overline{u}_i,E_i\times E_i)$ of TM. Then $(U_i,\overline{u}_i^{-1}\circ(u_i\times Id_{E_i}),E_i)_{i\in I}$ is a vector bundle atlas for (TM,π_M,M).

We did not require that the "fibre type" of the vector bundle is constant over the whole base manifold M, since this will not be the case for $C^\infty(X,Y)$.

We will meet C^∞_c-fibre bundles with structure groups later on too; these are defined in the obvious way along the lines explained so far. We will use these notions and all other well known notions from finite dimensional differential geometry without further notice in the "C^∞_c-complete-locally-convex"-setting if the generalization is obvious and without problems.

9.5 Our next aim is to investigate the (simple) example $J^\infty(X,Y)$, where X,Y are smooth manifolds without boundary, in order to get some feeling for the theory. We begin with some preparations.

1. Remember $J^\infty(U,V)$ where U is open in $\mathbb{R}^n$, V is open in $\mathbb{R}^m$ from 3.1. We had
$$\begin{aligned} J^\infty(U,V) &= U\times V\times \prod_{j\geq 0} L^j_{sym}(\mathbb{R}^n,\mathbb{R}^m) \\ &= U\times V\times P^\infty(n,m) \\ &= \varprojlim_k J^k(U,V). \end{aligned}$$

2. <u>Formal composition</u> comp: $P^\infty(n,m)\times P^\infty(k,n)\to P^\infty(k,m)$ <u>is jointly continuous</u>:
Let $A=(A^j)_{j\geq 1}\in P^\infty(n,m)=\prod_{j\geq 1} L^j_{sym}(\mathbb{R}^n,\mathbb{R}^m)$, let $B=(B^j)_{j\geq 1}\in P^\infty(k,n)$.

Then $(A\circ B)^j$ is a finite linear combination with universal constants (depending only on j,k,n,m) of expressions like $A^l \circ (B^{i_1},\dots,B^{i_r})$, $l,i_1,\dots,i_r \le j$, $r \le j$. Since each $A \to A^l$ is continuous (linear) the result follows.

3. Let E be a complete locally convex space, let $f\colon E \to P^\infty(n,m)$ be a mapping, $f = (f^j)_{j\ge1}$, $f^j\colon E \to L^j_{sym}(\mathbb{R}^n,\mathbb{R}^m)$ Then f is C^∞_c iff f^j is C^∞_c for each j.

Proof: $\frac{1}{\lambda}(f(x+\lambda y) - f(x)) = (\frac{f^j(x+\lambda y) - f^j(x)}{\lambda})_{j\ge1}$; this converges to $(df^j(x).y)_{j\ge1}$ in each coordinate, thus in $P^\infty(n,m)$. $(x,y) \to (df^j(x).y)_{j\ge1}$ is jointly continuous in x and y. So f is C^1_c and $Df = (df^j)_{j\ge1}$. By recursion f is C^∞_c. The other implication follows from the chain rule.

4. Let $f \in C^\infty(U,V)$, U open in $\mathbb{R}^n$, V open in $\mathbb{R}^m$. Then $j^\infty f\colon U \to J^\infty(U,V)$ is C^∞_c.

Proof: $j^\infty f = (Id_U, f, d^1f, d^2f, d^3f, \dots)\colon U \to J^\infty(U,V) = U \times V \times L^1_{sym} \times L^2_{sym} \times \dots$, and each coordinate mapping is C^∞. Now use 3.

5. Formal composition comp: $P^\infty(n,m) \times P^\infty(k,n) \to P^\infty(k,m)$ is (jointly) C^∞_c:

Proof: Repeat the proof of 2.; since each $A \to A^l$ is continuous linear, $(A,B) \to (A \circ B)^l$ is C^∞_c for each l. Now use 3.

6. If $g\colon U' \to U$ is a diffeomorphism between open subsets of $\mathbb{R}^n$, then for any open $V \subseteq \mathbb{R}^m$ the mapping $J^\infty(g,V)\colon J^\infty(U,V) \to J^\infty(U',V)$, given by $J^\infty(g,V)(j^\infty f)(x) = j^\infty(f \circ g)(g^{-1}(x))$, is a C^∞_c-diffeomorphism.

Proof: The inverse is of the same form, so it suffices to show that this mapping is smooth. Now this mapping has the following form:

$U \times V \times P^\infty(n,m) \to U' \times V \times P^\infty(n,m)$, $(x,y,A) \to (g^{-1}(x), y, A \circ (J^\infty g \circ g^{-1}(x)))$.

Now use 4., 5. and the chain rule.

7. If $h\colon V \to V'$ is a smooth mapping between open sets $V \subseteq \mathbb{R}^m$, $V' \subseteq \mathbb{R}^k$ resp., then $J^\infty(U,h)\colon J^\infty(U,V) \to J^\infty(U,V')$ is C^∞_c.

Proof: This mapping has the following form:

$U \times V \times P^\infty(n,m) \to U \times V' \times P^\infty(n,k)$, $(x,y,A) \to (x, h(x),$

$j^{\infty} h(x) \circ A)$. Use again 4., 5. and the chain rule.

8. Let X,Y be smooth manifolds without boundary. Then $J^{\infty}(X,Y)$ is a C^{∞}_c-manifold. $(J^{\infty}(X,Y), \pi^{\infty}_0, X \times Y, P^{\infty}(n,m))$ is a C^{∞}_c-fibre bundle (even with structure group, but we won't prove this).

Proof: Let (U_i, u_i) be an atlas of X, let (V_j, v_j) be an atlas of Y. Use $(J^{\infty}(U_i, V_j), J^{\infty}(u_i^{-1}, v_j), \mathbb{R}^n \times \mathbb{R}^m \times P^{\infty}(n,m))$ as a C^{∞}_c-compatible (by 6. and 7.) atlas of $J^{\infty}(X,Y)$. This atlas even gives a fibre bundle atlas $(U_i \times V_j, J^{\infty}(u_i, v_j^{-1}) \circ (u_i \times v_j \times Id_{P^{\infty}}), P^{\infty}(n,m))$ of $J^{\infty}(X,Y)$ (compare 1.10).

9. $P^{\infty}(n,m)$ is metrizable. If d^{∞} is a metric on it then $x \to d^{\infty}(0,x)$ is continuous, and does not factor over any projection $\pi^{\infty}_k : P^{\infty}(n,m) \to P^k(n,m)$ (truncation).
But: Let $f: P^{\infty}(n,m) \to \mathbb{R}$ be a C^1_c-function. Then for any $A \in P^{\infty}(n,m)$ there is an open neighbourhood U of A in $P^{\infty}(n,m)$ and a k such that $f|U: U \to \mathbb{R}$ factors over $\pi^{\infty}_k | U: U \to P^k(n,m)$:

$$\begin{tikzcd} U \arrow[rr, "f|U"] \arrow[dr, "\pi^{\infty}_k | U"'] & & \mathbb{R} \\ & \pi^{\infty}_k(U) \subseteq P^k(n,m) \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

Proof: $Df: P^{\infty} \times P^{\infty} \to \mathbb{R}$ is continuous, so $(Df)^{-1}\,]-1,1[$ is open and contains $(A,0)$. By the definition of the product topology there are open neighbourhoods U of A and V of 0 in P^{∞} of the form $\tilde{U} \times \prod_{j>k} L^j_{sym}$ and $\tilde{V} \times \prod_{j>k} L^j_{sym}$ resp. where $\tilde{U}$ is an open convex neighbourhood of $\pi^{\infty}_k(A)$ in P^k and $\tilde{V}$ is an open neighbourhood of 0 in P^k.

Let $B \in U$, $C \in P^{\infty}$ with $\pi^{\infty}_k(C) = 0$, then $t.C \in V$ for all $t \in \mathbb{R}$, so $|t.Df(B).C| = |Df(B)(t.C)| < 1$ for all $t \in \mathbb{R}$, so $Df(B).C = 0$.

Now let $B \in U$. Denote by $\overline{B}$ the element $(\pi^{\infty}_k B, 0) \in P^k \times \prod_{j>k} L^j_{sym}$.

Since U is convex we have by 8.4:
$f(B) - f(\overline{B}) = \int_0^1 Df(\overline{B} + tC).C\, dt$, where $C = B - \overline{B}$ satisfies

$\pi_k^\infty C = 0$ and all $\overline{B} + tC$, $0 \le t \le 1$, lie in U. So the integrand is 0, so $f(B) = f(\overline{B})$. This says that $f|U$ factors over π_k^∞. q.e.d.

10 Manifolds of mappings

10.1 Let X be a C^∞-manifold with corners (finite dimensional).

Definition: A *local addition* τ on X is a smooth mapping $\tau\colon {}^iTX \to X$ satisfying

(A1) $(\pi_X, \tau)\colon {}^iTX \to X \times X$ is a diffeomorphism onto an open neighbourhood of the diagonal in $X \times X$.

(A2) $\tau(0_x) = x$ for all x in X.

${}^iTX$ is no longer a manifold with corners (see 2.6), but it is so nice that one can still talk of differentiable mappings on it.

From the conditions above it follows immediately that $\tau_x = \tau|{}^iT_xX\colon {}^iT_xX \to X$ is a diffeomorphism of a quadrant ${}^iT_xX$ (see 2.6) onto an open neighbourhood of x in X.

Lemma: Any C^∞-*manifold with corners admits a local addition.*

Proof: Let $\exp\colon \mathfrak{D} \to X$ be an exponential mapping on X, where $\mathfrak{D}$ is an open neighbourhood of the zero section in ${}^iTX$ as we constructed in 2.10. Choose a fibre respecting diffeomorphism $h\colon TX \to V$ onto an open neighbourhood of the zero section in TX, $h(0_x) = 0_x$, such that $h({}^iTX) \subseteq \mathfrak{D}$. In the next lemma we will construct such a diffeomorphism. Then $\exp \circ (h|{}^iTX)\colon {}^iTX \to X$ is a local addition. q.e.d.

10.2 *Lemma: Let* X *be a* C^∞-*manifold with corners, let* (E,p,X,F) *be a vector bundle over* X, *let* V *be an open neighbourhood of the zero section in* E. *Then there is a diffeomorphism* $h\colon E \to h(E) \subseteq V$ *with* $h(0_x) = 0_x$, $p \circ h = p$,

$h({}^{i}TX) \subseteq V \cap {}^{i}TX$ <u>and</u> $h(T\partial^{j}X) \subseteq V \cap T\partial^{j}X$ <u>for all</u> j.

<u>Proof:</u> Let g be a (Riemannian) metric on E. Then there is a smooth function $\delta \in C^{\infty}(X,]0,\infty[)$ such that $U = \{\eta \in E: g(p(\eta))(\eta,\eta) < \delta(p(\eta))^2\} \subseteq V$. This can be proved as in lemma 3.2. Now let $h: E \to U$ be defined by $h(\eta) = \delta(p(\eta)).\eta/\sqrt{1+g(\eta,\eta)}$. Then $p \circ h = p$, $h(0_x) = 0_x$ and $h^{-1}(\mu) = \mu/\sqrt{\delta(p(\mu))^2 - g(\mu,\mu)}$. The last claims hold since h "contracts along rays entering from 0_x". q.e.d.

10.3 <u>Remark:</u> Construct the local addition as in lemma 10.1 but suppose furthermore, that the exponential mapping used comes from a tangential spray ξ (i.e. $\xi(T\partial^{j}X) \subseteq T^2\partial^{j}X$ for each j cf. 2.8), then by 2.10 we have $\exp_x^{-1}(\partial^{j}X) \subseteq T_x\partial^{j}X$ for $x \in \partial^{j}X$ for all j; conversely we only have $\exp_x(\mathfrak{D} \cap T_x\partial^{j}X) \subseteq \bigcup_{m \geq j} \partial^{m}X$. But for the local addition $\tau = \exp \circ (h|{}^{i}TX)$ we have too $\tau_x^{-1}(\partial^{j}X) = T_x\partial^{j}X$ if $x \in \partial^{j}X$, for each j. A local addition with this property will be called <u>boundary respecting</u>.

10.4 <u>Theorem:</u> <u>Let</u> X,Y <u>be</u> C^{∞}<u>-manifolds</u>, X <u>with corners</u>, Y <u>without boundary</u>. <u>Then</u> $(C^{\infty}(X,Y),(\mathcal{FD}))$ <u>is canonically a</u> C_c^{∞}<u>-manifold</u>, <u>modelled on nuclear and dually nuclear locally convex vector spaces</u> (<u>of the form</u> $\Gamma_c(f^*TY)$).

<u>Proof:</u> Let $\tau: TY \to Y$ be a local addition on Y; existence of such was asserted in 10.1.

Let $f \in C^{\infty}(X,Y)$.

Put $U_f = \{g \in C^{\infty}(X,Y): g \sim f,\ g(x) \in \tau_{f(x)}(T_{f(x)}Y)$ for for all $x \in X\}$

$= \{g \in C^{\infty}(X,Y): g \sim f,\ (f,g)(X) \subseteq (\pi_Y,\tau)(TY)\}$.

U_f is open in $(C^{\infty}(X,Y),(\mathcal{FD}))$.

Put $\mathfrak{D}_f(X,TY) = \{s \in C^{\infty}(X,TY): \pi_Y \circ s = f,\ s = 0$ off some compact in X, i.e. $s \sim 0_Y \circ f\}$, the space of all "vector fields along f with compact support".

Then $(\mathfrak{D}_f(X,TY),(\mathcal{FD}))$ is topological vector space, topologically and linearly isomorphic to $\Gamma_c(f^*TY)$, where f^*TY is the pullback onto X of the vector bundle TY

(cf. 1.17, 1.18). Furthermore put
$\varphi_f\colon U_f \to \mathfrak{D}_f(X,TY) \cong \Gamma_c(f^*TY)$, $\varphi_f(g) = (\pi_Y,\tau)^{-1} \circ (f,g)$, or $(\varphi_f(g))(x) = \tau_{f(x)}{}^{-1} g(x)$, and $\psi_f\colon \mathfrak{D}_f(X,TY) \to U_f$, given by $\psi_f(s) = \tau \circ s$. φ_f and ψ_f are continuous by 7.4.2, and φ_f and ψ_f are inverse to each other:

$$\psi_f\varphi_f(g) = \tau \circ (\pi_Y,\tau)^{-1}(f,g) = pr_2 \circ (\pi_Y,\tau) \circ (\pi_Y,\tau)^{-1} \circ (f,g) = \\ = pr_2 \circ (f,g) = g.$$

$$\varphi_f\psi_f(s) = (\pi_Y,\tau)^{-1} \circ (f,\tau \circ s) = (\pi_Y,\tau)^{-1}(\pi_Y \circ s, \tau \circ s) = \\ = (\pi_Y,\tau)^{-1} \circ (\pi_Y,\tau) \circ s = s.$$

We call $(U_f, \varphi_f, \mathfrak{D}_f\ (X,TY)) = (U_f, \varphi_f, \Gamma_c(f^*TY))$ the canonical chart of $C^\infty(X,Y)$, centered at f, induced by τ. The family $(U_f, \varphi_f, \Gamma_c(f^*TY))_{f \in C^\infty(X,Y)}$ is called the canonical atlas of $C^\infty(X,Y)$ induced by τ. It only remains to check that the chart change is C_c^∞. For this purpose we define for $f \in C^\infty(X,Y)$:

$$\tau_f\colon \begin{tikzcd} f^*TY \arrow[r] \arrow[dr] & X \times Y \arrow[d, "pr_1"] \\ & X \end{tikzcd}$$

by $\tau_f = (f^*\pi_Y, \tau)\colon f^*TY \to X \times Y$, $\tau_f(x, \mathfrak{v}_{f(x)}) = (x, \tau_{f(x)}\mathfrak{v}_{f(x)})$. Remember that $f^*TY = X \times_Y Y = \{(x,\eta)\colon f(x) = \pi_Y(\eta)\} \subseteq X \times TY$.

Then τ_f is a fibre respecting diffeomorphism onto an open subset of $X \times Y$, which is an open neighbourhood of the graph Γ_f of f in $X \times Y$.

Now choose $f,g \in C^\infty(X,Y)$ such that $U_f \cap U_g \neq \emptyset$, let $s \in \varphi_f(U_f \cap U_g) \subseteq \Gamma_c(f^*TY)$. Then we may compute as follows:

$$\begin{aligned} \varphi_g \circ \psi_f(s) &= (\pi_Y,\tau)^{-1} \circ (g, \tau \circ s) = \\ &= (\pi_Y,\tau)^{-1} \circ (g \times \tau) \circ (Id, s) = \\ &= \tau_g{}^{-1} \circ \tau_f \circ s = \\ &= (\tau_g{}^{-1} \circ \tau_f)_*(s). \end{aligned}$$

Here $\tau_g{}^{-1} \circ \tau_f\colon$
$$\begin{tikzcd} f^*TY \arrow[r] \arrow[dr] & X \times Y \arrow[r] \arrow[d] & g^*TY \arrow[dl] \\ & X & \end{tikzcd}$$

is a fibre respecting C^∞-mapping.

By the Ω-lemma 8.7 $\varphi_g \circ \varphi_f{}^{-1} = (\tau_g{}^{-1} \circ \tau_f)_*$ is a C_c^∞-mapping. So $C^\infty(X,Y)$ is a C_c^∞-manifold.

For completeness' sake we prove too:

Let τ, τ' be local additions, let $f \in C^\infty(X,Y)$, let $\varphi_f{}^\tau$, $\varphi_f{}^{\tau'}$ be the canonical chart mappings induced by τ and τ' respectively. Then $\varphi_f{}^\tau \circ (\varphi_f{}^{\tau'})^{-1}$ is C_c^∞ on its domain. For $\varphi^\tau f \circ (\varphi^{\tau'}{}_f)^{-1}(s) = (\pi_Y, \tau)^{-1} \circ (f, \tau' \circ s) =$

$$
\begin{aligned}
&= (\pi_Y, \tau)^{-1} \circ (f \times \tau') \circ (Id, s) = \\
&= \tau_f{}^{-1} \circ \tau_f' \circ s = \\
&= (\tau_f{}^{-1} \circ \tau_f')_*(s).
\end{aligned}
$$

So by the Ω-lemma again we are done. This shows that the C_c^∞-manifold structure on $C^\infty(X,Y)$ does not depend on the choice of the local additon τ. q.e.d.

10.5 <u>Proposition:</u> <u>Let</u> X,Y,Z <u>be</u> C^∞<u>-manifolds</u>, Y,Z <u>without boundary</u>. <u>Then the canonical identification</u> $C^\infty(X, Y \times Z) \cong C^\infty(X,Y) \times C^\infty(X,Z)$ <u>of</u> 4.7.9 <u>is of class</u> C_c^∞ <u>and even compatible with a suitable choice of canonical charts</u>.

<u>Proof:</u> Let $(f,g) \in C^\infty(X,Y) \times C^\infty(X,Z)$. We write again (f,g) for the corresponding element of $C^\infty(X, Y \times Z)$, which is given by $(f,g)(x) = (f(x), g(x))$. Let $\tau \colon TY \to Y$ and $\rho \colon TZ \to Z$ be local additions. Then $\tau \times \rho \colon TY \times TZ \to Y \times Z$ is a local addition on $Y \times Z$. Now we have:

$\Gamma_c((f,g)^*T(Y \times Z)) = \Gamma_c(f^*TY \oplus g^*TZ) = \Gamma_c(f^*TY) \oplus \Gamma_c(g^*TZ)$,

$U_{(f,g)} \cong U_f \times U_g$ for the canonical charts, and

$$
\begin{array}{ccc}
U_{(f,g)} & \xrightarrow{\varphi_{(f,g)}} & \Gamma_c((f,g)^*T(Y \times Z)) \\
\| & & \| \\
U_f \times U_g & \xrightarrow{\varphi_f \times \varphi_g} & \Gamma_c(f^*TY) \times \Gamma_c(g^*TZ).
\end{array}
$$
q.e.d.

10.6 For the next result we need some preparations.

<u>Definition:</u> Let X,Y be C^∞-manifolds without boundary, let $\tau \colon TY \to Y$ be a local addition. Suppose that X is a submanifold of Y. X is called <u>additively closed with respect to</u> τ in Y if $\tau(TX) \subseteq X$, i.e. τ induces a local addition on X.

(This notion is comparable to "geodesically closed" of Riemannian geometry).

Lemma: Let (E,p,B,F) be a vector bundle over a manifold without boundary B. Then there exists a local addition τ: $TE \to E$ with the following properties:

1. B, identified with the zero section in E, is additively closed in E with respect to τ.

2. Any vector subspace of each fibre $E_b = p^{-1}(b)$, $b \in B$, is additively closed in E with respect to τ.

Moreover τ induces on each fibre E_b the local addition coming from the affine structure of E_b: $\tau_{\mathfrak{b}}(\mathfrak{w}) = \mathfrak{b} + \mathfrak{w}$, $\mathfrak{b},\mathfrak{w} \in E_b$, $\mathfrak{w} \in T_{\mathfrak{b}}(E_b) = V(E)_{\mathfrak{b}}$.

Proof: Let (E',p',B,F') be a second vector bundle such that $E \oplus E'$ is trivial. Such a vector bundle exists, see HIRSCH (1976), p. 100. Then $E \oplus E'$ is isomorphic to $B \times \mathbb{R}^n$ for some u. Let τ_1 be a local addition on B, let τ_2 be the affine local addition on $\mathbb{R}^n$: $\tau_2(v_x) = x + v_x$, $v_x \in T_x\mathbb{R}^n \cong \mathbb{R}^n$. Then $\tau_1 \times \tau_2$ is a local addition on $B \times \mathbb{R}^n$, satisfying 1. and 2. Now transport $\tau_1 \times \tau_2$ back to $E \oplus E'$ via the isomorphism, then $\tau_1 \times \tau_2$ induces a local addition on the sub bundle E of $E \oplus E'$ by 2. q.e.d.

10.7 Definition: Let X be a submanifold of a manifold Y, both of them without boundary. A tubular neighbourhood of X in Y is an open neighbourhood U of X in Y together with a surjective submersion p: $U \to X$ such that:

1. $(U,p,X,\ldots)$ is a vector bundle
2. $X \to U$ is the zero section of this bundle.

Lemma: Let Y be a submanifold of a manifold X, both of them without boundary. Then there exists a tubular neighbourhood of Y in X.

This is a standard result of differential topology. A proof of it is contained in the proof of lemma 10.9 below.

10.8 Proposition: Let X,Y,Z be C^∞-manifolds, Y,Z without boundary. Let i: $Y \to Z$ be an embedding. Then $C^\infty(X,Y)$ is a splitting C^∞_c-submanifold of $C^\infty(X,Z)$ via

$i_*: C^\infty(X,Y) \to C^\infty(X,Z)$.

Proof: Let $Y \subseteq U \subseteq Z$ where U is a tubular neighbourhood of Y in Z, so (U,p,Y) is a vector bundle. By 10.6 there is a local addition τ on U, $\tau: TU \to U$, such that Y and all $p^{-1}(y)$, $y \in Y$ are additively closed in U.

Now let $g \in C^\infty(X,Y) \subseteq C^\infty(X,U)$.

Let $(U_g, \varphi_g, \Gamma_c(g^*TU))$ be the canonical chart of $C^\infty(X,U)$ centered at g which is induced by τ.

For any $f \in U_g$ we have: $f(X) \subseteq Y$, i.e. $f \in C^\infty(X,Y)$, iff $\varphi_g(f) = (\pi_U, \tau)^{-1}(g,f) \in \mathfrak{D}_g(X,TY) = \Gamma_c(g^*TY)$, since $\tau_{g(x)}(v) \in Y$ iff $v \in T_{g(x)}Y$. This says that $U_g \cap C^\infty(X,Y) = \varphi_g^{-1}(\Gamma_c(g^*TY))$, where $\Gamma_c(g^*TY)$ is a linear subspace of $\Gamma_c(g^*TU) = \Gamma_c(g^*TZ)$, even a direct summand, since

$$\begin{aligned}\Gamma_c(g^*TU) &= \Gamma_c(g^*(TU|Y)) = \Gamma_c(g^*(TY \oplus V(U)|Y)) = \\ &= \Gamma_c(g^*TY \oplus g^*V(U)|Y) = \\ &= \Gamma_c(g^*TY) \oplus \Gamma_c(g^*V(U)),\end{aligned}$$

where $V(U)$ is the vertical bundle of (U,p,Y).

$TU|Y = TY \oplus (V(U)|Y)$ can be seen by looking at the canonical chart change in 1.14. So $C^\infty(X,Y)$ is a splitting C_c^∞-submanifold of $C^\infty(X,U)$ which again is open in $C_c^\infty(X,Z)$. q.e.d.

10.9 Let X,Y be manifolds without boundary. $Q(X,Y)$ has been defined to be the set of all surjective submersions $X \to Y$ (5.6). $Q(X,Y)$ is open in $C^\infty(X,Y)$.

Definition: For $q \in Q(X,Y)$ let $S_q(Y,X)$ denote the space of all sections of q, i.e. $S_q(Y,X) = \{g \in C^\infty(Y,X): q \circ g = Id_Y\}$. Note that any $g \in S_q(Y,X)$ is an embedding.

Lemma: If q is as above, let $g \in S_q(Y,X)$. Then there exists a tubular neighbourhood $p_g: W_g \to g(Y)$ of $g(Y)$ in X whose projection p_g coincides with the restriction to W_g of the mapping $g \circ q: X \to g(Y)$.

Proof: Firstly let $i: X \to \mathbb{R}^n$ be an embedding of X into some $\mathbb{R}^n$. Then $g(Y)$ is a submanifold of $\mathbb{R}^n$ too via $i|g(Y)$. For $y \in g(Y) \subseteq \mathbb{R}^n$ let $P_y: \mathbb{R}^n \to T_{i(y)}(g(Y))$ be the orthogonal projection onto $T_{i(y)}(g(Y))$. $y \to P_y$ defines a C^∞-mapping

$g(Y) \to L(\mathbb{R}^n, \mathbb{R}^n)$ (this can be seen using the Gram-Schmidt-Orthonormalization process.)

$Id_{\mathbb{R}^n} - P_y$: $\mathbb{R}^n = T_{i(y)}\mathbb{R}^n \to (T_{i(y)}g(Y))^\perp$ is the associated orthonormal projection onto the orthonormal complement. $y \to (Id_{\mathbb{R}^n} - P_y)$ is again a C^∞-mapping $g(Y) \to L(\mathbb{R}^n, \mathbb{R}^n)$. Extend this mapping to a C^∞-mapping h: $X \to L(\mathbb{R}^n, \mathbb{R}^n)$ (this is possible since $g(Y)$ is a submanifold). Now let f: $X \to \mathbb{R}^n$ be defined by $f(x) = h(g \circ q(x))(i(x) - igq(x)) + igq(x)$. Then $f(q^{-1}(y)) \subseteq T_{ig(y)}(g(Y))^\perp + ig(y)$, $y \in Y$, and $f|g(Y) = i|g(Y)$, so there is an open neighbourhood V of $g(Y)$ in X such that $f|V$: $V \to \mathbb{R}^n$ is still an embedding. $q|V$: $V \to Y$ is still a surjective submersion, since V is open in X and $V \supseteq g(Y)$.

We will prove now that $g(Y)$ has a tubular neighbourhood in V, whose projection coincides with $g \circ q$. This proves the lemma then.

For that let W be an open neighbourhood of $f(V)$ in $\mathbb{R}^n$ and let r: $W \to f(V)$ be a normal tubular neighbourhood of $f(V)$ (i.e. $r^{-1}(x) = W \cap (T_x f(V)^\perp + x)$, $x \in f(V)$).

Proof, that this exists: Consider $Tf(V)^\perp$ in $T\mathbb{R}^n|f(V)$; this is a vector bundle over $f(V)$. Let l: $Tf(V)^\perp \to \mathbb{R}^n$ be defined by $l(v_x) = x + v_x$, $x \in f(V)$, $v_x \in T_x f(V)^\perp \subseteq \mathbb{R}^n$. Then $T_{0_x} l$ is the identity on $T_x f(V)$, where $f(V) \subseteq Tf(V)^\perp$ via the zero section, and $T_{0_x} l$ is the identity on $T_x f(V)^\perp$, so $T_{0_x} l$ is an isomorphism. So $l|$zero section is an embedding, $Tl|$zero section is fibre-wise an isomorphism, so there is an open neighbourhood of the zero section in $Tf(V)^\perp$ such that l is a diffeomorphism on it. Draw this diffeomorphism over the whole of $Tf(V)^\perp$ using 10.2 and get the looked for tubular neighbourhood W of $f(V)$ in $\mathbb{R}^n$ with normal projection.

Now equip $f(V)$ with the Riemannian metric induced from $\mathbb{R}^n$ and let $(E, p, fg(V) = ig(V))$ denote the normal bundle $Tfg(Y)^\perp \subseteq Tf(X)|fg(Y)$. Further let $U_y = \{v_y \in E_y: y + v_y \in W\}$, $y \in fg(Y)$ and put $U = \bigcup_{y \in fg(Y)} U_y$.

Then U is open in E (U is the inverse image of W under the mapping $\mathfrak{v}_y \to y + \mathfrak{v}_y$, $E \to \mathbb{R}^n$) and contains the zero section. Since $r: W \to f(V)$ is a normal projection and $U_y \subseteq T_y f(V)$ for any $y \in fg(Y)$ (i.e. U_y is tangential to $f(V)$ at y), the mapping $p: U \to f(V)$, $p(v_y) = r(y + v_y)$, is a diffeomorphism, at least in a neighbourhood of the zero section of U. Draw this neighbourhood over the whole of E using 10.2 again, and get the tubular neighbourhood $\bar{p}: \bar{W} \to fg(Y)$ of $fg(Y)$ in $f(V)$, whose fibres are orthogonal to $fg(Y)$ in $\mathbb{R}^n$. Transport back to V via f^{-1} and get the looked for tubular neighbourhood, whose projection coincides with $g \circ q$ by the construction of f. q.e.d.

Remark: To prove lemma ~~9~~.8 it suffices to choose any embedding $f: X \to \mathbb{R}^n$, $Y \subseteq X \subseteq \mathbb{R}^n$ and work in $f(X)$ instead of $f(V)$. /10.7

10.10 Proposition: Let $q \in Q(X,Y)$ be a surjective submersion between manifolds without boundary. Then the space $S_q(Y,X)$ of all sections of q is a splitting C^∞_c-submanifold of $C^\infty(Y,X)$.

Proof: Let $g \in S_q(Y,X)$. Then $g: Y \to X$ is an embedding and by 10.9 there exists a tubular neighbourhood $p_g: W_g \to g(Y)$ of $g(Y)$ in X such that $p_g = g \circ q|W_g$. $q|W_g$ is still a surjective submersion since $g(Y) \subseteq W_g$, W_g open. Let $\tau_g: TW_g \to W_g$ be a local addition satisfying 10.6, i.e. $g(Y) \subseteq W_g$ and each fibre $(W_g)_y$ is additively closed with respect to τ_g. Let $(U_g, \varphi_g, \Gamma_c(g^*TW_g))$ be a canonical chart of $C^\infty(Y,W_g)$ centered at g and induced by τ_g.

For $f \in U_g$ we have, by construction of τ_g: $f \in S_q(Y,W_g)$, i.e. $q \circ f = Id_Y$, iff $\varphi_g(f)(y) \in T_{g(y)}((W_g)_{g(y)}) \cong V(W_g)_{g(y)} \cong (W_g)_{g(y)}$, where $V(W_g)$ is the vertical bundle of W_g. This says that $S_q(Y,W_g) \cap U_g = \varphi_g^{-1}(\Gamma_c(g^*V(W_g)))$, and $\Gamma_c(g^*V(W_g))$ is a direct summand: $\Gamma_c(g^*V(W_g)) \oplus \Gamma_c(g^*T(g(Y))) =$
$= \Gamma_c(g^*(V(W_g)|g(Y) \oplus Tg(Y)) = \Gamma_c(g^*(TW_g|g(Y)) = \Gamma_c(g^*TW_g)$.
So $S_q(V,W_g) = S_q(Y,X) \cap C^\infty(Y,W_g)$ is a splitting C^∞_c-sub-

manifold of $C^\infty(Y,W_g)$ and $C^\infty(Y,W_g)$ is open in $C^\infty(Y,X)$. q.e.d.

10.11 Our next aim is to identify the tangent bundle of $C^\infty(X,Y)$. Again some preparations.

Let $\tau\colon TY \to Y$ be a local addition on Y, let $\varkappa = \varkappa_Y\colon T^2Y \to T^2Y$ be the canonical conjugation on Y, given locally by: $\varkappa_Y(x,y;\xi,\eta) = (x,\xi;y,\eta)$ (cf. 1.19).

<u>Lemma: In the situation above</u>, $T\tau \circ \varkappa_Y\colon T^2Y \to TY$ <u>is a local addition on</u> TY.

<u>Proof:</u> We have to check 10.1 (A1) and (A2).
$(\pi_Y,\tau)\colon TY \to Y\times Y$ is a diffeomorphism onto an open neighbourhood V of the diagonal in $Y\times Y$, so $T(\pi_Y,\tau)\colon T^2Y \to$ $\to T(Y\times Y)|V = (TY\times TY)|V$ is a diffeomorphism too. So $(\pi_{TY}, T\tau\circ\varkappa_Y) = (T\pi_Y\circ\varkappa_Y,\ T\tau\circ\varkappa_Y)$ (1.19) $= (T\pi_Y, T\tau)\circ\varkappa_Y = T(\pi_Y,\tau)\circ\varkappa_Y$ is a diffeomorphis too, and the image $(TY\times TY)|V$ is open in $TY\times TY$ and contains the diagonal. So (A1) holds.

To show (A2) we compute locally. Write $\tau\colon (x,y) \to \tau(x,y)$, $\tau(x,0) = x$.

Then $T\tau(x,y;\xi,\eta) = (\tau(x,y), d_1\tau(x,y).\xi + d_2\tau(x,y).\eta)$,
$T\tau\circ\varkappa(x,y;\xi,\eta) = (\tau(x,\xi), d_1\tau(x,\xi).y + d_2\tau(x,\xi).\eta)$.
$T\tau\circ\varkappa(x,y;0,0) = (\tau(x,0), d_1\tau(x,0).y + 0) = (x,y)$, since $\tau(x,0) = x$, so $d_1\tau(x,0) = \mathrm{Id}$. q.e.d.

<u>Remark 1:</u> Note that the image of $(T\tau\circ\varkappa)_{V_y} = T\tau\circ\varkappa|T_{V_y}TY$ contains the whole fibre T_yY in TY.

This follows from the proof of (A1) above or directly locally so:
$T\tau\circ\varkappa(x,y;0,\eta) = (\tau(x,0), d_1\tau(x,0).y + d_2\tau(x,0).\eta)$, and $d_2\tau(x,0)$ is invertible.

<u>Remark 2:</u> If ξ is a spray on Y and $\exp\xi$ its exponential map, then it can be proved that $\bar\xi = T\varkappa_Y\circ\varkappa_{TY}\circ T\xi\circ\varkappa_Y\colon$ $T^2Y \to T^3Y$ is again a spray and its exponential map is just $\exp\bar\xi = T\exp\xi\circ\varkappa_Y$ (compare with 1.20).

10.12 <u>The space</u> $\mathfrak{D}(X,TY)$:

Let X,Y be C^∞-manifolds, X with corners, Y without boundary. Let $\tau: TY \to Y$ be a local addition. Denote by $\bar{\tau} = T\tau \circ \kappa_Y: T^2Y \to TY$ the local addition investigated in 10.11. Let $0_Y: Y \to TY$ denote the zero section.

1. <u>Definition:</u> Let $\mathfrak{D}(X,TY)$ denote the space of all smooth mappings $s: X \to TY$ such that $s = 0$ off some compact in X, i.e. the space of all mappings $X \to TY$ "with compact support" With s the whole equivalence class of s is in $\mathfrak{D}(X,TY)$, so $\mathfrak{D}(X,TY)$ is a $(F\mathfrak{D})$-open subset of $C^\infty(X,TY)$ and inherits the canonical C^∞_c-manifold structure.

We will use the canonical atlas induced by the local addition $\bar{\tau}: T^2Y \to TY$.

Remark 1 of 10.11 shows that the charts $(U_{0_y \circ f}, \varphi_{0_Y \circ f}, \Gamma_c((0_Y \circ f)^*T(TY)))$ for $f \in C^\infty(X,Y)$ already cover the whole of $\mathfrak{D}(X,TY)$. We want to investigate this charts a little. Let $(U_{0_y \circ f}, \varphi_{0_Y \circ f}, \Gamma_c((0_Y \circ f)^*T(TY)))$ be such a chart, centered at $0_Y \circ f \in \mathfrak{D}(X,TY)$ for some $f \in C^\infty(X,Y)$.

2. <u>Claim:</u> $U_{0_Y \circ f} = \{s \in \mathfrak{D}(X,TY): \pi_Y \circ s \in U_f\}$, where $(U_f, \varphi_f, \Gamma_c(f^*TY))$ is the canonical chart of $C^\infty(X,Y)$, centered at f, induced by τ.

For by 10.4 we have $U_{0_Y \circ f} = \{s \in \mathfrak{D}(X,TY): (0_Y \circ f, s)(X) \subseteq$ $\subseteq \operatorname{Im}(\pi_{TY}, \bar{\tau})$ and $s \sim 0_Y \circ f\}$.

But $\operatorname{Im}(\pi_{TY}, \bar{\tau}) = \operatorname{Im}((T\pi_Y, T\tau) \circ \kappa_Y) = \operatorname{Im} T(\pi_Y, \tau) =$ $= (TY \times TY)/\operatorname{Im}(\pi_Y, \tau)$, as we saw already in the proof of 10.11. Now for $x \in X$ we have

$(0_{f(x)}, s(x)) \in (TY \times TY)/\operatorname{Im}(\pi_Y, \tau)$ iff

$(\pi_Y \times \pi_Y)(0_{f(x)}, s(x)) \in \operatorname{Im}(\pi_Y, \tau)$, i.e.

$(f, \pi_Y \circ s)(x) \in \operatorname{Im}(\pi_Y, \tau)$, and this is the case iff $\pi_Y \circ s \in U_f$, since for $s \in \mathfrak{D}(X,TY)$ clearly $s \sim 0_Y \circ f$ iff $\pi_Y \circ s \sim f$.

Now identify $\Gamma_c((0_Y \circ f)^*T(TY))$ with $\mathfrak{D}_{0_Y \circ f}(X, T(TY))$, then clearly $\mathfrak{D}_{0_Y \circ f}(X, T(TY)) = \mathfrak{D}_{0_Y \circ f}(X, T^2Y/Y)$, where

$T^2Y|Y = T^2Y|$zero section of TY.

But $T^2Y|Y$ splits canonically as $T^2Y|Y \cong TY\oplus TY$, given locally by $(y,0;b,c) \leftrightarrow ((y,b),(y,c))$ (cf. 1.15 - 1.19). So this splitting is described by the homomorphism of vector bundles over Y

$TY\oplus TY \xrightarrow{V_{TY}} V(TY) \xrightarrow{\varkappa_Y} T^2Y|Y$, where V is the vertical lift 1.15.3; its inverse is given by

$T^2Y|Y \xrightarrow{\varkappa_Y} V(TY) \xrightarrow{T_{TY}{}^{-1} = (\pi_{TY},\zeta_{TY})} TY\oplus TY$, where $\zeta_{TY}\colon V(TY)\to TY$ is the vertical projection of 1.15.4.

3. So by 1.18 we get an induced isomorphism of vector-bundles

$$\begin{tikzcd}
X\times_{(TY,0_Y\circ f,\pi_{TY})}T^2Y|Y \arrow[r, equal] \arrow[d, "S"', "(\mathrm{Id}_X\times\varkappa_Y)"] & (0_Y\circ f)^*(T^2Y/Y) \arrow[d, "f^*(\varkappa_Y)"] \\
X\times_{(Y,f,\pi_Y\circ\pi_{TY})}V(TY) \arrow[r, equal] \arrow[d, "S"', "(\mathrm{Id}_X\times {}_YV_{TY}{}^{-1})"] & f^*(V(TY)) \arrow[d, "f^*(V_{TY}{}^{-1})"] \\
X\times_{(Y,f,\pi_{TY\oplus TY})}(TY\oplus TY) \arrow[r, equal] \arrow[d, equal] & f^*(TY\oplus TY) \arrow[d, equal] \\
(X\times_{(Y,f,\pi_Y)}TY)\times_X(X\times_{(Y,f,\pi_Y)}TY) \arrow[r, phantom, "="] & (f^*TY)\oplus(f^*TY),
\end{tikzcd}$$

where we write $f^*(\varkappa_Y)$ by some abuse of notation.

But then clearly we have an isomorphism of topological vector spaces:

4. $(f^*(V_{TY}{}^{-1})\circ f^*(\varkappa_Y))_*\colon \Gamma_c((0_Y\circ f)^*(T^2Y|Y)) \to$
$\to \Gamma_c(f^*TY\oplus f^*TY) = \Gamma_c(f^*TY)\oplus\Gamma_c(f^*TY)$.

10.13 Theorem: Let X,Y be C^∞-manifolds, Y without boundary. Then $TC^\infty(X,Y)\cong\mathfrak{D}(X,TY)$ canonically as C^∞_c-manifolds and $\pi_{C^\infty(X,Y)} \cong (\pi_Y)_*\colon \mathfrak{D}(X,TY)\to C^\infty(X,Y)$.

Proof: Let $\tau\colon TY\to Y$ be a local addition; consider the canonical atlas $(U_f,\varphi_f,\Gamma_c(f^*TY))_{f\in C^\infty(X,Y)}$ of $C^\infty(X,Y)$, induced by τ. Remember the definition and the canonical atlas $(\overline{U}_f, \overline{\varphi}_f, \Gamma_c(f^*TY) \times \Gamma_c(f^*TY))$ of $TC^\infty(X,Y)$, explained in 9.3.

Now consider the canonical atlas of $\mathfrak{D}(X,TY)$, $(U_{0_Y \circ f}, \varphi_{0_Y \circ f}, \Gamma_c((0_Y \circ f)^*T^2Y/Y))_{f \in C^\infty(X,Y)}$, induced by $\overline{\tau} = T\tau \circ \varkappa_Y \colon T^2Y \to TY$. We claim that the isomorphisms $\Gamma_c((0_Y \circ f)^*T^2Y/Y) \to \Gamma_c(f^*TY) \times \Gamma_c(f^*TY)$ of 10.12.4 induce a natural identification of $\mathfrak{D}(X,TY)$ with $TC^\infty(X,Y)$.

In more detail: Let $f, g \in C^\infty(X,Y)$ with $U_f \cap U_g \neq \emptyset$. We claim that the following diagram commutes:

(1)

$$\begin{tikzcd}[column sep=small]
\overline{\varphi}_f(\overline{U}_f \cap \overline{U}_g) \arrow[rr, "\sim"', "\overline{\varphi}_g \circ (\overline{\varphi}_f)^{-1}"] \arrow[d, equal] & & \overline{\varphi}_g(\overline{U}_f \cap \overline{U}_g) \arrow[d, equal] \\
\varphi_f(U_f \cap U_g) \times \Gamma_c(f^*TY) \arrow[d, phantom, sloped, "\subset"] \arrow[dddd, dashed, bend right] & & \varphi_g(U_f \cap U_g) \times \Gamma_c(g^*TY) \arrow[d, phantom, sloped, "\subset"] \arrow[dddd, dashed, bend left] \\
\Gamma_c(f^*TY) \times \Gamma_c(f^*TY) \arrow[d, equal] & & \Gamma_c(g^*TY) \times \Gamma_c(g^*TY) \arrow[d, equal] \\
\Gamma_c(f^*TY \oplus f^*TY) \arrow[d, "\wr"', "(f^*(\varkappa_Y) \circ f^*(V_{TY}))_*"] & & \Gamma_c(g^*TY \oplus g^*TY) \arrow[d, "\wr"', "(g^*(\varkappa_Y) \circ g^*(V_{TY}))_*"] \\
\Gamma_c((0_Y \circ f)^*T^2Y/Y) & & \Gamma_c((0_Y \circ g)^*T^2Y/Y) \\
\varphi_{0_Y \circ f}(U_{0_Y \circ f} \cap U_{0_Y \circ g}) \arrow[u, phantom, sloped, "\subset"] \arrow[rr, "\sim", "\varphi_{0_Y \circ g} \circ (\varphi_{0_Y \circ f})^{-1}"'] & & \varphi_{0_Y \circ g}(U_{0_Y \circ f} \cap U_{0_Y \circ g}) \arrow[u, phantom, sloped, "\subset"]
\end{tikzcd}$$

If this is true, then we have an identification $\mathfrak{D}(X,TY) \cong TC^\infty(X,Y)$ as canonical as we can hope for.

First we check that $(f^*(\varkappa_Y) \circ f^*(V_{TY}))_*$ induces a bijection between the indicated subsets.

(2) <u>Claim:</u> Let $(r,s) \in \Gamma_c(f^*TY) \times \Gamma_c(f^*TY)$. Then $f^*(\varkappa_Y) \circ f^*(V_{TY}) \circ (r,s) \in \varphi_{0_Y \circ f}(U_{0_Y \circ f} \cap U_{0_Y \circ g})$ iff $r \in \varphi_f(U_f \cap U_g)$. $f^*(\varkappa_Y) \circ f^*(V_{TY}) \circ (r,s) \in \varphi_{0_Y \circ f}(U_{0_Y \circ f} \cap U_{0_Y \circ g})$ iff $\varphi_{0_Y \circ f}(f^*(\varkappa_Y) \circ f^*(V_{TY}) \circ (r,s)) \in U_{0_Y \circ g}$ since it is clearly in $U_{0_Y \circ f}$ for all (r,s).

This is the case, by 10.12.2, iff
$\pi_Y \circ \varphi_{O_Y \circ f}{}^{-1}(f^*(\varkappa_Y) \circ f^*(V_{TY}) \circ (r,s)) \in U_g$. Now
$\pi_Y \circ \varphi_{O_Y \circ f}{}^{-1}(f^*(\varkappa_Y) \circ f^*(V_{TY}) \circ (r,s)) =$
$= \pi_Y \circ \overline{\tau} \circ f^*(\varkappa_Y) \circ f^*(V_{TY}) \circ (r,s) =$
$= \pi_Y \circ T\tau \circ \varkappa_Y \circ \varkappa_Y \circ V_{TY} \circ (r,s) =$
$= \tau \circ \pi_{TY} \circ V_{TY} \circ (r,s) =$
$= \tau \circ r = \varphi_f{}^{-1}(r)$.

So the claim is proved. Clearly the same assertion holds for g.

Now we prove that the diagram commutes. Let $(r,s) \in \varphi_f(U_f \cap U_g) \times \Gamma_c(f^*TY)$. Then, by 9.3,
$\overline{\varphi}_g \circ (\overline{\varphi}_f)^{-1}(r,s) = (\varphi_g \circ \varphi_f{}^{-1}(r),\ D(\varphi_g \circ \varphi_f{}^{-1})(r).s) =$
$= (\tau_g{}^{-1} \circ \tau_f \circ r,\ d_F(\tau_g{}^{-1} \circ \tau_f) \circ (r,s))$ by 10.4,
$= (\tau_g{}^{-1} \circ \tau_f \circ r,\ \zeta_{g^*TY} \circ T(\tau_g{}^{-1} \circ \tau_f) \circ V_{f^*TY} \circ (r,s)$ by 1.16,
$= (\pi_{g^*TY}, \zeta_{g^*TY}) \circ T(\tau_g{}^{-1} \circ \tau_f) \circ V_{f^*TY} \circ (r,s)$
$= V_{g^*TY}{}^{-1} \circ T(\tau_g{}^{-1} \circ \tau_f) \circ V_{f^*TY} \circ (r,s)$.

(3) So $(g^*(\varkappa_Y) \circ g^*(V_{TY}))_*(\overline{\varphi}_g \circ (\overline{\varphi}_f)^{-1}(r,s)) =$
$= g^*\varkappa_Y \circ V_{g^*TY} \circ V_{g^*TY}{}^{-1} \circ T(\tau_g{}^{-1} \circ \tau_f) \circ V_{f^*TY} \circ (r,s) =$
$= g^*\varkappa_Y \circ T(\tau_g{}^{-1} \circ \tau_f) \circ V_{f^*TY} \circ (r,s)$.
$\varphi_{O_Y \circ g} \circ (\varphi_{O_Y \circ f})^{-1} \circ (f^*(\varkappa_Y) \circ f^*(V_{TY}))_*(r,s) =$
$= \overline{\tau}_{O_Y \circ g}{}^{-1} \circ \overline{\tau}_{O_Y \circ f} \circ f^*(\varkappa_Y) \circ f^*(V_{TY}) \circ (r,s)$.

(4)

$$\begin{tikzcd}[column sep=small]
f^*(TY\oplus TY) \arrow[rr, equal] \arrow[d, "f^*(V_{TY})=V_{f^*TY}", "\wr"'] & & X\times_{(Y,f,\pi_{TY\oplus TY})}(TY\oplus TY) \arrow[d, "\mathrm{Id}_X\times V_{TY}", "\wr"'] & \\
f^*(V(TY)) \arrow[rr, equal] \arrow[d, "f^*(\varkappa_Y)", "\wr"'] & & X\times_{(Y,f,\pi_Y\circ\pi_{TY})}V(TY) \arrow[d, "\mathrm{Id}_X\times\varkappa_Y", "\wr"'] & \\
(0_Y\circ f)^*(T^2Y/Y) \arrow[rr, equal] \arrow[dr, "f^*(\varkappa_Y)", "\wr"'] \arrow[dd, "\bar{\tau}_{0_Y\circ f}"'] & & X\times_{(TY,0_Y\circ f,\pi_{TY})}T^2Y/Y \arrow[d, "\mathrm{Id}_X\times\varkappa_Y", "\wr"'] \arrow[ddr, "{\mathrm{Id}_X\times\bar{\tau} = \mathrm{Id}_X\times(T\tau\circ\varkappa_Y)}"] & \\
& f^*(V(TY)) \arrow[r, equal] \arrow[dl, "(T\tau)_{0_Y\circ f}"] & X\times_{(Y,f,\pi_Y\circ\pi_{TY})}V(TY) \arrow[dr, "\mathrm{Id}_X\times T\tau"] & \\
X\times TY \arrow[rrr, equal] & & & X\times TY \\
& g^*(V(TY)) \arrow[r, equal] \arrow[ul, "(T\tau)_{0_Y\circ g}"'] & X\times_{(Y,g,\pi_Y\circ\pi_{TY})}V(T(Y)) \arrow[ur, "\mathrm{Id}_X\times T\tau"'] \arrow[d, "\mathrm{Id}_X\times\varkappa_Y", "\wr"'] & \\
(0_Y\circ g)^*(T^2Y/Y) \arrow[rr, equal] \arrow[ur, "g^*(\varkappa_Y)"', "\wr"] \arrow[uu, "\bar{\tau}_{0_Y\circ g}"] & & X\times_{(TY,0_Y\circ f,\pi_{TY})}T^2Y/Y \arrow[uur, bend right, "{\mathrm{Id}_X\times\bar{\tau} = \mathrm{Id}_X\times(T\tau\circ\varkappa_Y)}"'] &
\end{tikzcd}$$

So we may continue:

$= g^*(\varkappa_Y)\circ(T\tau)_{0_Y\circ g}{}^{-1}\circ(T\tau)_{0_Y\circ f}\circ f^*(\varkappa_Y)\circ f^*(\varkappa_Y)\circ f^*(V_{TY})\circ(r,s)$

$= g^*(\varkappa_Y)\circ(T\tau)_{0_Y\circ g}{}^{-1}\circ(T\tau)_{0_Y\circ f}\circ V_{f^*TY}\circ(r,s).$

It remains to show that

$(T\tau)_{0_Y\circ g}{}^{-1}\circ(T\tau)_{0_Y\circ f} = T(\tau_g^{-1}\circ\tau_f)\,|\,f^*V(TY).$

$$\begin{tikzcd}
f^*TY \arrow[r, "\tau_f"] \arrow[d, equal] & X\times Y \arrow[d, equal] & g^*TY \arrow[l, "\tau_g"'] \arrow[d, equal] \\
X\times_{(Y,f,\pi_Y)}TY \arrow[r, "\mathrm{Id}_X\times\tau"] & X\times Y & X\times_{(Y,g,\pi_Y)}TY \arrow[l, "\mathrm{Id}_X\times\tau"']
\end{tikzcd}$$

$$
\begin{tikzcd}
(5)\quad T(f^*TY) \arrow[r, "T(\tau_f)"] \arrow[d, equal] & T(X\times Y) \arrow[dd, equal] & T(g^*TY) \arrow[l, "T(\tau_g)"'] \arrow[d, equal] \\
T(X\times_{(Y,f,\pi_Y)}TY) \arrow[d, equal] & & T(X\times_{(Y,g,\pi\ )}TY) \arrow[d, equal] \\
TX\times_{(TY,Tf,T\pi_Y)}T^2Y \arrow[r, "Id_{TX}\times T\tau"] & TX\times TY & TX\times_{(TY,Tg,T\pi_y)}T^2Y \arrow[l, "Id_{TX}\times T\tau"'] \\
X\times_{(TY,0_y\circ f,T\pi_Y)}V(TY) \arrow[u, "0_X\times \mathrm{Incl.}"] \arrow[ur, "0_X\times T\tau"'] \arrow[d, equal] & & X\times_{(TY,0_y\circ g,T\pi_Y)}V(TY) \arrow[u, "0_X\times \mathrm{Incl.}"'] \arrow[ul, "0_X\times T\tau"] \arrow[d, equal] \\
X\times_{(Y,f,\pi_Y\circ\pi_{TY})}V(TY) \arrow[dr, "Id_X\times T\tau"] \arrow[d, equal] & & X\times_{(Y,g,\pi_Y\circ\pi_{TY})}V(TY) \arrow[dl, "Id_X\times T\tau"'] \arrow[d, equal] \\
f^*(V(TY)) \arrow[r, "(T\tau)_{0_Y\circ f}"'] & X\times TY \arrow[uuu] & g^*(V(TY)) \arrow[l, "(T\tau)_{0_Y\circ g}"]
\end{tikzcd}
$$

The above diagramm, which is clearly commutative, shows, that this is indeed the case.

So the theorem is proved (the second assertion is easily checked looking at diagram (4)). q.e.d.

10.14 Corollary: Let X,Y,Z be manifolds, without boundary, if necessary. Then the following holds:

1. If $f: Y\to Z$ is a C^∞-mapping, then $f_*: C^\infty(X,Y)\to C^\infty(X,Z)$ is C_c^∞ and its tangent mapping $T(f_*): TC^\infty(X,Y)\to TC^\infty(X,Z)$ is given by $\mathfrak{D}(X,Tf): \mathfrak{D}(X,TY)\to\mathfrak{D}(X,TZ)$.

2. if $g: Z\to X$ is a proper C^∞-mapping, then $g^*: C^\infty(X,Y)\to C^\infty(Z,Y)$ is C_c^∞ and its tangent mapping $T(g^*): TC^\infty(X,Y)\to TC^\infty(Z,Y)$ is given by $g^*=\mathfrak{D}(g,TY)$: $\mathfrak{D}(X,TY)\to\mathfrak{D}(Z,TY)$.

Proof: f_* is C_c^∞ by using the Ω-lemma 8.7 for the local representative of f_* in canonical charts. g^* is C_c^∞ since it induces continuous linear mappings between canonical charts (compare 10.4).

The form of $T(f_*)$ and $T(g^*)$ can be seen by looking at the canonical charts of $\mathfrak{D}(X,TY)$ and at the proof of 10.13; or, much easier, by applying the following lemma: see below. q.e.d.

10.15 Lemma: Let X,Y be C^∞-manifolds, Y without boundary. Let $c: \mathbb{R} \to C^\infty(X,Y)$ be a C^∞_c-curve. Then $\dot{c}(0) = \frac{d}{dt} c(t)|_{t=0} = 0$ in $T_{c(0)}C^\infty(X,Y) = \mathfrak{D}_{c(0)}(X,TY)$ iff $\frac{d}{dt} c(t,x) = 0$ in $T_{c(0)(x)}Y$ for all $x \in X$. In other words: $T_0 c = 0$ iff $T_0(ev_x \circ c) = 0$ for all $x \in X$, where $ev_x: C^\infty(X,Y) \to Y$ is evaluation at x.

Proof: $ev_x = (\hat{x})^*: C^\infty(X,Y) \to C^\infty(*,Y) = Y$, where $\hat{x}: * \to X$ is the mapping from the one-point-manifold $*$ to X with image $x \in X$. So ev_x is C^∞_c by 10.14. (It is easily seen that ev_x is continuous and linear in each chart.)

Not let $(U_{c(0)}, \varphi_{c(0)}, \Gamma_c(c(0)^*TY))$ be a canonical chart of $C^\infty(X,Y)$, centered at $c(0) \in C^\infty(X,Y)$, induced from a local addition $\tau: TY \to Y$. Then we have: $\frac{d}{dt} c(t)|_{t=0} = 0$ in $T_{c(0)}\, C^\infty(X,Y)$ iff $\frac{d}{dt} \varphi_{c(0)}(c(t))|_{t=0} = 0$ in $\Gamma_c(c(0)^*TY)$.

$\varphi_{c(0)}(c(t)) = (\pi_Y,\tau)^{-1} \circ (c(0),c(t))$. So

$$\frac{d}{dt} \varphi_{c(0)}(c(t))|_{t=0}(x) = \left[\lim_{t\to 0} \frac{\varphi_{c(0)}(c(t)) - \varphi_{c(0)}(c(0))}{t}\right](x) =$$

$= \left[\lim_{t\to 0} \frac{1}{t} \varphi_{c(0)}(c(t))\right](x)$ since $\varphi_{c(0)}(c(0))$ is the zero section.

$= \lim_{t\to 0} \frac{1}{t} \varphi_{c(0)}(c(t))(x)$, since evaluation at x is linear and continuous on $\Gamma_c(c(0)^*TY)$.

$= \lim_{t\to 0} \frac{1}{t} (\pi_Y,\tau)^{-1}(c(0,x),c(t,x))$

$= \frac{d}{dt} (\pi_Y,\tau)^{-1}(c(0,x),c(t,x))|_{t=0}$.

Since (π_Y,τ) is a diffeomorphism, this is 0 iff $\frac{d}{dt} c(t,x) = 0$ in TY. q.e.d.

Application: We compute the form of the tangent mapping of $f_*: C^\infty(X,Y) \to C^\infty(X,Z)$, i.e. we prove the rest of 10.14.1.

Let $c: \mathbb{R} \to C^\infty(X,Y)$ be a C^∞_c-curve, representing the tangent vector $\dot{c}(0) \in \mathfrak{D}_{c(0)}(X,TY) = T_{c(0)}C^\infty(X,Y)$ (each tangent vector may be represented in that form).

Then $T(f_*).\dot{c}(0) = \frac{d}{dt} f \circ c(t)|_{t=0}$.

$[T(f_*).\dot{c}(0)](x) = [\frac{d}{dt} f \circ c(t)|_{t=0}](x) =$

$= \frac{d}{dt} f(c(t,x))|_{t=0}$ by the lemma

$= Tf . \frac{d}{dt} c(t,x)|_{t=0}$

$= Tf . \dot{c}(0,x) = [Tf \circ \dot{c}(0)](x)$

So $T(f_*).\dot{c}(0) = (Tf)_*(\dot{c}(0))$. q.e.d.

Now we prove 10.14.2:

$T(g^*).\dot{c}(0) = \frac{d}{dt} g^*(c(0))|_{t=0} = \frac{d}{dt} c(0) \circ g|_{t=0}$.

$[T(g^*).\dot{c}(0)](x) = [\frac{d}{dt} c(0) \circ g|_{t=0}](x)$

$= \frac{d}{dt} c(0,g(x))|_{t=0}$ by the lemma

$= \dot{c}(0,g(x)) = [\dot{c}(0) \circ g](x)$

$= [g^*(\dot{c}(0))](x)$.

So $T(g^*).\dot{c}(0) = g^*(\dot{c}(0)) = \mathfrak{D}(g,TY).\dot{c}(0)$. q.e.d.

This method will be used a lot.

10.16 Up to now we have investigated the canonical manifold structure of $C^\infty(X,Y)$, if Y is a manifold without boundary.

1. Now let us suppose that Y is a manifold with corners too. Let τ: $TY \to Y$ be a boundary respecting (10.3) local addition on Y.

Let $f \in C^\infty(X,Y)$. Define again $U_f = \{g \in C^\infty(X,Y)$: $(f,g)(X) \subseteq (\pi_Y,\tau)({}^iTY), f \sim g\}$. This is again open in $C^\infty(X,Y)$. Define φ_f: $U_f \to \Gamma_c(f^*TY)$ by $\varphi_f(g) = (\pi_Y,\tau)^{-1} \circ (f,g) = \tau_f^{-1} \circ \Gamma_g$. This is again a continuous mapping, but $\varphi_f(U_f)$ does not coincide with the whole of $\Gamma_c(f^*TY)$. In fact,

$\varphi_f(U_f) = \{s \in \Gamma_c(f^*TY)\colon\ s(X) \subseteq f^*({}^iTY)\}$
$\qquad = \{s \in \mathfrak{D}_f(X,TY)\colon\ s(X) \subseteq {}^iTY\}$.

It can be proved, that this set is closed in $\Gamma_c(f^*TY)$. This is even a sort of quadrant in $\Gamma_c(f^*TY)$, but its boundary consists of "plane" pieces of infinite codimension, if f meets ∂Y in infinitely many points on X. We may call the chart change C^∞_c (compare with 10.4) and we may say cum grano salis, that $C^\infty(X,Y)$ is a "C^∞_c-manifold with corners".

2. We will not enter into this in full generality. But some further details are very interesting. Note first that the set $\varphi_f(U_f)$ contains a closed maximal linear subspace, ${}^t\mathfrak{D}_f(X,TY) = \mathfrak{D}_f(X,{}^tTY) = \{s \in \mathfrak{D}_f(X,TY) : s(X) \subseteq {}^tTY\}$, where we put ${}^tTY = \bigcup_{0\le j\le n} T\partial^jY \subseteq TY$. $(\partial^0 Y = Y\setminus\partial Y)$.

Let $s \in \mathfrak{D}_f(X,TY)$. Then $s \in {}^t\mathfrak{D}_f(X,TY)$ iff $s(f^{-1}(\partial^jY)) \subseteq T\partial^jY$ for each j.

So ${}^t\mathfrak{D}_f(X,TY) = \bigcap_{0\le j\le n} \bigcap_{x\in f^{-1}(\partial^jY)} \{s \in \mathfrak{D}_f(X,TY) : s(x) \in T_{f(x)}\partial^jY\}$

is indeed closed, since $s \to s(x)$ is a continuous linear functional.

How do the spaces ${}^t\mathfrak{D}_f(X,TY)$ behave under chart change? $\bar{s} := \varphi_g \circ \varphi_f^{-1}(s) = (\pi_Y,\tau)^{-1}(g,\tau\circ s)$.

Since τ is boundary respecting (10.3), the subspaces ${}^t\mathfrak{D}_f(X,TY)$ and ${}^t\mathfrak{D}_g(X,TY)$ map into each other under chart change, if we suppose that $f^{-1}(\partial^jY) = g^{-1}(\partial^jY)$ for all j. Define $C^\infty_{nice}(X,Y)$ to be the set of mappings $f \in C^\infty(X,Y)$ with $f^{-1}(\partial^jY) = \partial^jX$ for each j.

τ is boundary respecting, i.e. $\tau_y^{-1}(\partial^jY) = T_y\partial^jY$ for all $y \in \partial^jY$, for all j. So if f is nice, then $\varphi_f(U_f \cap C^\infty_{nice}(X,Y)) = {}^t\mathfrak{D}_f(X,TY)$.

So we have proved:

<u>Theorem:</u> <u>Let</u> X,Y <u>be manifolds with corners</u>. <u>Then the closed subset</u> $C^\infty_{nice}(X,Y)$ <u>of</u> $C^\infty(X,Y)$ <u>is a</u> C^∞_c-<u>manifold</u> (<u>without boundary</u>), <u>modelled on topological vector spaces of the form</u> ${}^t\mathfrak{D}_f(X,TY)$. <u>Since</u> Diff(X) <u>is open in</u> $C^\infty_{nice}(X,X)$, Diff(X) <u>is a</u> C^∞_c-<u>manifold</u> (<u>without boundary</u>) <u>too</u>.

11 Differentiability of certain mappings

11.1 Proposition: Let X,Y be C^∞-manifolds without boundary. Then for any $k \geq 0$ the mapping $j^k\colon C^\infty(X,Y) \to C^\infty(X,J^k(X,Y))$ is of class C^∞_c.

Proof: Let $\tau\colon TY \to Y$ and $\rho\colon TJ^k(X,Y) \to J^k(X,Y)$ be local additions, let $f \in C^\infty(X,Y)$. Let $(U_f, \varphi_f, \Gamma_c(f^*TY))$ be the canonical chart of $C^\infty(X,Y)$ centered at f and induced by τ; and let $(U_{j^kf}, \varphi_{j^kf}, \Gamma_c((j^kf)^*TJ^k(X,Y)))$ be the canonical chart of $C^\infty(X,J^k(X,Y))$ centered at j^kf and induced by ρ.

We have to check wether the mapping $\varphi_{j^kf} \circ j^k \circ \varphi_f^{-1}\colon \Gamma_c(f^*TY) \to \Gamma_c((j^kf)^*TJ^k(X,Y))$ (or rather a restriction to an open subset of it) is C^∞_c. For $s \in \Gamma_c(f^*TY)$ we have $\varphi_{j^kf} \circ j^k \circ \varphi_f^{-1}(s) = \rho_{j^kf}^{-1} \circ (\mathrm{Id}_X, j^k(\tau \circ s))$
$= \rho_{j^kf} \circ (\alpha, J^k(X, pr_2 \circ \tau_f)) \circ \varepsilon \circ \bar{j}^k(s)$
by the following diagram:

$$
\begin{tikzcd}
X \arrow[dr, "\bar{j}^k(s)"] \arrow[dd, "j^ks"'] \arrow[ddrr, bend left, "j^k(\tau \circ s)"] & & (j^kf)^*TJ^k(X,Y) \arrow[dd, "\rho_{j^kf}"] \\
& J^k(f^*TY) \arrow[dl, "\varepsilon"] & \\
J^k(X,f^*TY) \arrow[rr, "{(\alpha, J^k(X,pr_2 \circ \tau_f))}"] & & X \times J^k(X,Y)
\end{tikzcd}
$$

Here ε is the natural embedding of 1.12.2 and $f^*TY \xrightarrow{\tau_f} X \times Y \xrightarrow{pr_2} Y$ induces $J^k(X, pr_2 \circ \tau_f)$:

$J^k(X,f^*TY) \to J^k(X,Y)$. Now $\bar{j}^k: \Gamma_c(f^*TY) \to \Gamma_c(J^k(f^*TY))$ is continuous and linear (a linear partial differential operator) and $\rho_{j^kf}{}^{-1} \circ (\alpha, J^k(X, pr_2 \circ \tau_f)) \circ \varepsilon: J^k(f^*TY) \to$ $\to (j^kf)^*TJ^k(X,Y)$ is a smooth fibre respecting (over X) mapping, so by the chain rule and by the Ω-lemma 8.7 the mapping $\varphi_{j^kf} \circ \bar{j}^k \circ \varphi_f{}^{-1}$ is C_c^∞. q.e.d.

11.2 In order to compute $T(j^k)$ we need:

Lemma: $TJ^k(X,Y) = TX \times_{(X,\pi_X,\alpha)} J^k(X,TY)$.

Proof: If U,V are open in $\mathbb{R}^n,\mathbb{R}^m$ resp. then

$$\begin{aligned}
TJ^k(U,V) &= T(U\times V\times P(n,m)) = (U\times V\times P(n,m))\times(\mathbb{R}^n\times\mathbb{R}^m\times P(n,m))\\
&\cong (U\times\mathbb{R}^n)\times(V\times\mathbb{R}^m)\times P(n,2m)\\
&\cong (U\times\mathbb{R}^n)\times_U(U\times(V\times\mathbb{R}^m)\times P(n,2m))\\
&= TU\times_U J^k(U,TV),
\end{aligned}$$

since $L^j_{sym}(\mathbb{R}^n;\mathbb{R}^m)\times L^j_{sym}(\mathbb{R}^n;\mathbb{R}^m)\cong L^j_{sym}(\mathbb{R}^n\ \mathbb{R}^m\times\mathbb{R}^m)$ naturally. Since the above computation is natural (with respect to mappings as in 1.5, 1.6) the result holds globally too.

q.e.d.

11.3 <u>Proposition: Let</u> $j^k: C^\infty(X,Y)\to J^k(X,J^k(X,Y))$. <u>Then its tangent mapping</u> $T(j^k): TC^\infty(X,Y)\to TC^\infty(X,J^k(X,Y))$ <u>is given by the following sequence:</u>

$$\begin{tikzcd}
C^\infty(X,Y) \arrow[ddd, "j^k_{X,Y}"'] & TC^\infty(X,Y) \arrow[l, "\pi"'] \arrow[ddd, "T(j^k_{X,Y})"] \arrow[r, phantom, "\cong"] & \mathfrak{D}(X,TY) \arrow[d, "j^k_{X,TY}"] \\
 & & \mathfrak{D}(X,J^k(X,TY)) \arrow[d, "(0_X\circ\alpha,\mathrm{Id})_*"] \\
 & & \mathfrak{D}(X,TX\times_X J^k(X,TY)) \arrow[d, equal] \\
C^\infty(X,J^k(X,Y)) & TC^\infty(X,J^k(X,Y)) \arrow[l, "\pi"'] \arrow[r, phantom, "\cong"] & \mathfrak{D}(X,TJ^k(X,Y))
\end{tikzcd}$$

Proof: There is some abuse of notation in the diagram, $\mathfrak{D}(X,J^k(X,TY))$ is not defined properly. It is clear what is meant by it.

To prove it one could follow up the proof of 11.1 and could see $T(j^k_{X,Y})$ in chart-representative. We will compute explicitly using 10.15.

Let $c\colon \mathbb{R} \to C^\infty(X,Y)$ be a C^∞_c-curve. Then

$$[T_{c(0)}(j^k_{X,Y}) \cdot \dot c(0)](x) = [\tfrac{d}{dt}\, j^k_{X,Y} c(t)|_{t=0}](x) =$$
$$= \tfrac{d}{dt}\, j^k_{X,Y} c(t,.)(x)|_{t=0} = j^k_{X,TY}(\tfrac{d}{dt}\, c(t,.))(x)|_{t=0} =$$
$$= [j^k_{X,TY} \dot c(0)](x).$$

We may commute j^k and $\frac{d}{dt}$ since these are partial differential operators with respect to different variables. The rest is a question of natural embeddings. q.e.d.

11.4 <u>Theorem:</u> <u>Let</u> X <u>be a</u> C^∞-<u>manifold with corners, let</u> Y,Z <u>be</u> C^∞-<u>manifolds without boundary. Then the mapping</u> $\mathrm{Comp}\colon C^\infty(Y,Z) \times C^\infty_{prop}(X,Y) \to C^\infty(X,Z)$, <u>given by</u> $\mathrm{Comp}(g,f) = g \circ f$, <u>is a</u> C^∞_c-<u>mapping</u>.

<u>Proof:</u> Let $\tau\colon TY \to Y$ and $\rho\colon TZ \to Z$ be local additions. Let $g \in C^\infty(Y,Z)$, $f \in C^\infty_{prop}(X,Y)$ and let $(U_g, \varphi_g, \Gamma_c(g^*TZ))$ and $(U_f, \varphi_f, \Gamma_c(f^*TY))$ be the canonical charts of $C^\infty(Y,Z)$ and $C^\infty(X,Y)$, centered at g and f, induced by τ and ρ respectively. Further let $(U_{g\circ f}, \varphi_{g\circ f}, \Gamma_c((g\circ f)^*TZ))$ be the canonical chart of $C^\infty(X,Z)$ centered at $g\circ f$, induced by ρ. Let us assume that U_g and U_f are so small that $\mathrm{Comp}(U_g \times U_f) \subseteq U_{g\circ f}$ (this is possible by 7.3). Note that automatically $U_f \subseteq C^\infty_{prop}(X,Y)$ if f is proper ($h \sim f$, f proper implies h proper).

Consider the mapping

(1) $c = \varphi_{g\circ f} \circ \mathrm{Comp} \circ (\varphi_g^{-1} \times \varphi_f^{-1})\colon \Gamma_c(g^*TZ) \times \Gamma_c(f^*TY) \to$
$\to \Gamma_c((gt)^*TZ)$ (which is not globally defined – but we want to save notation).

Then

$$(2)\quad c(t,s) = \varphi_{g\circ f}(\varphi_g^{-1}(t) \circ \varphi_f^{-1}(s))$$
$$= \rho_{g\circ f}^{-1}(\mathrm{Id}_X, \rho t \tau s),$$

where $\rho_{g\circ f}\colon (gf)^*TZ \to X \times Z$ is the fibre respecting diffeo-

$$\begin{tikzcd}[column sep=small] (gf)^*TZ \arrow[rr] \arrow[dr] & & X \times Z \arrow[dl, "pr_1"] \\ & X & \end{tikzcd}$$

morphism into, given by $\rho_{g\circ f} = (\mathrm{Id}_X \times \rho)|X \times_{(Z,gf,\pi_Z)} TZ\colon (gf)^*TZ \to X \times Z$. First we investigate the partial mapping

(3) $t \to c(t,s)$, $s \in \Gamma_c(f^*TY)$ fixed, $t \in \Gamma_c(g^*TZ)$. Since $\tau \circ s$ is proper too. So the mapping

$(\tau s)^*\colon \Gamma_c(g^*TZ) \to \Gamma_c((g\tau s)^*TZ)$, $(\tau s)^*(t) = t\circ\tau\circ s$, is continuous and linear. Now consider the fibre respecting smooth diffeomorphism (not everywhere defined):

$$\begin{tikzcd}
(g\tau s)^*TZ \arrow[rr, "\rho_{g\tau s}"] \arrow[d, equal] & & X\times Z \arrow[d, equal] & & (gf)^*TZ \arrow[ll, "\rho_{gf}"'] \arrow[d, equal] \\
X\times_{(Z,g\tau s,\pi_Z)}TZ \arrow[dr, "i"] \arrow[ddrr] & & X\times Z \arrow[dd] & & X\times_{(Z,gf,\pi_Z)}TZ \arrow[ll, "Id_X\times\rho"'] \arrow[ddll] \\
& X\times TZ \arrow[ur, "Id_X\times\rho"] \arrow[dr] & & & \\
& & X & &
\end{tikzcd}$$

Then $(\rho_{gf}{}^{-1}\circ\rho_{g\tau s})_*(\tau s)^*t = \rho_{gf}{}^{-1}\circ(Id_X, pt\tau s) = c(t,s)$;
By the chain rule and by the Ω-lemma 8.7 the mapping $t\to c(t,s)$ is of class C_c^∞ and its derivative is given by

(4) $D_1 c(t,s).t' = D((\rho_{gf}{}^{-1}\circ\rho_{g\tau s})_*(\tau s)^*)(t).t' =$
$= D((\rho_{gf}{}^{-1}\circ\rho_{g\tau s})_*)((\tau s)^*t)((\tau s)^*t') =$
$= d_F(\rho_{gf}{}^{-1}\circ\rho_{g\tau s})\circ(t\tau s, t'\tau s)$ by 8.7
$= \zeta_{(gf)^*TZ}\circ T(\rho_{gf}{}^{-1}\circ\rho_{g\tau s})\circ V_{(g\tau s)^*TZ}\circ(t\tau s, t'\tau s)$ by. 1.16
$= \zeta_{(gf)^*TZ}\circ T(\rho_{gf}{}^{-1})\circ T(Id_X\times\rho)\circ(0_X, V_{TZ}\circ(p\circ t\tau s, p\circ t'\tau s))$
by 1.18, where $p\colon (g\tau s)^*TZ = X\times_Z TZ \to TZ$ is the restriction of the second projection,
$= \zeta_{(gf)^*TZ}\circ T(\rho_{gf}{}^{-1})\circ(0_X, T\rho\circ V_{TZ}\circ(pt\tau s, pt'\tau s))$. The last expression shows that $D_1 c(t,s).t'$ is jointly continuous in t,t',s (use 7.3, 7.4 and the fact that $s\to\tau s$, $\Gamma_c(f^*TY)\to C^\infty_{prop}(X,Y)$ is continuous).

Now we look at the mapping

(5) $s\to c(t,s)$, $t\in\Gamma_c(g^*TZ)$ fixed, $s\in\Gamma_c(f^*TY)$. For fixed t we define the mapping

(6) $\alpha(t)\colon f^*TY\to(gf)^*TZ$ by

$$\begin{tikzcd}
f^*TY \arrow[rrrr, "\alpha(t)"] \arrow[dr, "\tau_f"] \arrow[dd, equal] & & & & (gf)^*TZ \arrow[dl, "\rho_{gf}"'] \arrow[dd, equal] \\
& X\times Y \arrow[rr, "Id_X\times st"] \arrow[dr] & & X\times Z \arrow[dl] & \\
X\times_{(Y,f,\pi_Y)}TY \arrow[ur, "Id_X\times\tau"] \arrow[rr] & & X & & X\times_{(Z,gf,\pi_Z)}TZ \arrow[ul, "Id_X\times\rho"'] \arrow[ll]
\end{tikzcd}$$

Then $\alpha(t)_*(s) = \rho_{gf}{}^{-1} \circ (Id_X \times \rho t) \circ \tau_f \circ s$

$$= \rho_{gf}{}^{-1} \circ (Id_X, \rho t \tau s),$$

and $\alpha(t)$ is a smooth fibre respecting mapping (not everywhere defined).

So by the Ω-lemma 8.7 the mapping (5) is of class C^{∞}_c and we have

(7) $D_2 c(t,s).s' = D(\alpha(t)_*)(s).s' = d_F(\alpha(t)) \circ (s,s')$. But we are far from being done: $t \to \alpha(t)$ is not continuous in t (since τ_f does not have a closed image, so is not proper, and since $t \to Id_X \times \rho t$ is continuous iff X is compact). We will rewrite expression (7) in form where it is obvious that it is jointly continuous in t,s,s'.

First we compute as follows:

(8) $D_2 c(t,s).s' = d_F(\alpha(t)) \circ (s,s') =$

$= \zeta_{(gf)^*TZ} \circ T(\alpha(t)) \circ V_{f^*TY} \circ (s,s')$ by 1. 16

$= \zeta_{(gf)^*TZ} \circ T(\rho_{gf}{}^{-1}) \circ T(Id_X \times \rho t) \circ T(\tau_f) \circ V_{f^*TY} \circ (s,s')$

$= \zeta_{(gf)^*TZ} \circ T(\rho_{gf}{}^{-1}) \circ (0_X, T(\rho t \tau) \circ V_{TY} \circ (ps,ps'))$

where again $p\colon f^*TY = X \times_Y TY \to TY$ is the restriction of the second projection.

So it remains to show that

(9) $(t,s,s') \to T(\rho t \tau) V_{TY}(ps,ps')$ is jointly continuous.

For that we look at the manifold

$M := T^2Y \times_{(TY,\pi_{TY},\alpha)} J^1(TY,Y) \times_{(Y,\alpha)} J^1(Y,TZ) \times_{(Z,\omega,\alpha)} J^1(TZ,Z)$

Let $\gamma\colon M \to TZ$ be the C^{∞}-mapping, given by $\gamma(v,\sigma_1,\sigma_2,\sigma_3) = \sigma_3 \circ \sigma_2 \circ \sigma_1(v)$ (γ is just matrix multiplication locally). Then we have

(10) $T(\rho t \tau) V_{TY}(ps,ps') =$

$= \gamma \circ (V_{TY}(ps,ps'),\ j^1(\tau) \circ ps, j^1(t) \circ \tau ps,\ j^1(\rho) \circ t \tau ps)$

and this expression is jointly continuous in (t,s,s') ($\tau ps = \tau s$ as we have written before, $\tau\colon f^*TY \to Y$, τs is alway proper).

So we have seen that the partial derivatives $D_1 c(t,s).t'$ and $D_2 c(t,s).s'$ exist and are jointly continuous in (t,s,t',s'). So by 8.3 the mapping $(t,s) \to c(t,s)$ is a C^1_c-mapping, and $Dc(t,s)(t',s') = D_1 c(t,s)s'$. By (4) and (8), (10) the expressions $D_1c(t,s)t'$ and $D_2c(t,s)s'$ have a form similar to $c(t,s)$ itself; the part $j^1(t)$ of (10) is C^∞_c by 11.1. So by recursion c is C^2_c and C^∞_c.

q.e.d.

11.5 Remark: The question arises wether this result 11.4 remains valid if we admit corners in all manifolds. Remember 10.16 where we noted that $C^\infty_{prop}(X,Y)$ and $C^\infty(Y,Z)$ are not quite C^∞_c-manifolds since they have "corners of infinite index". The technical details of the proof offer no difficulty, since the Ω-lemma needs just convexity. We put:

Theorem: Let X,Y,Z be manifolds with corners. Then the mapping $\mathrm{Comp}\colon C^\infty_{nice}(Y,Z)\times C^\infty_{prop,nice}(X,Y)\to C^\infty_{nice}(X,Z)$ is of class C^∞_c.

Proof: Just note that $g\circ f$ is nice if g and f are nice. The proof is the same.

11.6 Corollary: Let X be a manifold with corners, let Y,Z be manifolds without boundary. Consider the C^∞_c-mapping $\mathrm{Comp}\colon C^\infty(Y,Z)\times C^\infty_{prop}(X,Y)\to C^\infty(X,Z)$; its tangent mapping $T\mathrm{Comp}\colon \mathfrak{D}(X,TZ)\times\mathfrak{D}_{prop}(X,TY)\to\mathfrak{D}(X,TZ)$ is given by

$T_{(g,f)}\mathrm{Comp}.(t,s)=(Tg)_*(s)+f^*(t)=j^1g\circ s+t\circ f.$

Proof: The finite dimensional proof for Lie-groups works here: Consider the mappings

$J_f\colon C^\infty(Y,Z)\to C^\infty(Y,Z)\times C^\infty_{prop}(X,Y)$,

$K_g\colon C^\infty_{prop}(X,Y)\to C^\infty(Y,Z)\times C^\infty_{prop}(X,Y)$, given by

$J_f(h)=(h,f)$, $K_g(h)=(g,h)$ respectively. Then clearly

$\mathrm{Comp}\circ J_f=f^*\colon C^\infty(Y,Z)\to C^\infty(X,Z)$

$\mathrm{Comp}\circ K_g=g_*\colon C^\infty_{prop}(X,Y)\to C^\infty(X,Z)$.

Therefore

$T_{(g,f)}\mathrm{Comp}.(t,s)=T_{(g,f)}\mathrm{Comp}(T_g(J_f).t+T_f(K_g).s)=$

$=T_{(g,f)}\mathrm{Comp}.T_g(J_f).t+T_{(g,f)}\mathrm{Comp}.T_f(K_g).s=$

$=T_g(f^*).t+T_f(g_*).s=f^*(t)+(T_g)_*(s)$ by 10.14. q.e.d.

Of course the same result is true in the setting of

11.5, as are all the following ones with the appropriate changes.

11.7 Corollary: Let Y be without boundary. Then evaluation $Ev: X \times C^\infty(X,Y) \to Y$, given by $Ev(x,f) = f(x)$, is a C^∞_c-mapping.

Proof: $X = C^\infty(*,X) = C^\infty_{prop}(*,X)$, where $*$ denotes the one-point-manifold. Thus
$Ev = Comp: C^\infty(X,Y) \times C^\infty_{prop}(*,X) \to C^\infty(*,Y) = Y$ is C^∞_c by 11.4. q.e.d.

11.8 Corollary: The canonical mapping
$\wedge: C^\infty_c(X, C^\infty(Y,Z)) \to (Z^Y)^X = Z^{Y\times X}$ takes values in $C^\infty(X \times Y, Z)$.

Proof: If $f \in C^\infty_c(X, C^\infty(Y,Z))$, then $\hat{f}(x,y) = f(x)(y)$, so $\hat{f} = Ev \circ (Id_Y \times f)$ is a C^∞_c-mapping $X \times Y \to Z$, i.e. a C^∞-mapping. q.e.d.
Remark: $f \to Id_Y \times f$ is not continuous in general.

11.9 Corollary: Let X,Y,Z be manifolds without boundary. Then via the canonical mapping of 11.8 we have always $C^\infty_c(X, C^\infty(Y,Z)) \subseteq C^\infty(X \times Y, Z)$; equality holds iff Y is compact or $\dim Z = 0$.

Proof: If $\dim Z = 0$ then $C^\infty(Y,Z) = Z^Y$ since any mapping into Z is smooth, and $C^\infty(Y,Z)$ is zero dimensional too (discrete). So let us assume that $\dim Z > 0$. $\subseteq$ was proved in 11.8.

Consider the mapping $\epsilon: X \to C^\infty(Y, Y\times X)$, $\epsilon(x)(y) = (y,x)$ (the insertion mapping). If Y is not compact, then ϵ is not continuous by 4.7.8. But $\hat{\epsilon} = Id_{Y\times X} \in C^\infty(Y\times X, Y\times X)$, so we have $\neq$.

Let us assume now that $\epsilon: X \to C^\infty(Y, Y\times X)$ is a C^∞_c-mapping. For any $f \in C^\infty(Y\times X, Z)$ we have $\hat{f} = f_* \circ \epsilon$: $X \to C^\infty(Y, \ )$, which is a C^∞_c-mapping then. So we have euquality for all Z.

It remains to show that ϵ is a C^∞_c-mapping, if Y is compact. We compute locally, of course: Let $x_0 \in X$ be

fixed, let $\tau: TX \to X$, $\rho: TY \to Y$ be local additions, let $(U_{\epsilon(x_o)}, \varphi_{\epsilon(x_o)}, \Gamma_c(\epsilon(x_o)^*T(Y\times X)))$ be the canonical chart of $C^\infty(Y, Y\times X)$, centered at $\epsilon(x_o)$, induced by $\rho\times\tau$. Let $(V, \tau_{x_o}^{-1}: V \to T_{x_o}X)$ be the chart at x_o of X. Since $\epsilon(x_o)(y) = (y, x_o)$ we have $\epsilon(x_o)^*T(Y\times X) = TY \times T_{x_o}X =$
$= TY \oplus (Y\times T_{x_o}X)$ as vector bundle over Y, so
$\Gamma_c(\epsilon(x_o)^*T(Y\times X)) = \Gamma_c(TY)\times\Gamma_c(Y\times T_{x_o}X) = \Gamma_c(TY)\times\mathfrak{D}(Y, T_{x_o}X)$.
For $x\in V$ and $y\in Y$ we have then

$$(\varphi_{\epsilon(x_o)}\circ\epsilon(x))(y) = (\rho\times\tau)_{\epsilon(x_o)(y)}{}^{-1}\epsilon(x)(y) =$$
$$= (\rho\times\tau)_{(y,x_o)}{}^{-1}(y,x) = (\rho_y{}^{-1}(y), \tau_{x_o}{}^{-1}(x)) = (0_y, \tau_{x_o}{}^{-1}(x)).$$

Therefore $\epsilon(V)\subseteq U_{\epsilon(x_o)}$ and
$\varphi_{\epsilon(x_o)}\circ\epsilon\circ\tau_{x_o} = (0, \mathrm{Id}): T_{x_o}X \to \Gamma_c(TY)\times T_{x_o}X \to \Gamma_c(TY)\times$ $\times\,\mathfrak{D}(Y, T_{x_o}X) = \Gamma_c(\epsilon(x)^*T(Y\times X))$, here $\mathfrak{D}(Y, T_{x_o}X)$ denotes the space of all smooth mappings with compact support $Y\to T_{x_o}X$, and $T_{x_o}X \to \mathfrak{D}(Y, T_{x_o}X)$ is a continuous linear embedding iff Y is compact (otherwise this mapping does not even taken values in $\mathfrak{D}(Y, T_{x_o}X)$ since the latter space does not contain constant mappings). q.e.d.

Remark: This result puts a definite end to all dreams of cartesian closedness in our setting. See GUTKNECHT (1977) for a slight extension of this result (Z may be a C^∞_c-manifolds too).

11.10 Corollary: Let X_1, X_2, Y_1, Y_2 be C^∞-manifolds Y_1, Y_2 without boundary.

1. If X_2 is compact, then for any fixed $f\in C^\infty(X_2, Y_2)$ the mapping $g\to g\times f$, $C^\infty(X_1, Y_1)\to C^\infty(X_1\times X_2, Y_1\times Y_2)$ is C^∞_c.

2. If X_1 and X_2 are compact, then the mapping $C^\infty(X_1, Y_1)\times C^\infty(X_2, Y_2)\to C^\infty(X_1\times X_2, Y_1\times Y_2)$, given by $(g,f)\to g\times f$, is C^∞_c.

Proof: If X_2 is compact, then the projection

$pr_1: X_1 \times X_2 \to X_1$ is smooth and proper, so
$(pr_1)^*: C^\infty(X_1,Y_1) \to C^\infty(X_1 \times X_2, Y_1)$ is C_c^∞ by 11.4, 10.14.
So $g \times f = (Id_{Y_1} \times f) \circ ((pr_1)^*(g), pr_2)$

$$= (Id_{Y_1} \times f)_*((pr_1)^*(g), pr_2) \text{ is } C_c^\infty \text{ in } g, \text{ by } 10.5$$

and 10.14.

If X_1 and X_2 are compact, then $g \times f = ((pr_1)^*(g), (pr_2)^*(f))$ is C_c^∞ in (g,f) by 11.4 or 10.14 and 10.5.

q.e.d.

11.11 <u>Theorem: Let</u> X <u>be a</u> C^∞ <u>-manifold</u> (<u>with or without corners</u>). <u>Then the inversion</u> Inv: $Diff(X) \to Diff(X)$ <u>is a</u> C_c^∞ <u>-mapping</u>. <u>So</u> Diff(X) <u>is a Lie-group in the</u> C_c^∞ <u>-sense</u>.

<u>Proof:</u> Let X have corners (for a change). It suffices to show, that Inv is C_c^∞ on an open neighbourhood U of Id_X. For if $f \in Diff(X)$ is arbitrary and g is near f then $g^{-1} = (f^{-1} \circ g)^{-1} \circ f^{-1} = [(f^{-1})^* \circ Inv|U \circ (f^{-1})_*](g)$, so Inv is C_c^∞ too on a neighbourhood of f by 11.4 (11.5).

Now let $\tau: TX \to X$ be a local addition which is boundary respecting (10.3), i.e. $\tau_x^{-1}(\partial^j X) = T_x\partial^j X$ if $x \in \partial^j X$, for all j. Let $(U = U_{Id}, \varphi = \varphi_{Id}, {}^t\Gamma_c(TX))$ be the canonical chart of $C^\infty_{nice}(X,X)$, centered at Id_X and induced by τ. We suppose that U so small as to be contained in Diff(X). Here ${}^t\Gamma_c(TX) = \{s \in \Gamma_c(TX): s(\partial^j X) \subseteq T\partial^j X \text{ for all } j\}$ is the closed subspace of all vector fields with compact support which are tangent to each boundary component $\partial^j X$.
Put

(1) $i = \varphi \circ Inv \circ \varphi^{-1}: {}^t\Gamma_c(TX) \to {}^t\Gamma_c(TX)$

(2) $c = \varphi \circ Comp \circ (\varphi^{-1} \times \varphi^{-1}): {}^t\Gamma_c(TX) \times {}^t\Gamma_c(TX) \to {}^t\Gamma_c(TX)$

(c is not everywhere defined in general).

Then $c(s,i(s)) = \varphi(Id) = 0$ for all s. Suppose that i is C_c^∞ and differentiate formally with respect to s:
$D_1c(s,i(s)) + D_2c(s,i(s)).Di(s) = 0$;
So we get the following ansatz:

(3) $Di(s) = -D_2c(s,i(s))^{-1} \bullet D_1c(s,i(s)): {}^t\Gamma_c(TX) \to {}^t\Gamma_c(TX)$.

From 11.4 (6) and (7) we know that $c(s,t) = \alpha(s)_*(t)$ and

$D_2c(s,t) = D(\alpha(s)_*(t) = (d_F\alpha(s))_*(t)$, where
$\alpha(s) = \tau_{Id}{}^{-1} \circ (Id \times \tau s) \circ \tau_{Id}\colon TX \to X \times X \to TX =$
$= (\pi_X, \tau)^{-1}(Id \times \tau s)(\pi_X, \tau)$.
$\tau s = \varphi^{-1}(s) \in Diff(X)$, so $(Id \times \tau s)$ is a diffeomorphism (when properly restricted), so $\alpha(s)$ is an invertible fibre respecting mapping (when properly restricted) and

$$\begin{aligned}\alpha(s)^{-1} &= (\pi_X,\tau)^{-1} \circ (Id \times (\varphi^{-1}(s)))^{-1} \circ (\pi_X,\tau) =\\ &= (\pi_X,\tau)^{-1} \circ (Id \times (\varphi^{-1}(s))^{-1}) \circ (\pi_X,\tau) =\\ &= (\pi_X,\tau)^{-1} \circ (Id \times \tau \circ i(s)) \circ (\pi_X,\tau) =\\ &= \alpha(i(s)).\end{aligned}$$

So $\alpha(s)_*$ is (locally) invertible and $(\alpha(s)_*)^{-1} = \alpha(i(s))_*$.

Thus $c(s,.)\colon {}^t\Gamma_c(TX) \to {}^t\Gamma_c(TX)$ is a (local) diffeomorphism and

(4) $c(s,.)^{-1} = (\alpha(s)_*)^{-1} = \alpha(i(s))_* = c(i(s),.)$. By the chain rule:

$$D_2c(s,t)^{-1} = [D(c(s,.))(t)]^{-1} = D(c(s,.)^{-1})(c(s,t)) =$$
$$= D(c(i(s),.))(c(s,t)) = D_2c(i(s),c(s,t)).$$

So (3) becomes the ansatz

$$\begin{aligned}(5)\quad Di(s)s' &= -D_2c(s,i(s))^{-1} \circ D_1c(s,i(s)).s'\\ &= -D_2c(i(s),c(s,i(s))) \circ D_1c(s,i(s)).s'\\ &= -D_2c(i(s),0_X) \circ D_1c(s,i(s)).s'.\end{aligned}$$

From 11.4 we conclude that $Di(s)s'$ is jointly continuous in s and s' (i is continuous by 7.6) if it is of the form (5). It remains to show that

$$(6)\quad \lim_{\lambda\to 0} \frac{i(s+\lambda s') - i(s)}{\lambda} = -D_2c(s,i(s))^{-1} \circ D_1c(s,i(s)).s'$$

holds in ${}^t\Gamma_c(TX)$. We will prove this in the following lemma.

Suppose that (6) holds. Then i is of class C^1_c. Look again at (5): Di is smoothly expressed in terms of i; so Di is C^1_c, so i is C^2_c; by recursion i is C^∞_c. q.e.d.

11.12 <u>Lemma: In the setting of 11.11 we have for any</u> $s,s' \in {}^t\Gamma_c(TX)$, $\lambda \in \mathbb{R}$:

$$\lim_{\lambda\to 0} \frac{i(s+\lambda s') - i(s)}{\lambda} = -D_2c(s,i(s))^{-1} \circ D_1c(s,i(s)).s'$$
in ${}^t\Gamma_c(TX)$.

Proof: By 11.11. (2) we have
$c(s+\lambda s', i(s+\lambda s')) = 0$, $c(s,i(s)) = 0$. Using lemma 8.4 we compute as follows: Let $\lambda \neq 0$, λ near zero.

$$0 = \frac{1}{\lambda} [c(s+\lambda s', i(s+\lambda s')) - c(s,i(s))] =$$
$$= \frac{1}{\lambda} \int_0^1 \frac{d}{d\mu} c(s+\mu\lambda s', i(s) + \mu(i(s+\lambda s') - i(s)))d\mu =$$
$$= \frac{1}{\lambda} \int_0^1 Dc(s+\mu\lambda s', i(s) + \mu(i(s+\lambda s') - i(s))).$$
$$. (\lambda s', i(s+\lambda s') - i(s))d\mu =$$
$$= \frac{1}{\lambda} \int_0^1 D_1c(s+\mu\lambda s', i(s) + \mu(i(s+\lambda s') - i(s))).(\lambda s')d\mu +$$
$$+ \frac{1}{\lambda} \int_0^1 D_2c(s+\mu\lambda s', i(s) + \mu(i(s+\lambda s') - i(s))).(i(s+\lambda s') -$$
$$- i(s))d\mu.$$

For $\lambda \to 0$ the first summand converges visibly to $\int_0^1 D_1c(s,i(s)).s'd\mu = D_1c(s,i(s)).s'$. Therefore we have

$$(1) \quad \lim_{\lambda\to 0} \int_0^1 D_2c(s+\mu\lambda s', i(s) + \mu(i(s+\lambda s') - i(s))).$$
$$. \frac{i(s+\lambda s') - i(s)}{\lambda} d\mu = -D_1c(s,i(s)).s' \text{ in } {}^t\Gamma_c(TX).$$

Suppose that we know already that the set
$M := \{\frac{i(s+\lambda s') - i(s)}{\lambda}: 0 < |\lambda| \leq 1, \lambda \in \mathbb{R}\}$ is bounded in ${}^t\Gamma_c(TX)$. Then M is relatively compact, since ${}^t\Gamma_c(TX)$ is a Montel space (as a closed subspace of the Montel space $\Gamma_c(TX)$). $M = M_+ \cup M_-$, where $M_+ = \{\frac{i(s+\lambda s') - i(s)}{\lambda}, 0 < \lambda \leq 1\}$, M_- is defined similarly by $-1 \leq \lambda < 0$. Then M_+ and M_- are continuous curves in ${}^t\Gamma_c(TX)$ with "one end open" each (for $\lambda \to 0$). So there are cluster points of M not lying in M.

Let t be a cluster point of M, not lying in M. Then there is a net (Moore-Smith-sequence)
$t_\alpha = (\frac{i(s+\lambda_\alpha s') - i(s)}{\lambda_\alpha})$ in M such that $\lim_\alpha t_\alpha = t$ in ${}^t\Gamma_c(TX)$ and $\lim_\alpha \lambda_\alpha = 0$. By the joint continuity of D_2c in all variables we see that

$$\lim_{\alpha} \int_0^1 D_2c(s+\mu\lambda s', i(s)+\mu(i(s+\lambda_\alpha s')-i(s)).t_\alpha d\mu =$$
$$= D_2c(s,i(s)).t.$$

But this limit equals $-D_1c(s,i(s)).s'$ by (1), so $t = -D_2c(s,i(s))^{-1}.D_1c(s,i(s)).s'$ ($D_2c(s,i(s))$ is invertible by 11.11. (4)). Since this holds for any cluster point of M for $\lambda \to 0$ we get the desired result

$$\lim_{\lambda\to 0} \frac{i(s+\lambda s')-i(s)}{\lambda} = -D_2c(s,i(s))^{-1}.D_1c(s,i(s)).s'.$$

It remains to show that M is bounded. This will be proved using (1) again. Since ${}^t\Gamma_c(TX)$ is closed in $\Gamma_c(TX)$ it suffices to show that M is bounded in $\Gamma_c(TX)$.

$\Gamma_c(TX) = \varinjlim_K \Gamma_K(TX)$ where $\Gamma_K(TX) = \{s \in \Gamma(TX) : \text{supp } s \subseteq K\}$, and where K runs through all compact sets in X (cf. 4.8). In order to show that M is bounded we have to show that the following two conditions are fulfilled:

(2) There is a compact set $K \subseteq X$ such that $\text{supp } \frac{i(s+\lambda s')-i(s)}{\lambda} \subseteq K$ for all $0 < |\lambda| \leq 1$.

(3) For any $k \geq 0$ the mapping $(\lambda, x) \to \bar{j}^k \left(\frac{i(s+\lambda s')-i(s)}{\lambda}\right)(x)$ is "uniformly bounded" in $(\lambda, x) \in ([-1,0) \cup (0,1]) \times K$ (with respect to any metric on the bundle $J^k(TX)$).

It is easy to show that (2) holds:

$\{i(s+\lambda s')-i(s) : -1 \leq \lambda \leq 1\}$ is a compact piece of a continuous curve in ${}^t\Gamma_c(TX)$, so it is bounded, so there is some compact set $K \subseteq X$ such that $\text{supp}(i(s+\lambda s')-i(s)) \subseteq K$ for all λ, $-1 \leq \lambda \leq 1$. Then clearly

$\text{supp } \frac{i(s+\lambda s')-i(s)}{\lambda} = \text{supp}(i(s+\lambda s')-i(s)) \subseteq K$ for $0 < |\lambda| \leq 1$.

It remains to show that for each k the expression (4) $\bar{j}^k \left(\frac{i(s+\lambda s')-i(s)}{\lambda}\right)(x)$ is uniformly bounded for $(x,\lambda) \in K \times ([-1,1] \setminus \{0\})$. Since K is compact it suffices to show that for any $x_0 \in K$ expression (4) is uniformly bounded for $(x,\lambda) \in U_{x_0} \times ([-1,1] \setminus \{0\})$, where U_{x_0} is a neighbourhood of x_0 in X.

Choose the neighbourhood $U = U_{x_0}$ so small that $TX|U$ is trivial, $\cong X \times \mathbb{R}^n$. So we assume that we are in an open set in $\mathbb{R}^n$: We use the same notation for the local representatives in $\mathbb{R}^n$. So $s,s': U\to\mathbb{R}^n$, $\tilde{\alpha}=\alpha(s): U\times\mathbb{R}^n\to\mathbb{R}^n$, $i(s)$, $i(s+\lambda s'): U\to\mathbb{R}^n$. Now (1) takes the following form:

$$(1')\ \lim_{\lambda\to 0}\int_0^1 d_2[\alpha(s+\mu\lambda s')](x,i(s)(x)+\mu[i(s+\lambda s')(x)- - i(s)(x)]) \cdot \frac{i(s+\lambda s')(x)-i(s)(x)}{\lambda}\,d\mu = -[D_1 c(s,i(s)).s'](x)$$

The limit is uniform with respect to $x\in U$ and any derivative with respect to x converges too. Now $d_2[\alpha(s+\mu\lambda s')](\ldots): \mathbb{R}^n\to\mathbb{R}^n$ is invertible for all $x\in\overline{U}$, $\mu\in[0,1]$, $\lambda\in[-1,1]$. So there is an $\varepsilon>0$ such that $|d_2[\alpha(s+\mu\lambda s')](\ldots).v|\geq\varepsilon.|v|$ for all those x,μ,λ, by compactness. So we get: If $\left|\frac{i(s+\lambda s')(x)-i(s)(x)}{\lambda}\right|\to\infty$, then the norm of the integral converges to ∞ too, a contradiction to (1').

Repeat this argument for each derivative with respect to x of (1') and get the desired result: (3) is true.

q.e.d.

11.13 <u>Proposition: Let</u> X <u>be a</u> C^∞-<u>manifold with or without corners</u>. <u>Then the tangent mapping</u> $T\ \mathrm{Inv}: {}^t\mathfrak{D}_{\mathrm{Diff}(X)}(X,TX)\to{}^t\mathfrak{D}_{\mathrm{Diff}(X)}(X,TX)$, <u>is given by</u> $T_f(\mathrm{Inv}).s=-(Tf^{-1}).s\circ f^{-1}=-(f^{-1})^*s=-f_*s$.

<u>Proof:</u> Again the usual finite-dimensional proof is applicable. By 11.6 we have $T_{(g,f)}\mathrm{Comp}(t,s)=Tg.s+t\circ f$. Since $\mathrm{Comp}(f,\mathrm{Inv}(f))=\mathrm{Id}$ we have (by the chain rule):

$0=T_{f,\mathrm{Inv}(f)}\mathrm{Comp}.(s,T_f\ \mathrm{Inv}.s)=Tf\circ(T_f\mathrm{Inv}.s)+s\circ\mathrm{Inv}(f)=$
$=Tf\circ(T_f\mathrm{Inv}\ s)+s\circ f^{-1}$.

So $T_f\mathrm{Inv}.s=-Tf^{-1}\circ s\circ f^{-1}$. q.e.d.

12 Some tangent mappings

From now on all manifolds are supposed to be without boundary.

12.1 Local topological affine spaces: Let (E,p,X) be a vector bundle. Consider the space $\Gamma(E)$ of all smooth sections of the vector bundle E, equipped with the $(\mathcal{FD})$-topology (4.7). We used to call two sections s_1, s_2 equivalent, $s_1 \sim s_2$, if they coincide off some compact set in X, i.e. if $s_1 - s_2$ has compact support, $s_1 - s_2 \in \Gamma_c(E)$.

Each equivalence class in $\Gamma(E)$ is a topological affine space whose model vector space is the topological vector space $\Gamma_c(E)$. So $\Gamma(E)$ is the disjoint union of topological affine spaces, whose model space is $\Gamma_c(E)$. So we call $\Gamma(E)$ a local topological affine space with model $\Gamma_c(E)$. Consequently the manifold structure of $(\Gamma(E),(\mathcal{FD}))$ is very simple: $T\Gamma(E) = \Gamma(E) \times \Gamma_c(E)$. Any mapping f from $\Gamma(E)$ into some topological vector space can be differentiated as if $\Gamma(E)$ was a topological vector space too:
$T_{s_1}(f).s_2 = Df(s_1).s_2 = \lim_{t\to 0} \frac{1}{t}\,(f(s_1+ts_2) - f(s_1))$ for any $s_1 \in \Gamma(E)$, $s_2 \in \Gamma_c(E)$, $t \in \mathbb{R}$.

12.2 1. Let $\omega \in \Gamma(\overset{k}{\otimes} T^*Y)$ be a k-times covariant tensor-field on Y. Let $P\omega\colon C^\infty(X,Y) \to \Gamma(\overset{k}{\otimes} T^*X)$ be the mapping given by $(P\omega)(f) = f^*\omega = (\omega \circ f)(Tf \otimes \dots \otimes Tf)$. The chain rule and the results of § 11 imply that $P\omega$ is a C_c^∞-mapping. So it has a tangent mapping:
$T(P\omega)\colon TC^\infty(X,Y) = \mathcal{D}(X,TY) \to T\Gamma(\overset{k}{\otimes} T^*X) = \Gamma(\overset{k}{\otimes} T^*X) \times \Gamma_c(\overset{k}{\otimes} T^*X)$.

2. Definition: Let $s \in \mathfrak{D}_f(X,TY) = T_f C^\infty(X,Y)$ be a vectorfield along f with compact support. For any $\omega \in \Gamma(\overset{k}{\otimes} T^*Y)$ we define $\mathfrak{L}_s\omega := T_f(P\omega).s$, we call $\mathfrak{L}_s\omega$ the Lie-derivative of ω along s.

3. Since $T\Gamma(\overset{k}{\otimes} T^*X) = \Gamma(\overset{k}{\otimes} T^*X) \times \Gamma_c(\overset{k}{\otimes} T^*X)$ is trivial, we may compute $\mathfrak{L}_s\omega$ as follows: Let $t \to f_t$ be any smooth curve in $C^\infty(X,Y)$ through f (i.e. $f_o = f$) such that $\frac{\partial}{\partial t} f_t = s \in \mathfrak{D}_t(X,TY)$, then $\mathfrak{L}_s\omega = T_f(P\omega).s = \frac{d}{dt}(P\omega)(f_t)|_{t=0} =$

$$= \frac{d}{dt}(f_t{}^*\omega)|_{t=0} \quad \text{(cf. 10.15)}.$$

4. Now if $f = Id_X$ and $\omega \in \Gamma(\overset{k}{\otimes} T^*X)$, then $s \in \Gamma_c(TX)$ is a vectorfield with compact support, so it has a global flow $t \to f_t$. We may use this global flow to compute $\mathfrak{L}_s\omega$ and we see that in this case we got the usual Lie-derivative of ω along the vectorfield with compact support s. Hence the name Lie-derivative.

5. It is not possible to give a more detailed expression for $\mathfrak{L}_s\omega$ in general: interpret $\omega\colon \overset{k}{\otimes} TY \to \mathbb{R}$ as a C^∞-mapping. Then clearly $\mathfrak{L}_s\omega = pr_2 \circ T\omega \circ (\frac{d}{dt} Tf_t \otimes \dots \otimes Tf_t|_{t=0})$: $\overset{k}{\otimes} TX \to T(\overset{k}{\otimes} TY) \to T\mathbb{R} = \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and the first mapping does not take values in the vertical bundle.

6. Lemma: For $\omega \in \Gamma(\overset{k}{\otimes} T^*Y)$, $\psi \in \Gamma(\overset{l}{\otimes} T^*Y)$ we have for any $s \in \mathfrak{D}_f(X,TY)$: $\mathfrak{L}_s(\omega \otimes \psi) = \mathfrak{L}_s\omega \otimes f^*\psi + f^*\omega \otimes \mathfrak{L}_s\psi$. So $\mathfrak{L}_s\colon \Gamma(\overset{k}{\otimes} T^*Y) \to \Gamma(\overset{k}{\otimes} T^*X)$ is a derivation over the algebra homomorphism $f^*\colon \Gamma(\otimes T^*Y) \to \Gamma(\otimes T^*X)$.

Proof: $\mathfrak{L}_s$ is clearly linear. Now $f^*(\omega \otimes \psi) = f^*\omega \otimes f^*\psi$, so $P(\omega \otimes \psi) = (.\otimes.) \circ (P\omega, P\psi)\colon C^\infty(X,Y) \to \Gamma(\overset{k}{\otimes} T^*X) \times \Gamma(\overset{l}{\otimes} T^*X) \to$ $\to \Gamma(\overset{k+l}{\otimes} T^*X)$ and the last mapping is "bilinear" on the product of the local topological affine spaces, so we use the remark at the end of 12.1 for the tangent mapping of $P(\omega \otimes \psi)$. q.e.d.

7. Let us finally compute a local formula for $\mathfrak{L}_s\omega$.

Let (U,u) be a chart on X, (V,v) be a chart on Y such that $f(\overline{U}) \subseteq V$. We use the same letters for the local representatives of all objects. So we assume that $\omega(y)$ is a k-linear mapping on $(\mathbb{R}^m)^k$ (where $m = \dim Y$, $n = \dim X$) for

each $y \in v(V) \subseteq \mathbb{R}^m$, we denote the action by $\langle \omega(y);\ w_1 \times \dots \times w_k \rangle$. Then locally we have for $v_i \in \mathbb{R}^n$:
$\langle ((P\omega)f)(x);\ v_1 \times \dots \times v_k \rangle =$
$= \langle \omega(f(x)); df(x).v_1 \times \dots \times df(x).v_k \rangle$. Now let $t \to f_t$ be a smooth curve through f with $\frac{d}{dt} f_t|_{t=0} = s$. Then $f_t(\overline{U}) \subseteq V$ for small t. So we have to compute:

$$\frac{d}{dt} \langle \omega(f_t(x));\ df_t(x).v_1 \times \dots \times df_t(x).v_k \rangle|_{t=0} =$$
$$= \langle d\omega(f(x)).s(x);\ df(x).v_1 \times \dots \times df(x).v_k \rangle +$$
$$+ \langle \omega(f(x));\ ds(x).v_1 \times df(x).v_2 \times \dots \times df(x).v_k \rangle +$$
$$+ \dots$$
$$+ \langle \omega(f(x));\ df(x).v_1 \times \dots \times df(x).v_{k-1} \times ds(x).v_k \rangle,$$

since $\frac{d}{dt} df_t(x).v_j|_{t=0} = d(\frac{d}{dt} f_t)(x).v_j|_{t=0} = ds(x).v_j$.

12.3 <u>Definition:</u> If $s \in C_f^\infty(X,TY)$ is a vector field along f (not necessarily with compact support) and $\omega \in \Gamma(\overset{k}{\otimes} T^*Y)$, let the <u>contraction</u> of ω along s be defined by $s \lrcorner \omega$, where
$(s \lrcorner \omega)(x)(\xi_1,\dots,\xi_{k-1}) =$
$= \omega(f(x))(s(x), T_x f.\xi_1,\dots,\overline{d}_x f.\xi_{k-1})$. It is clear that $s \lrcorner .: \Gamma(\overset{k}{\otimes} T^*Y) \to \Gamma(\overset{k-1}{\otimes} T^*X)$ is a linear (affine) mapping which is $\mathcal{F}\mathfrak{D}$-continuous iff f is proper.

12.4 <u>Lemma: Let</u> $\omega \in \Omega^p(Y)$ <u>be a differential form on</u> Y <u>and let</u> $s \in \mathfrak{D}_f(X,TY)$ <u>be a vector field along</u> f. <u>Then we have</u>:

1. $\mathfrak{L}_s(\omega \wedge \psi) = \mathfrak{L}_s\omega \wedge f^*\psi + f^*\omega \wedge \mathfrak{L}_s\psi$ <u>for any</u> $\psi \in \Omega^q(Y)$.
2. $\mathfrak{L}_s\omega = \delta(s \lrcorner \omega) + s \lrcorner (\delta\omega)$ (<u>where</u> δ <u>is exteriour differentiation on</u> X,Y).

<u>Remark:</u> If s does not have compact support we may use 2. to define $\mathfrak{L}_s\omega$ in general. This can also be done by the formula $\mathfrak{L}_s\omega = \frac{d}{dt} f_t{}^*\omega|_{t=0}$.
<u>Proof:</u> 1. is clear from 12.2.6. To prove 2. let $\xi_1,\dots,\xi_p$ be vector fields on X. Let us assume as in 12.2.7, that (U,u), (V,v) are local charts on X,Y resp. with $f(\overline{U}) \subseteq V$. We denote again local representatives by

the same letters. Then $\delta\omega$ has the following local expression (see LANG, 1972).

$$\langle \delta\omega(y);\ w_o \times \dots \times w_p \rangle =$$
$$= \sum_{i=0}^{p} (-1)^i < d\omega(y).w_i;\ w_o \times \dots \times \hat{w}_i \times \dots \times w_p \rangle$$

$s \lrcorner \delta\omega$ has the following expression:

$$\langle (s \lrcorner \delta\omega)(x);\ v_1 \times \dots \times v_p \rangle =$$
$$= \langle \delta\omega(f(x));\ s(x) \times df(x).v_1 \times \dots \times df(x).v_p \rangle =$$
$$= \langle d\omega(f(x)).s(x);\ df(x).v_1 \times \dots \times df(x).v_p \rangle +$$
$$+ \sum_{i=1}^{p} (-1)^i \langle d\omega(f(x)).df(x).v_i;\ s(x) \times df(x).v_1 \times \dots$$
$$\dots \times df(x).v_i \times \dots \times df(x).v_p \rangle.$$

$\delta(s \lrcorner \omega)$ has the following expression:

$$\langle \delta(s \lrcorner \omega)(x);\ v_1 \times \dots \times v_p \rangle =$$
$$= \sum_{i=1}^{p} (-1)^{i-1} \langle d(s \lrcorner \omega)(x).v_i;\ v_1 \times \dots \times \hat{v}_i \times \dots \times v_p \rangle =$$
$$= \sum_{i=1}^{p} (-1)^{i-1} \langle d[(\omega \circ f)(s \times \hat{df} \times \dots \times df)](x).v_i;\ v_1 \times \dots$$
$$\dots \times \hat{v}_i \times \dots \times v_p \rangle$$
$$= \sum_{i=1}^{p} (-1)^{i-1} \langle d\omega(f(x)).df(x).v_i;\ s(x) \times df(x).v_1 \times \dots$$
$$\dots \times df(x).v_i \times \dots \times df(x).v_p \rangle$$
$$+ \sum_{i=1}^{p} (-1)^{i-1} \langle \omega(f(x));\ ds(x).v_i \times df(x).v_1 \times \dots$$
$$\dots \times df(x).v_i \times \dots \times df(x).v_p \rangle$$
$$+ \sum_{i=1}^{p} \sum_{j<i} (-1)^{i-1} \langle \omega(f(x));\ s(x) \times df(x).v_1 \times \dots$$
$$\dots \times d^2f(x).(v_j,v_i) \times \dots \times df(x).v_i \times \dots \times df(x).v_p \rangle$$
$$+ \sum_{i=1}^{p} \sum_{j>i} (-1)^{i-1} \langle \omega(f(x));\ s(x) \times df(x).v_1 \times \dots \times df(x).v_i \times \dots$$
$$\dots \times d^2f(x).(v_j,v_i) \times \dots \times df(x).v_p \rangle.$$

The last two sums cancel since $d^2f(x).(v_j,v_i)$ is symmetric in i,j, and if we transport this element to the first place we get a sign $(-1)^{i-1+j-1}$ for the first sum and a sign $(-1)^{i-1+j-2}$ for the second sum.

The first sum cancels with the second sum of $s \lrcorner \delta\omega$

above. So we get the following expression:

$\langle [s \lrcorner \delta\omega + \delta(s \lrcorner \omega)](x),\ v_1 \times \dots \times v_p \rangle =$

$= \langle d\omega(f(x)).s(x);\ df(x).v_1 \times \dots \times df(x).v_p \rangle +$

$+ \sum_{i=1}^{p} (-1)^{i-1} \langle \omega(f(x));\ ds(x).v_i \times df(x).v_1 \times \dots$

$\dots \times \widehat{df(x).v_i} \times \dots \times df(x).v_p \rangle .$

Transport $ds(x).v_i$ to the i'-th place, then the sign in the last sum disappears and the local formula of 12.2.7 remains. q.e.d.

12.5 Lemma: Let $s \in \mathfrak{D}_f(X,TY)$, $g \in C^\infty(Y,Z)$ and $\omega \in \Gamma(\overset{k}{\otimes} T^*Z)$, then we have:

1. $s \lrcorner (g^*\omega) = (Tg)_* s \lrcorner \omega$, where $(Tg)_* s = Tg \circ s$.
2. $\mathfrak{L}_s(g^*\omega) = \mathfrak{L}_{(Tg)_* s}\omega$.

Proof: 1. is a trivial computation. 2. can be seen as follows:

$T_f((P\omega) \circ g_*).s = T_{gf}(P\omega) \circ T_f(g_*).s =$

$= T_{gf}(P\omega) \circ (Tg)_*.s = \mathfrak{L}_{(Tg)_* s}\omega$. But

$(P\omega) \circ g_*(f) = (P\omega)(g \circ f) = (g \circ f)^*\omega =$

$= f^* \circ g^*\omega = P(g^*\omega)(f)$, so $\mathfrak{L}_s(g^*\omega) = T_f(P(g^*\omega)).s =$

$= T_f((P\omega) \circ g_*).s = \mathfrak{L}_{(Tg)_* s}\omega$. q.e.d.

Remark: 1. is obviously true if s does not have compact support. and 2. can be shown to be true in this case too with some care:

12.6 Application: The lemma of Poincaré (J. Moser, A. Weinstein). Let (E,p,X) be a vector bundle. We want a homotopy operator I: $\Omega^p(E) \to \Omega^{p-1}(E)$.

Let M_t denote the multiplication operator with $t \in \mathbb{R}$, $M_t: E \to E$.

Let $\mu_t = \frac{d}{ds} M_s|_{s=t}$ be the vectorfield along M_t for each t.

Let $\beta \in \Omega^p(E)$ be a differential form. By the general principle of computing tangents we have by 12.2:

$\frac{d}{ds} M_s{}^*\beta|_{s=t} = T_{M_t}(P\beta).\mu_t = \mathfrak{L}_{\mu_t}\beta$ (note that the second expression is not well defined, since μ_t does not have compact support in general: the first expression however equals the last one by the computation in 12.2.7, where compact support was never used). So

$\frac{d}{dt} M_t{}^*\beta = \mathfrak{L}_{\mu_t}\beta = \delta(\mu_t \lrcorner \beta) + \mu_t \lrcorner \delta\beta$ by 12.4. Now

$M_1 = Id_E$, $M_0 : E \to E$ is the projection onto the zero section of E, so $M_0 = 0_E \circ p : E \to X \to E$.

Thus we have

$\beta - M_0{}^*\beta = \beta - p^*(0_E{}^*\beta) = \int_0^1 [\delta(\mu_t \lrcorner \beta) + \mu_t \lrcorner \delta\beta]dt.$

(We can evaluate this integral pointwise on X). Put

$I(\beta) = \int_0^1 (\mu_t \lrcorner \beta)dt.$

Then $$\boxed{\beta - M_0{}^*\beta = I(\delta\beta) + \delta(I\beta)} \quad .$$

<u>Remark:</u> If β is closed, $\delta\beta = 0$, and $0_E{}^*\beta$ is exact, then $\beta = -\delta p^*\varphi + \delta I(\beta)$. If $X = \{*\}$ is a point, then $0_E{}^*\beta = 0$ since $TX = 0$, so β closed implies $\beta = \delta I(\beta)$ on a vector space. This is the lemma of Poincaré.

12.7 Let X be a manifold without boundary. Let $(Vol(X), p, X)$ denote the line bundle of all <u>densities</u> on X, the volume bundle, which is defined by the transition mappings $\psi_{ij} : U_j \times \mathbb{R} \to U_i \times \mathbb{R}$,

$\psi_{ij}(x,a) = (x, |\det d(u_i \circ u_j{}^{-1})(u_j(x))| . a)$, where (U_i, u_i) is any atlas on X.

This bundle is always trivial, but not canonically so. If X is orientable, then $\Lambda^n T^*X$ is isomorphic to $Vol(X)$ (two isomorphisms, one for each choice of orientation). Any element $\sigma \in Vol(X)_x$ can be visualized as a "non-oriented" volume function on T_xX, assigning to each n-tuple $(\xi_1, \ldots, \xi_n)$ of vectors in T_xX ($n = \dim X$) a number $\sigma(\xi_1, \ldots, \xi_n)$ which is positively homogeneous and subadditive in each variable (sort of absolute value of a determinant function).

See DIEUDONNÉ, Vol. 7, 23.4.1-3 for further information.

12.8 Let $\Gamma(S^2T^*X)_+$ denote the space of all Riemannian metrics on X (positive definite sections of the bundle S^2T^*X of symmetric 2-tensors on X), an open "convex" subset of the local topological affine space $\Gamma(S^2T^*X)$.

For a Riemannian metric $g \in \Gamma(S^2T^*X)_+$ denote by $\mathrm{vol}(g)$ the density on X determined by g as follows: For $\xi_1,\dots,\xi_n \in T_xX$ let $\mathrm{vol}(g)_x(\xi_1,\dots,\xi_n) = \sqrt{\det(g_x(\xi_i,\xi_j)_{i,j})}$. Since g is positive definite, the determinant is always > 0 for an n-frame (ξ_i). So we get a mapping $\mathrm{vol}\colon \Gamma(S^2T^*X)_+ \to \Gamma\,\mathrm{Vol}(X)$, which is C_c^∞ since it is the composition of the following mappings: $\Gamma(S^2T^*X)_+ \to \Gamma\, P^2(F_n(TX),\, X\times\mathbb{R}^{n^2})_+$, the space of sections of the bundle of fibre respecting fibrewise quadratic mappings from the n-frame-bundle $F_n(TX)$ of TX into the trivial bundle $X\times\mathbb{R}^{n^2}$, taking positive definite values only, $\xrightarrow{\det_*} \Gamma\, P^{2n}(F_nTX,\, X\times\mathbb{R}^+) \xrightarrow{*} \Gamma\,\mathrm{Vol}(X)$. The first mapping is C_c^∞ since it is only fibrewise vector operations with g, so one may use the Ω-lemma. The rest is clear.

So there is a tangent mapping $T\,\mathrm{vol}\colon \Gamma(S^2T^*X)_+ \times \Gamma_c(S^2T^*X) \longrightarrow \Gamma\,\mathrm{Vol}(X)\times\Gamma_c\mathrm{Vol}(X)$. We want to compute this mapping. We do this locally. Let $(U,x^1,\dots,x^n)$ be a local coordinate system on X such that $g(x) = g_{ij}dx^i\otimes dx^j$. Then $dx^1\wedge\dots\wedge dx^n$ is a smooth Lebesgue measure (a density) on U and we have $\mathrm{vol}(g)(x) = \sqrt{\det(g_{ij}(x))}\,.\,dx^1\wedge\dots\wedge dx^n$ on U.

We need the derivative of the determinant function $\det\colon \mathbb{R}^{n^2}\to\mathbb{R}$. If $X = (X^i_j)\in\mathbb{R}^{n^2}$, let $C(X)$ denote the transposed matrix of the signed algebraic complements of X, so that $X.C(X) = \det(X).\mathrm{Id}$. With this notation we have: $d(\det)(X).Y = \mathrm{trace}\,(C(X)\circ Y)$. If X is invertible, then $X^{-1} = \frac{1}{\det X}\,.\,C(X)$, so $d(\det)(X).Y = \det(X).\mathrm{trace}\,(X^{-1}\circ Y)$.

Now let g be a Riemannian metric, let $k\in T_g\,\Gamma(S^2T^*X)_+ = \Gamma_c(S^2T^*X)$. Choose a smooth curve $t\to g_t$ in $\Gamma(S^2T^*X)_+$ through g with $\frac{d}{dt}\,g_t|_{t=0} = k$. By 10.15 we have

$[T_g(\mathrm{vol})k](x) = \frac{d}{dt}\,\mathrm{vol}(g_t)(x)|_{t=0}$. If x is in U, then we can continue:

$$\frac{d}{dt}\,\mathrm{vol}(g_t)(x)|_{t=0} = \frac{d}{dt}\sqrt{\det(g_t)_{ij}(x)}\; dx^1 \wedge \ldots \wedge dx^n|_{t=0} =$$
$$= \frac{1}{2\sqrt{\det(g_0)_{ij}(x)}} \,.\, d(\det)(g(x)).(\tfrac{d}{dt}\, g_t(x))_{t=0} dx^1 \wedge \ldots \wedge dx^n$$
$$= \frac{1}{2}\,\frac{}{\sqrt{\det\, g_{ij}(x)}}\,\det(g_{ij}(x)).\mathrm{trace}(g(x)^{-1} \circ k(x)). \\ .dx^1 \wedge \ldots \wedge dx^n =$$
$$= \tfrac{1}{2}\,\mathrm{trace}(g(x)^{-1} \circ k(x)).\mathrm{vol}(g)(x).$$

In the last expression the trace is indeed invariantly defined:

$T_xX \xrightarrow{g(x)} T_x{}^*X$ is invertible, so $g(x)^{-1} \circ k(x)$: $T_xX \to T_x{}^*X \to T_xX$. We have denoted the mappings associated to g with the same letter ($g^{\#}, k^{b}$ is sometimes usual).

We have shown:

12.9 <u>Theorem: The mapping</u> $\mathrm{vol}\colon \Gamma(S^2T^*X)_+ \to \Gamma\,\mathrm{Vol}(X)$ <u>is a</u> C^∞_c <u>mapping and its tangent mapping</u>
$T\,\mathrm{vol}\colon \Gamma(S^2T^*X)_+ \times \Gamma_c(S^2T^*X) \to \Gamma\,\mathrm{Vol}(X) \times \Gamma_c\,\mathrm{Vol}(X)$ <u>is given by</u> $Tg\,\mathrm{vol}.k = \frac{1}{2}\,\mathrm{trace}\,(g^{-1} \circ k).\mathrm{vol}(g)$, <u>where the trace is taken from the fibre linear mapping</u>
$$T_x{}^*X \xrightarrow{g^{-1}(x)} T_xX \xrightarrow{k(x)} T_x{}^*X.$$

12.10 <u>Remark:</u> Nearly all constructions from differential geometry have somewhere a mapping between manifolds of mappings at their base. The tangent mappings of these mappings are very interesting objects to study; many of them are already well known from variational calculus. The main example is of course the following: The space of all (linear) connections of a vector bundle is a local topological affine space with model vector space $\Gamma_c(E^* \otimes T^*X \otimes E)$. If $E = TX$, then the mapping $\nabla\colon \Gamma(S^2T^*X)_+ \to$ space of connections, which associates to each g its Levi-Civita connection ∇_g, is clearly C^∞_c. It is straightforward to compute locally its tangent mapping (as in 12.8), but the resulting formulae are very complicated and admit no obvious global interpretation.

13 The principal bundle of embeddings

13.1 Let X and Y be C^∞-manifolds, X possibly with corners, Y without boundary. Let us suppose furthermore that $\dim X < \dim Y$.

Let $E(X,Y)$ be the space of all C^∞ embeddings, $E_{prop}(X,Y)$ be the space of all proper embeddings, i.e. closed embeddings. These two spaces are open in $(C^\infty(X,Y),(\mathcal{FD}))$ (cf. 5.3, 5.4), so they are C_c^∞-manifolds.

Consider the following C_c^∞-mappings:

ρ: $\mathrm{Diff}(X) \times E(X,Y) \to E(X,Y)$, $\rho(g,i) = i \circ g$.

ρ: $\mathrm{Diff}(X) \times E_{prop}(X,Y) \to E_{prop}(X,Y)$; i.e. ρ denotes the right action of the C_c^∞-Lie-group on $E(X,Y)$, $E_{prop}(X,Y)$ respectively. Any $g \in \mathrm{Diff}(X)$ induces a C_c^∞-diffeomorphism $\rho(g,.)$ of $E(X,Y)$ and $E_{prop}(X,Y)$, whose inverse is $\rho(g^{-1},.)$. Since each element of $E(X,Y)$ is injective, the action of $\mathrm{Diff}(X)$ on $E(X,Y)$ is free: $i \circ g_1 = i \circ g_2$ for some $i \in E(X,Y)$ implies $g_1 = g_2$ in $\mathrm{Diff}(X)$.

Therefore $\rho(.,i)$: $\mathrm{Diff}(X) \to E(X,Y)$ is a bijection onto the orbit $i \circ \mathrm{Diff}(X)$ of i. If i is proper, then the whole orbit is contained in $E_{prop}(X,Y)$. (We will see later that $\rho(.,i)$ is even a diffeomorphism onto the orbit).

13.2 Definition: Let $U(X,Y) = E(X,Y)/\mathrm{Diff}(X)$ denote the orbit space, equipped with the quotient topology; let u: $E(X,Y) \to U(X,Y)$ denote the quotient mapping.

$U(X,Y)$ is, heuristically speaking, the space of all "submanifolds of type X in Y".

13.3 Lemma: Let $i \in E(X,Y)$. Write $L = i(X)$, a submanifold of Y. Then the following hold:

1. The orbit $i \circ \mathrm{Diff}(X)$ coincides with $\mathrm{Diff}(X,L)$ as subset of $E(X,Y)$.

2. The inclusion $\mathrm{Diff}(X,L) \to E(X,Y)$ is a splitting C_c^∞-submanifold.

3. The mapping $\rho(.,i)\colon \mathrm{Diff}(X) \to i \circ \mathrm{Diff}(X) = \mathrm{Diff}(X,L)$ is a C_c^∞-diffeomorphism.

4. If i is proper and X is without boundary, then the orbit of i is closed in $E_{prop}(X,Y)$.

5. If X is without boundary and has only finitely many connected components, then $\mathrm{Diff}(X,L) = E_{prop}(X,L)$.

Proof: 1. is clear. 2. follows from 10.8 and 5.7. 3. follows from 11.4. 4. Let (g_α) be a net in $\mathrm{Diff}(X)$ such that $i \circ g_\alpha = \rho(g_\alpha, i)$ converges to $f \in E_{prop}(X,Y)$. Then $i \circ g_\alpha(X) = i(X) = L$ for all α, and since L is closed in Y (i is proper) we have $f(X) \subseteq L$.

Let X_j be a connected component of X, then X_j is open and closed in X, so $f(X_j)$ is open (since f is an immersion and X is without boundary) and closed (since f is proper) in L, so $f(X_j)$ is a connected component of L.

Now let L_k be any connected component of L, then for some α_0 and some component X_j of X we have $i \circ g_\alpha(X_j) = L_k$ for all $\alpha \geq \alpha_0$. Therefore $f(X_j) = L_k$, so f is surjective, $f \in \mathrm{Diff}(X,L)$.

5. Let $X_1, \ldots, X_k$ be the connected components of X, let $f \in E_{prop}(X,L)$. From the proof of 4. we see that $f(X_1), \ldots, f(X_k)$ are different connected components of L. Since $L = i(X)$ has as many connected components as X the assertion follows. q.e.d.

13.4 Now let X,Y be both without boundary, fix $i \in E(X,Y)$ and $i(X) = L$. Let (W_L, p_L, L) be a tubular neighbourhood of L in Y (10.7).

Lemma: Let $j \in C^\infty(X, W_L)$ such that $p_L \circ j \in E(X,Y)$. Then j is an embedding with inverse $(p_L \circ j)^{-1} \circ (p_L | j(X)) : j(X) \to X$. Furthermore for any $x \in X$ we have $(T_x j)(T_x X) \oplus T_{j(x)}(p_L^{-1}(p_L j(x))) = T_{j(x)} W_L = T_{j(x)} Y$, i.e. j is transversal to the fibres of $p_L: W_L \to L$.

Proof: $p_L \circ j$ is injective, so $j: X \to W_L$ is injective, with inverse $(p_L \circ j)^{-1} \circ (p_L | j(X))$. This inverse is continuous, so j is a topological embedding. For x in X we have:
$(T_{j(x)} p_L)(T_x j)(T_x X) = T_x(p_L \circ j)(T_x X) = T_{p_L j(x)} L =$
$= (T_{j(x)} p_L)(T_{j(x)} W_L)$, thus $\dim T_x j(T_x X) \geq \dim T_{p_L j(x)} L =$
$= \dim T_x X$, so j is an immersion, so $j \in E(X,Y)$.

Furthermore the kernel of $T_{j(x)} p_L : T_{j(x)} W_L \to T_{p_L j(x)} L$ is just $T_{j(x)}(p_L^{-1}(p_L j(x)))$, the tangent space to the fibre of p_L through $j(x)$, so the second assertion follows.

q.e.d.

13.5 Let the data of 13.4 be given. Then we define:
$$Q_i = \{j \in C^\infty(X, W_L) : p_L \circ j = i,\ j \sim i\} =$$
$$= (p_L)_*^{-1}(i) \cap \{j : j \sim i\}.$$

By 13.4 we have $Q_i \subseteq E(X, W_L)$.

Lemma: 1. For the quotient mapping $u: E(X,Y) \to U(X,Y)$ we have: $u/Q_i : Q_i \to U(X,Y)$ is injective.

2. Let $V \subseteq \mathrm{Diff}(X)$ be open. Then $Q_i \circ V$ is open in $E(X,Y)$.

Proof: 1. Let $j, j' \in Q_i$, $u(j) = u(j')$, i.e. $j = j' \circ g$ for some $g \in \mathrm{Diff}(X)$. Then $i = p_L \circ j = p_L \circ (j' \circ g) = (p_L \circ j') \circ g = i \circ g$, so $g = Id_X$ and $j = j'$.

2. Let us suppose first that $V \subseteq \{g \in \mathrm{Diff}(X) : g \sim Id_X\}$, the open subgroup of diffeomorphisms with compact support.
$(p_L)_* : E(X, W_L) \to C^\infty(X, L)$ is continuous, $i \circ \mathrm{Diff}(X) =$
$= \mathrm{Diff}(X, L)$ is open in $C^\infty(X, L)$, $\rho(\,.\,, i) : \mathrm{Diff}(X) \to \mathrm{Diff}(X, L)$ is a diffeomorphism, so $(p_L)_*^{-1}(\rho(\,.\,, i)(V))$ is open in $E(X, W_L)$ and in $E(X,Y)$.

Claim: $(p_L)_*^{-1}(\rho(\,.\,, i)(V)) \cap \{j \in E(X,Y) : j \sim i\} = Q_i \circ V$. This proves the lemma in the special case.

If $j \in (p_L)_*^{-1}(\rho(.,i)(V))$ and $j \sim i$, then $p_L \circ j \in i \circ V$, so $p_L \circ j = i \circ g$ for some $g \in V$, $g \sim Id_X$. Then $j \circ g^{-1} \in E(X, W_L)$, $p_L \circ (j \circ g^{-1}) = i \circ g \circ g^{-1} = i$, $j \circ g^{-1} \sim i$, so $j \circ g^{-1} \in Q_i$ and $j = (j \circ g^{-1}) \circ g \in Q_i \circ V$.

Let conversely $j \in Q_i$, $g \in V$. Then $p_L \circ j = i$, $j \sim i$, so $p_L \circ (j \circ g) = i \circ g \in \rho(.,i)(V)$ and $j \circ g \sim i$, so $j \circ g \in (p_L)_*^{-1}(\rho(.,i)(V)) \cap \{j: j \sim i\}$, and the claim follows.

Now let V be an arbitrary open subset in Diff(X). Decompose V into the disjoint union of all non empty intersections of V with open equivalence classes $\{g \in \mathrm{Diff}(X): g \sim f\}$ for all $f \in \mathrm{Diff}(X)$. Call these non empty intersections V_α. For any α choose $g_\alpha \in V_\alpha$, then $V_\alpha \circ g_\alpha^{-1}$ is an open subset of $\{g \in \mathrm{Diff}(X), g \sim Id_X\}$, so $Q_i \circ (V_\alpha \circ g_\alpha^{-1})$ is open in $E(X,Y)$ by the first part of the proof. But then $Q_i \circ V_\alpha = \rho(g_\alpha, .)(Q_i \circ (V_\alpha \circ g_\alpha^{-1}))$ is open too and $Q_i \circ V = \bigcup_\alpha Q_i \circ V_\alpha$ also. q.e.d.

13.6 <u>Corollary:</u> $u(Q_i)$ <u>is open in</u> $U(X,Y) = E(X,Y)/\mathrm{Diff}(X)$ <u>in the quotient topology</u>.

<u>Proof:</u> By 13.5.2 the full inverse image $Q_i \circ \mathrm{Diff}(X)$ of $u(Q_i)$ under u is open in $E(X,Y)$, so $u(Q_i)$ is open in the quotient topology.

13.7 Let again X,Y be C^∞-manifolds without boundary, $i \in E(X,Y)$, $L = i(X)$, (W_L, p_L, L) a tubular neighbourhood of L in Y. Furthermore let $\tau_L: TW_L \to W_L$ be a local addition for the vector bundle W_L as constructed in lemma 10.6, i.e. the zero section $L \subseteq W_L$ and each vector subspace of each fibre is additively closed in W_L with respect to τ_L.

Decompose $TW_L|L = TL \oplus V(W_L)|L$ as in 10.12. Write $V_L = V(W_L)|L$ (we do not identify $V(W_L)|L$ with W_L itself to get more clarity).

The mapping $\tau_L|(V_L)_y: (V_L)_y = T_y(p_L^{-1}(y)) \to (W_L)_y = p_L^{-1}(y)$ is a diffeomorphism onto by the construction of τ_L in lemma 10.6, for each $y \in L$.

Therefore $\tau_L|V_L: V_L \to W_L$ is a fibre respecting diffeomorphism (onto).

13.8 Lemma: In the setting of 13.7 the set Q_i from 13.5 is a splitting C_c^∞-submanifold of $E(X,Y)$.

Proof: We will show that Q_i is a splitting C_c^∞-sumanifold of the open subset $E(W,W_L)$ of $E(X,Y)$.

Let $(U_i, \varphi_i, \Gamma_c(i^*TW_L))$ be the canonical chart of $E(X,W_L)$ centered at i, induced by the local addition τ_L from 13.7, i.e.
$U_i = \{j \in E(X,W_L): (i,j)(X) \subseteq (\pi_W, \tau_L)(TW_L),\ j \sim i\}$.
$Q_i = \{j \in E(X,W_L): p_L \circ j = i,\ j \sim i\} \subseteq U_i$ since $\tau_L | V_L: V_L \to W_L$ is a fibre respecting diffeomorphism onto (by 13.7).

$j \in Q_i$ iff $p_L \circ j = i$ and $j \sim i$, i.e. $j(x) \in (p_L)^{-1}(i(x))$. Since $(p_L)^{-1}(i(x))$ is additively closed with respect to τ_L we see that $j(x) \in p_L^{-1}(i(x))$ iff $(\tau_L)_{i(x)}^{-1}(j(x)) \in (V_L)_{i(x)}$. So for $j \in U_i$ we have:
$j \in Q_i$ iff $\varphi_i(j) = (\tau_L)_i^{-1} \circ (Id_X, j) \in \Gamma_c(i^*V_L)$. So
$\varphi_i | Q_i: Q_i \to \Gamma_c(i^*V_L)$ is a bijection and $\Gamma_c(i^*V_L)$ is a direct summand in $\Gamma_c(i^*TW_L)$ since $\Gamma_c(i^*TW_L) = \Gamma_c(i^*TW_L|L) = \Gamma_c(i^*(TL \oplus V_L)) = \Gamma_c(i^*TL \oplus i^*V_L) = \Gamma_c(i^*TL) \oplus \Gamma_c(i^*V_L)$.

q.e.d.

13.9 Lemma: Let X,Y be both without boundary. Then $U(X,Y) = E(X,Y)/Diff(X)$ is a Hausdorff space in the quotient topoloy.

Proof: Let $i,j \in E(X,Y)$ with $u(i) \neq u(j)$. Then $i(X) \neq j(X)$ in Y for otherwise put $i(X) = j(X) = L$, a submanifold of Y; $i^{-1} \circ j: X \to L \to X$ is a diffeomorphism of X and $j = i \circ (i^{-1} \circ j) \in i \circ Diff(X)$, so $u(i) = u(j)$, contrary to the assumption.

Now we distinguish two cases:

Case 1: We may find a point $y_o \in i(X) \setminus j(X)$, say, which is not a cluster point of $j(X)$. Choose an open neighbourhood V of y_o in Y and an open neighbourhhod W of $j(X)$ in Y such that $V \cap W = \emptyset$. Let $\mathfrak{B} = \{k \in E(X,Y): k(X) \cap V \neq \emptyset\}$ and $\mathfrak{W} = \{k \in E(X,Y): k(X) \subseteq W\}$. Then $\mathfrak{W}$ is visibly WO-open and $\mathfrak{B}$ is CO^o-open (if $k \in \mathfrak{B}$ choose $x \in X$ with $k(x) \in V$. Then

$\{l \in E(X,Y): l(x) \in V\}$ is a CO^0-open neighbourhood of k in $\mathfrak{B}$). Furthermore $\mathfrak{B}$ and $\mathfrak{W}$ are Diff(X)-saturated, $i \in \mathfrak{B}$, $j \in \mathfrak{W}$, and $\mathfrak{B} \cap \mathfrak{W} = \emptyset$. So $u(\mathfrak{B})$ and $u(\mathfrak{W})$ separate $u(i)$ and $u(j)$ in $U(X,Y)$.

Case 2: $i(X) \subseteq \overline{j(X)}$, $j(X) \subseteq \overline{i(X)}$. Let $y \in i(X)$ for instance. Let (V,v) be a chart of Y centered at y which maps $i(X) \cap V$ into a linear subspace, $v(i(X) \cap V) \subseteq \mathbb{R}^n \cap v(V) \subseteq \mathbb{R}^m$ ($n = \dim X$, $m = \dim Y$). Since $j(X) \subseteq \overline{i(X)}$ we conclude that $v((i(X) \cup j(X)) \cap V) \subseteq \mathbb{R}^n \cap v(V)$ too. So we see that $i(X) \cup j(X)$ is a submanifold of Y of the same dimension as X.

Put $M := i(X) \cup j(X)$. Let (W_M, p_M, M) be a tubular neighbourhood of M in Y. Then $W_M | i(X)$ is a tubular neighbourhood of $i(X)$ in Y, $W_M | j(X)$ is one of $j(X)$. Let Q_i, Q_j be defined as in 13.5, using these tubular neighbourhoods of $i(X)$, $j(X)$. There is some $y_0 \in i(X) \setminus j(X)$, say. By 13.5.2 $Q_i \circ \mathrm{Diff}(X)$ and $Q_j \circ \mathrm{Diff}(X)$ are open and Diff(X) saturated in $E(X,Y)$, containing i and j resp., and $Q_i \circ \mathrm{Diff}(X) \cap Q_j \circ \mathrm{Diff}(X) = \emptyset$ since for any $k \in Q_i \circ \mathrm{Diff}(X)$ the set $k(X)$ meets $p_M^{-1}(y_0)$, and for all $k \in Q_j \circ \mathrm{Diff}(X)$ it does not. So $u(Q_i) \cap u(Q_j) = \emptyset$ in $U(X,Y)$, they are open neighbourhoods separating $u(i)$ and $u(j)$. q.e.d.

13.10 Corollary: Each orbit $i \circ \mathrm{Diff}(X)$ is closed in $E(X,Y)$. (This is better than 13.3.4).

13.11 We make now a first assault on the fibre bundle structure of $E(X,Y)$. Let X,Y be C^∞-manifolds without boundary, $i \in E(X,Y)$, $L = i(X)$. Write $\hat{i} := u(i) \in U(X,Y)$. Then $\hat{Q}_i := u(Q_i)$ is an open neighbourhood of $\hat{i}$ in $U(X,Y)$. We will show that $E(X,Y) | \hat{Q}_i$ is trivial.

1. Define $s_i : \hat{Q}_i \to E(X,Y)$ by $s_i = (u | Q_i)^{-1}$. s_i is well defined, since $u | Q_i$ is injective (13.5.1). So s_i is a local section of u.

2. Then fibres of $u: E(X,Y) \to U(X,Y)$ (i.e. the Diff(X)-orbits) over $\hat{Q}_i$ meet Q_i in exactly one point each (13.5.1). Since the action ρ of Diff(X) on $E(X,Y)$ is free,

the mapping $\rho|\mathrm{Diff}(X)\times Q_i$: $\mathrm{Diff}(X)\times Q_i \to u^{-1}(\hat{Q}_i)$ is bijective, so there is an inverse mapping
$(\rho|\mathrm{Diff}(X)\times Q_i)^{-1} = (\gamma_i,\delta_i)$: $u^{-1}(\hat{Q}_i)\to \mathrm{Diff}(X)\times Q_i$; so
γ_i: $u^{-1}(\hat{Q}_i)\to\mathrm{Diff}(X)$, δ_i: $u^{-1}(\hat{Q}_i)\to Q_i$ and we have
$\delta_i(j)\circ\gamma_i(j) = \rho(\gamma_i(j),\delta_i(j)) = j$ for each $j\in u^{-1}(\hat{Q}_i)$, and
$\delta_i(j)\sim i$, $p_L\delta_i(j) = i$.

3. <u>Claim:</u> γ_i: $u^{-1}(\hat{Q}_i)\to\mathrm{Diff}(X)$ is a C_c^∞-mapping. We have $i\circ\gamma_i(j) = p_L\circ\delta_i(j)\circ\gamma_i(j) = p_L\circ j$ (so $p_L\circ j$ is defined), so $\gamma_i(j) = \rho(.,i)^{-1}\circ(p_L)_*(j)$ or
$\gamma_i \equiv \rho(.,i)^{-1}\circ(p_L)_*$: $u^{-1}(\hat{Q}_i)\to\mathrm{Diff}(X)$ which is C_c^∞.

4. <u>Claim:</u> δ_i: $u^{-1}(\hat{Q}_i)\to Q_i$ is C_c^∞. We have
$\delta_i(j)\circ\gamma_i(j) = j$, so $\delta_i(j) = j\circ\gamma_i(j)^{-1}$, so
$\delta_i = \rho\circ(\mathrm{Inv}\circ\gamma_i,\mathrm{Id})$: $u^{-1}(\hat{Q}_i)\to Q_i$, which is C_c^∞.

5. Therefore ρ: $\mathrm{Diff}(X)\times Q_i\to u^{-1}(\hat{Q}_i)$ is a C_c^∞-diffeomorphism. This mapping will serve as trivialising mapping.

6. <u>Claim:</u> s_i: $\hat{Q}_i\to Q_i$ (from 1.) is continuous (so a homeomorphism).

For $\hat{y}\in\hat{Q}_i$ we have $\{s_i(\hat{y})\} = \delta_i(u^{-1}(\hat{y}))$ by constructions. Let $V\subseteq Q_i$ be open, then $\delta_i{}^{-1}(V)$ is open in $u^{-1}(\hat{Q}_i)$ by 4. $u^{-1}(\hat{Q}_i)$ is open in $E(X,Y)$, so $u^{-1}(s_i{}^{-1}(V)) = u^{-1}(u(V)) =$ $= \delta_i{}^{-1}(V)$ is open in $E(X,Y)$. By definition of the quotient topology $s_i{}^{-1}(V)$ is open in $U(X,Y)$.

We have proved the following:

<u>Theorem: Let</u> X,Y <u>be</u> C^∞-<u>manifolds without boundary,</u> $\dim X<\dim Y$. <u>Then</u> $(E(X,Y),\ u,\ U(X,Y),\ \mathrm{Diff}(X))$ <u>is a topological principal fibre bundle</u>, <u>trivial over the open neighbourhoods</u> $\hat{Q}_i$ <u>of</u> $\hat{i}$ <u>in</u> $U(X,Y)$ <u>for each</u> $i\in E(X,Y)$; <u>a trivializing mapping is given by</u>:
$\mathrm{Diff}(X)\times\hat{Q}_i\to u^{-1}(\hat{Q}_i)$, $(g,\hat{y})\to s_i(\hat{y})\circ g$.

13.12 <u>Theorem: In the setting of</u> 13.7, $U(X,Y)$ <u>is a</u> C_c^∞-<u>manifold</u>.

<u>Proof:</u> For any $i\in E(X,Y)$ the open neighbourhood $\hat{Q}_i$ of $\hat{i}$ in $U(X,Y)$ is homeomorphic to the splitting C_c^∞-submanifold Q_i of $E(X,Y)$ (cf. 13.8, 13.11.6); so it remains to check whether these submanifolds fit together nicely.

In other words: We use the mappings $(\varphi_i|Q_i)\circ s_i\colon \hat{Q}_i \to \Gamma_c(i^*V_L)$ (in the notation of 13.7, 13.8) as charts for $U(X,Y)$. $U(X,Y)$ is a Hausdorff space by 13.9. So it remains to check wether the chart change is C_c^∞. Let $i,k \in E(X,Y)$ so that $\hat{Q}_i \cap \hat{Q}_k \neq \emptyset$.

Suppose first that i and k lie on the same $\mathrm{Diff}(X)$-orbit in $E(X,Y)$, i.e. there is some $g \in \mathrm{Diff}(X)$ with $i = k\circ g$.

Then $L = i(X) = k(X)$ in Y and

$$\begin{aligned}
Q_i &= \{j \in E(X,W_L)\colon p_L\circ j = i,\ j\sim i\} = \\
&= \{j \in E(X,W_L)\colon p_L\circ j = k\circ g,\ j\sim k\circ g\} = \\
&= \{j\circ g\colon j\in E(X,W_L),\ p_L\circ j = k,\ j\sim k\} = \\
&= Q_k\circ g = \rho(g,.)(Q_k).
\end{aligned}$$

So Q_i and Q_k are $\mathrm{Diff}(X)$-translates of each other, $\hat{Q}_i = \hat{Q}_j$ and we have $((\varphi_k|Q_k)\circ s_k)\circ((\varphi_i|Q_i)\circ s_i)^{-1} =$
$= (\varphi_k|Q_k)\circ s_k\circ (u|Q_i)\circ(\varphi_i|Q_i)^{-1} =$
$= (\varphi_k|Q_k)\circ(\rho(g,.)|Q_i)\circ(\varphi_i|Q_i)^{-1}$.

The last mapping is a C_c^∞-diffeomorphism by 13.1 and 13.7.

Now consider $i,k\in E(X,Y)$ with $\hat{Q}_i\cap\hat{Q}_k \neq \emptyset$, but not lying on the same orbit. Let $L = i(X)$, $K = k(X)$. Then $L\neq K$ (13.9). We have $s_k(\hat{Q}_i\cap\hat{Q}_k) = s_k(\hat{Q}_k)\cap u^{-1}(\hat{Q}_i) =$
$= Q_k \cap u^{-1}(\hat{Q}_i)$ by construction.

For $j\in Q_k$ we have $p_K\circ j = k$ and $j\sim k$, so $j = \tau_K\circ t =$
$= \varphi_k^{-1}(t)$ for some $t\in\Gamma_c(k^*V_K)$. If furthermore $j\in u^{-1}(\hat{Q}_i)$, then $j = \delta_i(j)\circ\gamma_i(j)$ for $\delta_i(j)\in Q_i$ and $\gamma_i(j)\in\mathrm{Diff}(X)$, by 13.11. If $t\in(\varphi_k|Q_k)\circ s_k(\hat{Q}_i\cap\hat{Q}_k)\subseteq\Gamma_c(k^*V_K)$, then
$((\varphi_i|Q_i)\circ s_i)\circ((\varphi_k|Q_k)\circ s_k)^{-1}(t) =$
$= (\varphi_i|Q_i)\circ s_i\circ s_k^{-1}\circ(\varphi_k|Q_k)^{-1}(t) = (\varphi_i|Q_i)\circ s_i\circ u(j) =$
$= (\varphi_i|Q_i)(s_i(\hat{j})) = (\varphi_i|Q_i)(\delta_i(j))$ (cf. 13.11.6)
$= (\varphi_i|Q_i)\circ\delta_i\circ(\varphi_k|Q_k)^{-1}(t)$.

The last expression is C_c^∞ by 13.8 and 13.11.4. q.e.d.

13.13 <u>Lemma:</u> <u>In the setting of</u> 13.7, <u>the mapping</u> $u\colon E(X,Y)\to U(X,Y)$ <u>is a submersion</u>, <u>i.e. for each</u> $i\in E(X,Y)$ <u>the mapping</u> $T_i u\colon T_iE(X,Y)\cong\mathfrak{D}_i(X,TY)\to T_{\hat{i}}\,U(X,Y)\cong\Gamma_c(i^*V_L)$

is surjective and moreover a topological linear quotient mapping with splitting kernel.

Proof: The kernel of $T_i u$ is $T_i\ (i \circ \mathrm{Diff}(X)) = \Gamma_c(i^*TL)$, and this is a splitting subspace of $\Gamma_c(i^*TY)$ as we already proved in 13.8.

The rest follows from the construction of the canonical charts for $U(X,Y)$. q.e.d.

13.14 Theorem: Let X,Y be C^∞-manifolds without boundary, $\dim X < \dim Y$. Then $(E(X,Y),u,U(X,Y),\ \mathrm{Diff}(X))$ is a C^∞_c principal fibre bundle.

Proof: $s_i : \hat{Q}_i \to Q_i$ is a C^∞_c-diffeomorphism by construction of the charts for $U(X,Y)$. Then the mappings $\mathrm{Diff}(X) \times \hat{Q}_i \to u^{-1}(\hat{Q}_i)$, $(g,\hat{y}) \to s_i(\hat{y}) \circ g$ are C^∞_c diffeomorphisms defining the local product structure of the principal bundle. q.e.d.

13.15 Let $U_{prop}(X,Y) = u(E_{prop}(X,Y))$ denote the space of all proper orbits (i.e. to say the "space of all closed submanifolds of type X in Y").

Corollary: $(E_{prop}(X,Y),u,U_{prop}(X,Y),\ \mathrm{Diff}(X))$ is a C^∞_c-principal fibre bundle too, in fact $E_{prop}(X,Y) = E(X,Y)/U_{prop}(X,Y)$, the restriction of the principal fibre bundle $E(X,Y)$ to the open subset $U_{prop}(X,Y)$ of $U(X,Y)$.

13.16 Remark: 1. If X has corners, then $Q_i \circ \mathrm{Diff}(X)$ is no longer open in $E(X,Y)$, so 13.5.2 does not hold: one may stretch or shrink $i(X)$ in Y by moving the corners of $i(X)$ tangentially to $i(X)$ outwardly or inwardly. This is a continuous curve in $E(X,Y)$ which cannot be absorbed into a local product structure.

2. If X does not have corners and $\dim X = \dim Y$, then each orbit $\mathrm{Diff}(X,L) = i \circ \mathrm{Diff}(X)$, $i(X) = L$, is open in $E(X,Y)$, so $U(X,Y)$ is discrete. Thus $(E(X,Y),u,U(X,Y),\ \mathrm{Diff}(X))$ is a C^∞_c principal fibre bundle too, but trivially so.

14 Lie groups of symplectic diffeomorphisms

This section is based on ideas of A. WEINSTEIN (1971).

14.1 Let X be a smooth C^{∞}-manifold without boundary and let Ω be a symplectic structure on X (i.e. a non degenerate closed 2-form on X). We say, that (X,Ω) is a <u>symplectic manifold</u>.

The dimension of X is necessarily even.

First we need to state some wellknown facts:
An <u>isotropic submanifold</u> $Y \subseteq X$ is a submanifold Y such that $\Omega|TY = 0$; it follows that $\dim Y \leq \frac{1}{2} \dim X$. A <u>Lagrangian submanifold</u> is a maximal isotopic submanifold; each isotropic submanifold is contained in a Lagrangian one, and an isotropic submanifold Y is Lagrangian iff $\dim Y = \frac{1}{2} \dim X$.

14.2 If (X,Ω) is a symplectic manifold, let $\tau\colon T^*X \to X$ be a <u>"local addition"</u> (these are defined only for $TX \to X$, but carry such one to T^*X via an identification $T^*X \cong TX$, induced by a Riemannian metric on X or by the symplectic form). Then $\tau\colon T^*X \to X$ has the following properties:

(A1) $(\pi_X{}^*,\tau)\colon T^*X \to X \times X$ is a diffeomorphism onto an open neighbourhood of the diagonal Δ_X in $X \times X$

(A2) $\tau(0_x) = x$ for all $x \in X$.

Now consider the symplectic structure $\Omega^X := pr_1{}^*\Omega - pr_2{}^*\Omega$ on $X \times X$, where $pr_1\colon X \times X \to X$, $pr_2\colon X \times X \to X$ are the first and second projection resp.

We consider two symplectic structures on T^*X: the first one is $\tilde{\Omega}:=(\pi_X^*,\tau)^*\Omega^x$.

The second one is the canonical symplectic structure ω on T^*X.

For completeness sake we repeat its construction:

Denote by α the canonical 1-form on T^*X, which is characterized by the following property: If $s\in\Gamma(T^*X)$ is any 1-form on X, then $s^*\alpha=s$. α can be defined directly as follows: Let $\xi\in T_\eta(T^*X)$ for $\eta\in T^*X$, then $(\xi,\alpha(\eta))=$ $=(T_\eta(\pi_X^*).\xi,\eta)$, where π_X^*: $T^*X\to X$ is the canonical projection.

If $s\in\Gamma(T^*X)$, $\zeta\in T_xX$, then $(\zeta,s^*\alpha(x))=$

$$=(\zeta,(T_xs)^*\alpha(s(x)))=(T_xs.\zeta,\alpha(s(x)))=(T_{s(x)}(\pi_X^*).T_xs.\zeta,\ s(x))=$$

$$=(T_x(\pi_X^*\circ s).\zeta,s(x))=(\zeta,s(x));$$ therefore $s^*\alpha=s$.

Having constructed the 1-form α on T^*X we define $\omega=-\delta\alpha$.

Now let us denote the zero section of T^*X by Z_X. Then $\alpha|TZ_X=0$, so $\omega|TZ_X=0$. Furthermore $\Omega^x|T\Delta_X=$ $=(pr_1^*\Omega-pr_2^*\Omega)|T\Delta_X=0$ and (π_X^*,τ): $Z_X\to\Delta_X$, so $\tilde{\Omega}$, $TZ_X=((\pi_X^*,\tau)^*\Omega^x)|TZ_X=((\pi_X^*,\tau)|Z_X)^*(\Omega^x|T\Delta_X)=0$.

Let us summarize this discussion in the following lemma.

Lemma: Let (X,Ω) be a symplectic manifold, let τ: $T^*X\to X$ be a local addition. Put $\Omega^x=pr_1^*\Omega-pr_2^*\Omega$: this is a symplectic structure on $X\times X$, and let $\tilde{\Omega}=(\pi_X^*,\tau)^*\Omega^x$. Then $\tilde{\Omega}$ is a symplectic structure on T^*X and $\tilde{\Omega}|TZ_X=0$. We have $\omega|TZ_X=0$ too for the canonical symplectic structure on T^*X.

14.3 Now we will construct a local diffeomorphism f: $T^*X\to T^*X$ with $f|Z_X=Id$ such that $f^*\omega=\tilde{\Omega}$. So f: $(T^*X,\tilde{\Omega})\to(T^*X,\omega)$ will be a symplectomorphism, $f|Z_X=Id_{Z_X}$.

First we solve the problem in $T(T^*X)|Z_X$. Z_X is a Lagrangian submanifold for each of the two symplectic structures, $\omega, \tilde{\Omega}$, on T^*X. By linear algebra of symplectic linear spaces there is a vector bundle isomorphism $\gamma\colon T(T^*X)|Z_X \to T(T^*X)|Z_X$ over the identity on Z_X mapping the symplectic structure $\tilde{\Omega}$ (on each fibre) to the symplectic structure ω and leaving TZ_X pointwise fixed (on TZ_X both structures vanish).

There is a diffeomorphism $h\colon U \to V$ between open neighbourhoods U, V of Z_X in T^*X such that $T_\alpha h = \gamma_\alpha$ for $\alpha \in Z_X$ (so even $h|Z_X = \mathrm{Id}$). This implies $h^*(\omega|Z_X) = \tilde{\Omega}|Z_X$. Such a diffeomorphism may be constructed using a tubular neighbourhood of $Z_X{}^*$ in T^*X. Put $\bar{\Omega} = (h^{-1})^*(\tilde{\Omega}|U)$. This is a symplectic structure on V such that $\bar{\Omega}|Z_X = \omega|Z_X$.

Now we solve the problem in a neighbourhood of Z_X. Put $\bar{\omega} = \bar{\Omega} - \omega$ on V.
$\omega_t = (1-t)\omega + t\bar{\Omega} = \omega + t\bar{\omega}$, $t \in \mathbb{R}$. ω_t is a 2-form for all $t \in \mathbb{R}$. $\delta\omega_t = (1-t)\delta\omega + t\delta\bar{\Omega} = 0$. $\omega_t|Z_X = \omega|Z_X = \bar{\Omega}|Z_X$ for all t, so ω_t is non degenerate in the fibres over Z_X. So all ω_t for $t \in [0,1]$ are non degenerate in the fibres over an open fibrewise convex neighbourhood $W \subseteq V$ of Z_X in T^*X, so $\omega_t{}^{\#}\colon T_\alpha(T^*X) \to T_\alpha{}^*(T^*X)$ is invertible for all $\alpha \in W$ and $t \in [0,1]$, where $(\xi, \omega_t{}^{\#}(\zeta)) = \omega_t(\xi, \zeta)$, $\xi, \zeta \in T_\alpha(T^*X)$. Let $I\colon \Omega^2(W) \to \Omega^1(W)$ be the homotopy operator constructed in 12.6, put $\varphi = I(\bar{\omega}) \in \Omega^1(W)$.

Since $\bar{\omega}|Z_X = 0$ (even $\bar{\omega}|TZ_X = 0$ would suffice) we have $\bar{\omega} = I(\delta\bar{\omega}) + \delta I(\bar{\omega}) = 0 + \delta\varphi$ by 12.6. Put $\xi_t = -(\omega_t{}^{\#})^{-1}.\varphi$, then ξ_t is a time dependent vector field on W. Let g_t denote the local flow of ξ_t, i.e. $\frac{d}{dt}\, g_t = \xi_t \circ g_t$.

Since $\bar{\omega}|Z_X = 0$ (here we need it!) we have $\xi_t|Z_X = 0$ (see 12.6), so there is a neighbourhood W_1 of Z_X in W such that g_t exists for $t \in [0,1]$ in W_1.

Now we compute:

$$\frac{d}{ds}\,(g_s{}^*\omega_s)|_{s=t} = \frac{d}{ds}\,(g_s{}^*\omega_t)|_{s=t} + \frac{d}{ds}\,(g_t{}^*\omega_s)|_{s=t} =$$
$$= \mathfrak{L}_{\xi_t \circ g_t}\omega_t + g_t{}^*\,(\frac{d}{ds}\,\omega_s)|_{s=t}$$

by 12.2 (note that $\xi_t \circ g_t$ does not have compact support). $g_t{}^*$ commutes with $\frac{d}{ds}$ since $g_t{}^*$ acts linearly and continuously on $\Omega^2(W_1)$.

$= \delta(\xi_t \circ g_t \lrcorner \omega_t) + \xi_t \circ g_t \lrcorner \delta\omega_t$ by 12.4.2

$+ g_t^* \overline{\omega}$

$= g_t^* \delta(\xi_t \lrcorner \omega_t) + 0 + g_t^* \delta\varphi$

$= g_t^* \delta(\omega_t^{\#}(\xi_t)) + g_t^* \delta\varphi$

$= g_t^* \delta(-\varphi + \varphi) = 0$

Therefore $g_s^* \omega_s$ is constant in s, so $g_1^* \omega_1 = g_1^* \overline{\Omega} = g_0^* \omega_0 = \omega$. So if we put $f = h^{-1} \circ g_1$ we get $f^* \widetilde{\Omega} = g_1^* (h^{-1})^* \widetilde{\Omega} = g_1^* \overline{\Omega} = \omega$ in an open neighbourhood of Z_X in T^*X. We summarize:

Lemma: In the setting of 14.2 there exists a diffeomorphism $f: U \to V$ between open neighbourhoods of Z_X in T^*X such that $f^* \widetilde{\Omega} = \omega$ and $f | Z_X = Id_{Z_X}$.

14.4 Theorem: Let (X, Ω) be a symplectic manifold. Then the group Diff $\Omega(X)$ of all symplectic diffeomorphisms of X is a (splitting if X is compact) C_c^∞-submanifold of Diff(X). So it is a C_c^∞-Lie-group itself.

Warning: It is not clear whether $T_{Id} \Omega$ Diff(X) coincides with the space of vector-fields with compact support on X such that $\mathfrak{L}_\xi \Omega = 0$. We only know that these lie in the Lie-algebra.

Proof: Let τ: $T^*X \to X$ be a local addition as in 14.2, construct Ω^X on $X \times X$, $\widetilde{\Omega}$ and ω on T^*X as in 14.2. Let f be the diffeomorphism of a neighbourhood of Z_X in T^*X onto another with $f^* \widetilde{\Omega} = \omega$, $f | Z_X = Id_{Z_X}$.

Let $\rho = (\pi_X^*, \tau) \circ f$: $U \subseteq T^*X \to X \times X$, a diffeomorphism of an open neighbourhood U of Z_X in T^*X onto an open neighbourhood V of Δ_X in $X \times X$. Then $\rho(Z_X) = \Delta_X$ by construction and $\rho^* \Omega^X = f^* (\pi_X^*, \tau)^* \Omega^X = f^* \widetilde{\Omega} = \omega$.

Now let $U_0 \subseteq$ Diff(X) be the open neighbourhood of Id_X given by all $g \in$ Diff(X) such the graph Γ_g of g lies in V in $X \times X$ and $g \sim Id_X$. Since V is open, U_0 is open in the $(\mathfrak{FD})$-topology.

Claim 1: Let $g \in U_o$. Then $g \in \Omega\,\mathrm{Diff}(X)$ (i.e. $g^*\Omega = \Omega$) iff Γ_g is a Lagrangian submanifold of $(X \times X, \Omega^X)$.

Proof: $x \to (x, g(x))$ is the natural embedding of X onto Γ_g. Therefore $T\Gamma_g = \Gamma_{Tg} \subseteq TX \times TX$, so $T\Gamma_g = \Gamma_{Tg} = \{(\xi, Tg.\xi),\ \xi \in TX\} \subseteq TX \times TX$. Now $\dim \Gamma_g = \dim X = \frac{1}{2}\dim(X \times X)$, so Γ_g is Lagrangian in $X \times X$ iff $\Omega^X | T\Gamma_g = 0$. This is the case iff $pr_1{}^*\Omega - pr_2{}^*\Omega | \Gamma_{Tg} = 0$ or $(pr_1{}^*\Omega - pr_2{}^*\Omega)((\xi, Tg.\xi), (\eta, Tg.\eta)) = 0$ for all $\xi, \eta \in T_xX$, $x \in X$, i.e. $\Omega_x(\xi,\eta) = \Omega_{gx}(Tg.\xi, Tg.\eta)$ for all $\xi,\eta \in T_xX$, $x \in X$. But this means $\Omega = g^*\Omega$. So the claim is proved.

Now, since $\rho^*\Omega^X = \omega$, a submanifold $M \subseteq V \subseteq X \times X$ of $\dim M = \dim X$ is a Lagrangian submanifold of $(X \times X, \Omega^X)$ iff $\rho^{-1}(M)$ is a Lagrangian submanifold of (T^*X, ω).

Note that a submanifold $N \subseteq T^*X$ is the image of a 1-form iff $\pi_X{}^*|N\colon N \to X$ is a diffeomorphism.

Claim 2: Let $\varphi \in \Gamma(T^*X)$ be a 1-form on X. Then $\varphi(X) \subseteq (T^*X, \omega)$ is Lagrangian iff $\delta\varphi = 0$.

Proof: Remember the canonical 1-form α from 14.2: for any 1-form $\varphi \in \Gamma(T^*X)$ we have $\varphi^*\alpha = \varphi$. Now $\varphi(X)$ is Lagrangian iff $(\omega = -\delta\alpha) | T(\varphi(X)) = 0$, i.e. $-\varphi^*\delta\alpha = 0$. But $\varphi^*\delta\alpha = \delta\varphi^*\alpha = \delta\varphi$. So $\varphi(X)$ is Lagrangian iff $\delta\varphi = 0$.

Now let $U_o \subseteq \mathrm{Diff}(X)$ be so small that for each $g \in U_o$ the submanifold $\rho^{-1}(\Gamma_g)$ is the image of a 1-form (with compact support). Since $\rho^{-1}(\Gamma_{Id}) = \rho^{-1}(\Delta_X) = Z_X$ this is still a $(\mathcal{F}\mathcal{D})$-neighbourhood of Id_X. Then for $g \in U_o$ we have: $g \in \Omega\,\mathrm{Diff}(X)$ iff Γ_g is Lagrangian in $(X \times X, \Omega^X)$ iff the one form s whose image is $\rho^{-1}(\Gamma_g)$ is closed. So let $\mu\colon U_o \to \Gamma_c(T^*X)$ be the mapping assigning to each g the 1-form s with $s(X) = \rho^{-1}(\Gamma_g)$.

Claim 3: $\mu\colon U_o \to \Gamma_c(T^*X)$ is C_c^∞.

Proof: If $g \in U_o \subseteq \mathrm{Diff}(X)$, then $\alpha(g)\colon x \to (x, gx) \to \rho^{-1}(x,gx) \to \pi_X{}^*\rho^{-1}(x,gx)$ is a diffeomorphism: it is clearly smooth, it is immersive since $\pi_X | \rho^{-1}(\Gamma_g)$ is a diffeomorphism, and it is bijective since $\rho^{-1}(\Gamma_g) = \mu(g)(X)$. (There is no obvious relation between $\alpha(g)$ and g.) By construction, $\alpha\colon U_o \to \mathrm{Diff}(X)$ is C_c^∞ by the Ω-lemma and the chain rule.

Let $\beta(g)$ be the mapping: $x \to (x,gx) \to \rho^{-1}(x,gx)$, $\beta(g)\colon X \to T^*X$. Then $\beta\colon U_o \to C^\infty(X,T^*X)$ is C^∞_c by the Ω-lemma and the chain rule. We have $\alpha(g) = (\pi_X{}^*)_*(\beta(g))$. Clearly
$$\begin{aligned}\mu(g) &= \beta(g) \circ \alpha(g)^{-1} = \\ &= \beta(g) \circ \mathrm{Inv}(\alpha(g)) = \\ &= \mathrm{Comp}(\beta(g),\ \mathrm{Inv}(\alpha(g))).\end{aligned}$$
Since composition and Inversion are C^∞_c, the mapping μ is C^∞_c and the claim is proved.

<u>Claim 4:</u> $\mu^{-1}\colon \mu(U_o) =: V_o \subseteq \Gamma_c(T^*X) \to \mathrm{Diff}(X)$ is C^∞_c too.

<u>Proof:</u> For $s \in V_o$ consider the mapping $\gamma(s)\colon x \to s(g) \to \rho(s(x)) \to pr_1 \circ \rho \circ s(x)$. Then $\gamma\colon V_o \to \mathrm{Diff}(X)$ is a C^∞_c-mapping by the Ω-lemma.

Let $\nu\colon V_o \to \mathrm{Diff}(X)$ be given by $\nu(s)\colon x \to s(x) \to \rho(s(x)) \to pr_2 \circ \rho \circ s(x)$. Then ν is C^∞_c too.

It is clear that $\mu^{-1}(s) = \nu(s) \circ \gamma(s)^{-1} = \mathrm{Comp}(\nu(s), \mathrm{Inv}(\gamma(s)))$; so μ^{-1} is C^∞_c and $\mu\colon U_o \to V_o \subseteq \Gamma_c(T^*X)$ is a C^∞_c diffeomorphism.

Now $Z^1(\Gamma_c(T^*X)) = \mathrm{kernel}(\delta\colon \Gamma_c(T^*X) \to \Gamma_c(\Lambda^2 T^*X))$, the space of all closed 1-forms with compact support, is a closed linear subspace of $\Gamma_c(T^*X)$ since the exteriour derivative is a continuous linear differential operator on $\Gamma_c(T^*X)$. If the manifold X is compact then the theorem of Hodge says that the space of smooth closed 1-forms is a direct summand in the space of all 1-forms.

So we get: $g \in U_o \cap \mathrm{Diff}_\Omega(X)$ iff $\delta(\mu(g)) = 0$, or $\mu(g) \in V_o \cap \mathrm{kernel}\ \delta$. Thus $U_o \cap \mathrm{Diff}_\Omega(X)$ is a C^∞_c-submanifold of Diff(X); it is splitting if X is compact. Since $\mathrm{Diff}_\Omega(X)$ is a group one may transport around the open set U_o and finish the proof of the theorem. q.e.d

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List of symbols

§ 1

$j^k f$	k-jet extension of f	1
$J^k(X,Y)$	k-jet bundle	1
$\alpha\colon J^k(X,Y) \to X$	source mapping	1
$\omega\colon J^k(X,Y) \to Y$	target mapping	1
$J^k(n,m) = J^k(\mathbb{R}^n,\mathbb{R}^m)$		2
$P^k(n,m)$	polynomial mappings $\mathbb{R}^n \to \mathbb{R}^m$ of degree $\leq k$ without constant term	2
$L^j_{sym}(\mathbb{R}^n,\mathbb{R}^m)$	symmetric j-linear mappings	2
$\pi^k_{k'}\colon J^k(X,Y) \to J^{k'}(X,Y)$	projection	5
$\Gamma(E)$	space of smooth sections	7
$\Gamma_c(E)$	space of sections with compact support	7
$J^k(E)$	k-jet bundle of a vector bundle	8
$V(E)$	vertical bundle	10
$V_E\colon E \oplus E \to V(E)$	vertical lift	11
$\zeta_E\colon V(E) \to E$	vertical projection	11
$d_F\varphi$	fibre derivative of φ	12
f^*E	pullback of a vector bundle	12
$\varkappa_X\colon T^2X \to T^2X$	canonical flip mapping	13
exp	exponential mapping	16

§ 2

$\partial^j X$	manifold of corners of index j	19
${}^iTX$	space of inner tangent vectors	20

${}^{i}T^{2}X$	space of inner second tangent vectors	22

§ 3

$C(X,Y)$	space of continuous mappings	26
CO	compact open topology	26
Γ_f	graph of f	26
$W(U)$		26
WO	wholly open topology	26
WO^o	graph topolo y	26
$M(U)$		26
$N(f,\varepsilon)$		27
LO	locally finite open topology	29
LO^o-topology		29
$M(L,U)$, $N(f,L,\varepsilon)$, $N(f,\varphi)$		29

§ 4

$J^{\infty}(X,Y)$	∞-jet bundle	32
CO^k	compact C^k-topology	33
WO^k	Whitney C^k-topology	33
$W(U)$, $N(f,k,\varepsilon)$		34
$\mathfrak{D}$, $\mathfrak{D}_K$	spaces of test functions	35
$\mathfrak{D}^F$	space of test functions	36
$\mathfrak{D}$-topology		36
$M(L,U)$, $M'(L,U)$, $N(f,L,\varepsilon)$, $N(f,\varphi)$		37
$f \sim g$	equivalent mappings, f and g coincide off some compact set	40
$(F\mathfrak{D})$-topology, fine $\mathfrak{D}$-topology		40

§ 5

$Imm^r(X,Y)$	space of immersions	42
$Sub^r(X,Y)$	space of submersions	42
$C^r_{prop}(X,Y)$	space of proper mappings	42
$E^r(X,Y)$	space of embeddings	43
$E^r_{prop}(X,Y)$	space of closed embeddings	46
$Q^r(X,Y)$	space of surjective submersions	47
$Diff^r(X)$	group of diffeomorphisms	48

$C^r_\partial(X,Y)$	space of boundary respecting mapping	48
DX, Df	double of a manifold, mapping	48
§ 6		
$\pitchfork$	transversal	50
$X_1 \times_{(Y,f_1,f_2)} X_2$	topological pullback	51
$(s)J^k(X,Y)$	multijet bundle	58
$(s)j^k f$	multijet extension of f	58
§ 7		
Comp	composition	68
$f^* = C^\infty(f,Y)$		69
$h_* = C^\infty(X,h)$		69
Inv	inversion	70
$\mathrm{Diff}_c(X)$	group of diffeomorphisms with compact support	71
§ 8		
C^1_c	differentiability class	74
Df	derivative	74
$D_1 f$, $D_2 f$	partial derivatives	75
$D^2 f$	second derivative	76
C^p_c	differentiability class	77
§ 9		
$P^\infty(n,m)$	space of formal power series without constant terms	86
§ 10		
$\mathfrak{D}_f(X,TY)$	vector fields along f with compact support	91
$(U_f, \varphi_f, \Gamma_c(f^*TY))$	canonical chart of $C^\infty(X,Y)$ centered at f	92
$\psi_f = \varphi_f^{-1}$		92
$\tau_f\colon f^*TY \to X \times Y$		92
$S_q(Y,X)$	space of sections of q	95

$\mathfrak{D}(X,TY)$		99
ev_x	evaluation at x	105
$C^\infty_{nice}(X,Y)$	nice mappings	107

§ 11

$Ev: X\times C^\infty(X,Y)\to Y$	evaluation	114

§ 12

$P\omega$	pullback mapping	121
$\mathfrak{L}_s\omega$	Lie-derivative along s	122
$s \lrcorner \omega$	contraction along s	123
$Vol(X)$	bundle of densitites	126
$\Gamma(S^2T^*X)_+$	space of Riemannian metrics	127
$vol(g)$	density induced by a metric	127

§ 13

$\rho: Diff(X)\times E(X,Y)\to E(X,Y)$	action	129
$U(X,Y)=E(X,Y)/Diff(X)$	orbit space	129
$u: E(X,Y)\to U(X,Y)$		129

§ 14

Z_X	zero section of X	139
$Diff_\Omega(X)$	group of symplectic diffeomorphisms	141

Index

additively closed 93

border 18
border faithful mapping 48
boundary respecting local addition 91

cartesisn closed 115
C_c^∞-manifold 83
compact C^k-topology 33
compact open topology 26
contraction 123
corner 18

density 126
double of a manifold 48

elementary transversality theorem 57
equivalent mappings 40
exponential mapping 16,23

fibre bundle 4
fibre bundle with structure group 5
fibre derivative 12
fine $\mathfrak{D}$-topology 40

graph topology 26

index of a quadrant	18
inner spray	22
inner tangent vector	20
isotropic submanifold	138
jet	1
jet bundle	1,24
Kelley topological space	65
k-jet	1,25
Lagrangian submanifold	138
Lie-derivative	122
Lindelöf space	39,78
lemma of Poincaré	125
local addition	90
locally finite open topology	28
local topological affine space	41,121
manifold with corners	19
multijet bundle	58
multijet transversality theorem	59
neat submanifold	19
non-splitting C_c^∞ - submanifold	84
nuclear space	39,78
partition of unity	78
proper mapping	42
pullback of vector bundle	12
quadrant	18
section	7
source mapping	1
splitting C_c^∞ -submanifold	84
spray	14
strictly inner tangent vector	20
submanifold with corners	19

tangential spray	22
target mapping	1
Thom's transversality theorem	55
transversal	50
truncated composition	2
tubular neighbourhood	94
vertical bundle	10,11
vertical lift	11
vertical projection	11
Whitney C^k-topology	33
wholly open topology	26