MOMENTUM MAPPINGS

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The momentum map is essentially due to Lie, [5], pp. 300–343. The modern notion is due to Kostant [3], Souriau [9], and Kirillov [2].

The setting for the moment mapping is a smooth *symplectic manifold* (M,ω) or even a *Poisson manifold* (M,P) with the Poisson bracket on functions $\{f,g\} = P(df,dg)$ (where $P=\omega^{-1}: T^*M \to TM$ is the Poisson tensor). To each function f there is the associated Hamiltonian vector field $H_f = P(df) \in \mathfrak{X}(M,P)$, where $\mathfrak{X}(M,P)$ is the Lie algebra of all *locally Hamiltonian vector fields* $Y\in\mathfrak{X}(M)$ satisfying $\mathcal{L}_Y P = 0$ for the Lie derivative.

Let (M,ω) be a symplectic manifold for some time. Then this can be subsumed into the following exact sequence of Lie algebra homomorphisms

$$0 \to H^0(M) \to C^\infty(M) \xrightarrow{X} \mathfrak{X}(M,\omega) \xrightarrow{\gamma} H^1(M) \to 0,$$

where $\gamma(Y) = [i_Y\omega]$, the De Rham cohomology class of the contraction of Y into ω, and where the brackets not yet mentioned are all 0.

A Lie group G can act from the right on M by $\alpha : M\times G\to M$ in a way which respects ω, so that we get a homomorphism $\alpha' : \mathfrak{g} \to \mathfrak{X}(M,\omega)$, where $\mathfrak{g}$ is the Lie algebra of G. (For a left action we get an anti homomorphism of Lie algebras). One can lift α' to a linear mapping $j : \mathfrak{g} \to C^\infty(M)$ if $\gamma\circ\alpha' = 0$; if not we replace $\mathfrak{g}$ by its Lie subalgebra $\ker(\gamma\circ\alpha')\subset\mathfrak{g}$. The question is whether one can change j into an homomorphism of Lie algebras. The map $\mathfrak{g}\ni X,Y\mapsto \{jX,jY\}-j([X,Y])$ then induces a Chevalley 2-cocycle in $H^2(\mathfrak{g},H^0(M))$. If it vanishes one can change j as desired. If not, the cocycle describes a central extension of $\mathfrak{g}$ on which one may change j to a homomorphism of Lie algebras.

In any case, even for a Poisson manifold, for a homomorphism of Lie algebras $j:\mathfrak{g}\to C^\infty(M)$ (or more generally, if j is just a linear mapping), by flipping coordinates we get a *momentum mapping J of the $\mathfrak{g}$-action α'* from M into the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$,

$$J : M\to\mathfrak{g}^*, \qquad \langle J(x),X\rangle = j(X)(x), \qquad H_{j(X)} = \alpha'(X), \qquad x\in M, X\in\mathfrak{g},$$

where $\langle\ ,\ \rangle$ is the duality pairing.

For a particle in Euclidean 3-space and the rotation group acting on $T^*\mathbb{R}^3$ this is just the *angular momentum*, hence its name. The momentum map is infinitesimally equivariant for the $\mathfrak{g}$-actions if j is a homomorphism of Lie algebras. It is a Poisson morphism for the canonical Poisson structure on $\mathfrak{g}^*$, whose symplectic leaves are

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the coadjoint orbits. The momentum map can be used to reduce the number of coordinates of the original mechanical problem, hence plays an important role in the theory of *reductions of Hamiltonian systems*. [6], [4] and [7] are convenient references, [7] has a large and updated bibliography. The momentum map has a strong tendency to have *convex image*, and is important for *representation theory*, see [2] and [8]. Recently, there is also a proposal for a group-valued momentum mapping, see [1].

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