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# $n$-TRANSITIVITY OF CERTAIN DIFFEOMORPHISM GROUPS

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Abstract. It is shown that some groups of diffeomorphisms of a manifold act $n$-transitively for each finite $n$.

Let $M$ be a connected smooth manifold of dimension $\dim M \geq 2$. We say that a subgroup $G$ of the group $\operatorname{Diff}(M)$ of all smooth diffeomorphisms acts *$n$-transitively on $M$*, if for any two ordered sets of $n$ different points $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ in $M$ there is a smooth diffeomorphism $f \in G$ such that $f(x_i) = y_i$ for each $i$.

**Theorem.** *Let $M$ be a connected smooth (or real analytic) manifold of dimension $\dim M \geq 2$. Then the following subgroups of the group $\operatorname{Diff}(M)$ of all smooth diffeomorphisms with compact support act $n$-transitively on $M$, for each finite $n$:*

(1) *The group $\operatorname{Diff}_c(M)$ of all smooth diffeomorphisms with compact support.*
(2) *The group $\operatorname{Diff}^\omega(M)$ of all real analytic diffeomorphisms.*
(3) *If $(M, \sigma)$ is a symplectic manifold, the group $\operatorname{Diff}_c(M, \sigma)$ of all symplectic diffeomorphisms with compact support, and even the subgroup of all globally Hamiltonian symplectomorphisms.*
(4) *If $(M, \sigma)$ is a real analytic symplectic manifold, the group $\operatorname{Diff}^\omega(M, \sigma)$ of all real analytic symplectic diffeomorphisms, and even the subgroup of all globally Hamiltonian real analytic symplectomorphisms.*
(5) *If $(M, \mu)$ is a manifold with a smooth volume density, the group $\operatorname{Diff}_c(M, \mu)$ of all volume preserving diffeomorphisms with compact support.*
(6) *If $(M, \mu)$ is a manifold with a real analytic volume density, the group $\operatorname{Diff}^\omega(M, \mu)$ of all real analytic volume preserving diffeomorphisms.*
(7) *If $(M, \alpha)$ is a contact manifold, the group $\operatorname{Diff}_c(M, \alpha)$ of all contact diffeomorphisms with compact support.*
(8) *If $(M, \alpha)$ is a real analytic contact manifold, the group $\operatorname{Diff}^\omega(M, \alpha)$ of all real analytic contact diffeomorphisms.*

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Result (1) is folklore, the first trace is in [8]. The results (3), (5), and (7) are due to [3] for 1-transitivity, and to [1] in the general case. Result (2) is from [7]. We shall give here a short uniform proof, following an argument from [7]. That this argument suffices to prove all results was noted by the referee, many thanks to him.

*Proof.* Let us fix a finite $n \in \mathbb{N}$. Let $M^{(n)}$ denote the open submanifold of all $n$-tuples $(x_1, \dots, x_n) \in M^n$ of pairwise distinct points. Since $M$ is connected and of dimension $\geq 2$, each $M^{(n)}$ is connected.

The group $\operatorname{Diff}(M)$ acts on $M^{(n)}$ by the diagonal action, and we have to show, that any of the subgroups $G$ described above acts transitively. We shall show below that for each $G$ the $G$-orbit through any $n$-tuple $(x_1, \dots, x_n) \in M^{(n)}$ contains an open neighborhood of $(x_1, \dots, x_n)$ in $M^{(n)}$, thus any orbit is open; since $M^{(n)}$ is connected, there can then be only one orbit. □

**Lemma.** *Let $M$ be a real analytic manifold. Then for any real analytic vector bundle $E \to M$ the space $C^\omega(E)$ of real analytic sections of $E$ is dense in the space $C^\infty(E)$ of smooth sections. In particular the space $\mathfrak{X}^\omega(M)$ of real analytic vector fields is dense in the space $\mathfrak{X}(M)$ of smooth vector fields, in the Whitney $C^\infty$-topology.*

*Proof.* For functions instead of sections this is [2], proposition 8. Using results from [2] it can easily be extended to sections, as is done in [6], 7.5. □

**The cases** (2) **and** (1)**.** We choose a complete Riemannian metric $g$ on $M$ and we let $(Y_{ij})_{j=1}^m$ be an orthonormal basis of $T_{x_i}M$ with respect to $g$, for all $i$. Then we choose real analytic vector fields $X_k$ for $1 \leq k \leq N = nm$ which satisfy the following conditions:

$$\begin{aligned} &|X_k(x_i) - Y_{ij}|_g < \varepsilon &&\text{for } k = (i-1)m + j, \\ &|X_k(x_i)|_g < \varepsilon &&\text{for all } k \notin [(i-1)m+1, im], \\ &|X_k(x)|_g < 2 &&\text{for all } x \in M \text{ and all } k. \end{aligned} \tag{9}$$

Since these conditions describe a Whitney $C^0$ open set, such vector fields exist by the lemma. The fields are bounded with respect to a complete Riemannian metric, so they have complete real analytic flows $\mathrm{Fl}^{X_k}$, see e.g. [4]. We consider the real analytic mapping

$$f : \mathbb{R}^N \to M^{(n)}$$
$$f(t_1, \dots, t_N) := \begin{pmatrix} (\mathrm{Fl}^{X_1}_{t_1} \circ \dots \circ \mathrm{Fl}^{X_N}_{t_N})(x_1) \\ \dots \\ (\mathrm{Fl}^{X_1}_{t_1} \circ \dots \circ \mathrm{Fl}^{X_N}_{t_N})(x_n) \end{pmatrix}$$

which has values in the $\operatorname{Diff}^\omega(M)$-orbit through $(x_1, \dots, x_n)$. To get the tangent mapping at 0 of $f$ we consider the partial derivatives

$$\frac{\partial}{\partial t_k}|_0 f(0, \dots, 0, t_k, 0, \dots, 0) = (X_k(x_1), \dots, X_k(x_n)).$$

If $\varepsilon > 0$ is small enough, this is near an orthonormal basis of $T_{(x_1,\dots,x_n)}M^{(n)}$ with respect to the product metric $g \times \dots \times g$. So $T_0 f$ is invertible and the image of $f$ contains thus an open subset.

In case (1), we can choose smooth vector fields $X_k$ with compact support which satisfy conditions (9). $\square$

For the remaining cases we just indicate the changes which are necessary in this proof.

**The cases** (4) **and** (3)**.** Let $(M, \sigma)$ be a connected real analytic symplectic smooth manifold of dimension $m \geq 2$. We choose real analytic functions $f_k$ for $1 \leq k \leq N = nm$ whose Hamiltonian vector fields $X_k = \operatorname{grad}^\sigma(f_k)$ satisfy conditions (9). Since these conditions describe Whitney $C^1$ open subsets, such functions exist by [2], proposition 8. Now we may finish the proof as above. $\square$

**Contact manifolds.** Let $M$ be a smooth manifold of dimension $m = 2n+1 \geq 3$. A *contact form* on $M$ is a 1-form $\alpha \in \Omega^1(M)$ such that $\alpha \wedge (d\alpha)^n \in \Omega^{2n+1}(M)$ is nowhere zero. This is sometimes called an *exact* contact structure. The pair $(M, \alpha)$ is called a *contact manifold* (see [5]). The *contact vector field* $X_\alpha \in \mathfrak{X}(M)$ is the unique vector field satisfying $i_{X_\alpha}\alpha = 1$ and $i_{X_\alpha} d\alpha = 0$.

A diffeomorphism $f \in \operatorname{Diff}(M)$ with $f^*\alpha = \lambda_f.\alpha$ for a nowhere vanishing function $\lambda_f \in C^\infty(M, \mathbb{R} \setminus 0)$ is called a *contact diffeomorphism*. Note that then $\lambda_f = i_{X_\alpha}(\lambda_f.\alpha) = i_{X_\alpha} f^*\alpha = f^*(i_{(f^{-1})^* X_\alpha}\alpha) = f^*(i_{f_* X_\alpha}\alpha)$. The group of all contact diffeomorphisms will be denoted by $\operatorname{Diff}(M, \alpha)$.

A vector field $X \in \mathfrak{X}(M)$ is called a contact vector field if $\mathcal{L}_X \alpha = \mu_X.\alpha$ for a smooth function $\mu_X \in C^\infty(M, \mathbb{R})$. The linear space of all contact vector fields will be denoted by $\mathfrak{X}_\alpha(M)$ and it is clearly a Lie algebra. Contraction with $\alpha$ is a linear mapping again denoted by $\alpha : \mathfrak{X}_\alpha(M) \to C^\infty(M, \mathbb{R})$. It is bijective since we may apply $i_{X_\alpha}$ to the equation $\mathcal{L}_X\alpha = i_X\, d\alpha + d\alpha(X) = \mu_X.\alpha$ and get $0 + i_{X_\alpha} d\alpha(X) = \mu_X$; but the equation uniquely determines $X$ from $\alpha(X)$ and $\mu_X$. The inverse $f \mapsto \operatorname{grad}^\alpha(f)$ of $\alpha : \mathfrak{X}_\alpha(M) \to C^\infty(M, \mathbb{R})$ is a linear differential operator of order 1.

**The cases** (8) **and** (7)**.** Let $(M, \alpha)$ be a connected real analytic contact manifold of dimension $m \geq 2$. We choose real analytic functions $f_k$ for $1 \leq k \leq N = nm$ such that their contact vector fields $X_k = \operatorname{grad}^\alpha(f_k)$ satisfy conditions (9). Since these conditions describe Whitney $C^1$ open subsets, such functions exist by [2], proposition 8. Now we may finish the proof as above. $\square$

**The cases** (6) **and** (5)**.** Let $(M, \mu)$ be a connected real analytic manifold of dimension $m \geq 2$ with a real analytic positive volume density. We can find a real analytic Riemannian metric $\gamma$ on $M$ whose volume density is $\mu$. We also choose a complete Riemannian metric $g$.

First we assume that $M$ is orientable. Then the divergence of a vector field $X \in \operatorname{Vect}(M)$ is $\operatorname{div} X = *d * X^\flat$, where $X^\flat = \gamma(X) \in \Omega^1(M)$ (here we view $\gamma : TM \to T^*M$) and $*$ is the Hodge star operator of $\gamma$. We choose real analytic $(m-2)$-forms $\beta_k$ for $1 \leq k \leq N = nm$ such that the vector fields $X_k = (-1)^{m+1}\gamma^{-1} * d\beta_k$ satisfy conditions (9). Since these conditions describe Whitney $C^1$ open subsets, such $(m-2)$-forms exist by the lemma. The real analytic vector fields $X_k$ are then divergence free since $\operatorname{div} X_k = *d*\gamma X_k = *dd\beta_k = 0$. Now we may finish the proof as usual.

For non-orientable $M$, we let $\pi : \tilde M \to M$ be the real analytic connected oriented double cover of $M$, and let $\varphi : \tilde M \to \tilde M$ be the real analytic involutive covering map. We let $\pi^{-1}(x_i) = \{x_i^1, x_i^2\}$, and we pull back both metrics to $\tilde M$, so $\tilde\gamma := \pi^*\gamma$ and $\tilde g := \pi^* g$. We choose real analytic $(m-2)$-forms $\beta_k \in \Omega^{m-2}(\tilde M)$ for $1 \le k \le N = nm$ whose vector fields $X_{\beta_k} = (-1)^{m+1}\tilde\gamma^{-1} * d\beta_k$ satisfy the following conditions, where we put $Y_{ij}^p := T_{x_{ij}^p}\pi^{-1}.Y_{ij}$ for $p = 1, 2$:

$$\begin{aligned} &|X_{\beta_k}(x_i^p) - Y_{ij}^p|_{\tilde g} < \varepsilon && \text{for } k = (i-1)m + j, p = 1, 2,\\ &|X_{\beta_k}(x_i^p)|_{\tilde g} < \varepsilon && \text{for all } k \notin [(i-1)m + 1, im], p = 1, 2,\\ &|X_{\beta_k}|_{\tilde g} < 2 && \text{for all } x \in \tilde M \text{ and all } k. \end{aligned} \tag{10}$$

Since these conditions describe Whitney $C^1$ open subsets, such $(m-2)$-forms exist by the lemma. Then the vector fields $\frac{1}{2}(X_{\beta_k} + \varphi_* X_{\beta_k})$ still satisfy the conditions (10), are still divergence free and induce divergence free vector fields $Z_{\beta_k} \in \mathfrak{X}(M)$, so that $\mathcal{L}_{Z_{\beta_k}}\mu$ is the zero density, which satisfy the conditions (9) on $M$ as in the oriented case, and we may finish the proof as above. □

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